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Results in Mathematics

Banach–Mazur Distance from the Parallelogram to the Affine-Regular Hexagon and Other Affine-Regular Even-Gons

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In memory of Heinrich Wefelscheid

Abstract. We show that the Banach–Mazur distance between the parallelogram and the affine-regular hexagon is $\frac{3}{2}$ and we conclude that the diameter of the family of centrally-symmetric planar convex bodies is just $\frac{3}{2}$. A proof of this fact does not seem to be published earlier. Asplund announced this without a proof in his paper proving that the Banach–Mazur distance of any planar centrally-symmetric bodies is at most $\frac{3}{2}$. Analogously, we deal with the Banach–Mazur distances between the parallelogram and the remaining affine-regular even-gons.

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1. Introduction

Denote by $\mathcal{M}^d$ the family of all centrally symmetric convex bodies of the Euclidean space E^d centered at the origin o of E^d. Below always when we say on a homothety, we mean that its center is at o. For any $K \in \mathcal{M}^d$ and any positive λ, by λK we denote the homothety image of K with ratio λ.

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The *Banach–Mazur distance* (or shortly, the BM-*distance*) of $C, D \in \mathcal{M}^d$ is defined as

$$\delta_{BM}(C,D) = \inf_{a,\lambda}\{\lambda;\, a(C) \subset D \subset \lambda a(C)\},$$

where a stands for an affine transformation.

This definition is presented by Banach [3] in behalf of him and Mazur. From the role of the affine transformation in this definition we see that a different formulation of this definition is formally more precise, namely when we consider the Banach–Mazur distance of the equivalence classes of centrally symmetric bodies with respect to affine transformations. But in this note it is more convenient to deal with the BM-distance of convex bodies from $\mathcal{M}^d$. It is well known that the Banach–Mazur distance is a multiplicative metric, i.e., that $\log \delta_{BM}$ is a metric. In particular, we have $\delta_{BM}(C,D) = \delta_{BM}(D,C)$ and the multiplicative triangle inequality $\delta_{BM}(C,D)\cdot\delta_{BM}(D,E) \geq \delta_{BM}(C,E)$ for every $C, D, E \in \mathcal{M}^d$. Later the notion of the BM-distance has been extended to BM-distance of pairs of arbitrary convex bodies C, D. For a survey of results on the BM-distance we propose the books by Tomczak-Jaegerman [8], Toth [9], and Aubrun and Szarek [2].

By P_n we denote the regular n-gon with vertices $(\cos\frac{2j}{n}\pi, \sin\frac{2j}{n}\pi)$ for $j = 0, \dots, n-1$.

Our main aim is to show that $\delta_{BM}(P_4, P_6) = \frac{3}{2}$ and to establish all the optimal positions of the parallelograms $a(P_4)$ with respect to the hexagon P_6. Recall that the value $\frac{3}{2}$ is stated in Theorem 1 of [1] by Asplund. It says that $\delta_{BM}(P_4, C) \leq \frac{3}{2}$ for any $C \in \mathcal{M}^2$ with equality only for $C = P_6$. But the proof of Lemma 3 of [1] that $\delta_{BM}(P_4, P_6) = \frac{3}{2}$ is omitted. This statement is quoted by Stromquist on p. 206 of [7] and illustrated in Fig. 1 there. Since the hexagon from this Fig. 1 is not inscribed in the larger parallelogram, then looking to the second thesis of our Proposition we get further questions. The basic task of [7] is to construct a center R of $\mathcal{M}^2$ (see p. 207 and compare Fig. 2). With the help of Fig. 3 it is shown that both P_4 and P_6 are in the distance $\sqrt{3/2}$ from R. This implies $\delta_{BM}(P_4, P_6) \leq \frac{3}{2}$, but does not imply the equality. These doubts mobilized the writer to present a detailed proof of the equality $\delta_{BM}(P_4, P_6) = \frac{3}{2}$ in the present note in order to be sure that the diameter of $\mathcal{M}^2$ is $\frac{3}{2}$.

An additional aim is explained here. The idea of the proof of our Theorem 1 that $\delta_{BM}(P_4, P_6) = \frac{3}{2}$ encouraged the author to consider the analogous task for all the regular even-gons with more vertices in place of P_6. Theorem 2 establishes the Banach–Mazur distances from P_4 to all P_{8j} and P_{8j+4}. This task for the remaining even-gons, so to P_{8j+2} and P_{8j+6} appears to be more complicated. We estimate and conjecture the values of these distances.

2. Pairs of Homothetic Parallelograms Contained and Containing a Centrally Symmetric Convex Body

By an inscribed parallelogram in a convex body C we mean a parallelogram with all vertices in the boundary of C and by a circumscribed parallelogram about C we mean a parallelogram containing C whose all sides have non-empty intersections with C.

By the width $\mathrm{width}(S)$ of a strip S between two parallel hyperplanes of E^d we mean the distance of these hyperplanes. We omit an easy proof of the following lemma, whose two-dimensional version is applied in the proofs of Theorems 1 and 2.

Lemma. *Let H^1_+, H^1_- and H^2_+, H^2_- be two pairs of hyperplanes of E^d and let all of them be parallel. Assume that each pair is symmetric with respect to o and that all these hyperplanes do not contain o. Let L be a straight line through o not parallel to these hyperplanes. Assume that $H^2_+, H^1_+, H^1_-, H^2_-$ intersect L in this order. Denote by $(a^i_1, \dots, a^i_n)$ the points of intersection of L with H^i_+, where $i \in \{1, 2\}$. Consider the strips $S^i = \mathrm{conv}(H^i_+ \cup H^i_-)$, where $i \in \{1, 2\}$. We claim that* $\mathrm{width}(S^2)/\mathrm{width}(S^1) = a^2_j/a^1_j$ *for any $j \in \{1, \dots, n\}$ if $a^1_j \neq 0$.*

The following proposition is applied later a few times when evaluating particular BM-distances.

Proposition. *Let $C \in \mathcal{M}^2$. Assume that $P \subset C \subset \mu P$ for a parallelogram $P \in \mathcal{M}^2$ and a positive μ. Then there exists a parallelogram P' inscribed in C such that $\mu' P'$ is circumscribed about C for a $\mu' \leq \mu$.*

Proof. Since the thesis is obvious for $\mu = 1$, below we assume that $\mu > 1$.

If P in the part of P' fulfills the thesis, there is nothing to prove. In the opposite case we intend to construct the required P'.

Of course, there is a homothetic image $P_\alpha \subset P$ of P such that for least one pair of opposite sides Z^+, Z^- of P_α the sides $\mu Z^+, \mu Z^-$ of μP_α touch C from outside. Denote the other pair of sides of P_α by W^+, W^- and assume that their order is $W^+, Z^+, W^-, Z^-$when we move counterclockwise.

If $\mu W^-, \mu W^+$ do not touch C, then we lessen P_α up to P_β by moving W^+, W^- such that their vertices remain in Z^+, Z^- symmetrically closer to o up to the position F^+, F^- when both μF^+ and μF^- touch C. The other two sides of the parallelogram P_β are denoted by G^+, G^-. Clearly, $G^+ \subset Z^+$ and $G^- \subset Z^-$. The parallelogram μP_β is circumscribed about C. Of course, when we go counterclockwise, then the order of the sides of P_β is F^+, G^+, F^-, G^-.

Clearly P_β is a subset of C, but it may be not inscribed in C.

Take the homothetic copy $P_\gamma \subset C$ of P_β such that at least one pair of opposite vertices t, v of P_γ is in $\mathrm{bd}(C)$. Denote the other two vertices of P_γ by u, w such that t, u, v, w be in this order on $\mathrm{bd}(C)$.

Let $a, b \in \mathrm{bd}(C)$ be the points such that $w \in ta$ and $w \in vb$

If u, w are not in $\mathrm{bd}(C)$, let us cleverly enlarge P_γ up to a parallelogram P_δ (not obligatory homothetic) inscribed in C such that a homothetic copy of P_δ with a ratio at most μ is circumscribed about C.

Here we explain how to provide this task. For every boundary point c of $\mathrm{bd}(C)$ on the arc $\widehat{ab}$ provide the straight lines $K_1^+(c), K_2^+(c)$ containing cv and ct, respectively. Let $K_1^-(c), K_2^-(c)$ be the line symmetric to $K_1^+(c), K_2^+(c)$, respectively. Moreover, for $i = 1, 2$ provide the two pairs of the straight lines $L_i^+(c)$ and $L_i^-(c)$ supporting C and parallel to the pairs $K_i^+(c)$ and $K_i^-(c)$ such that the order of them is $L_i^+(c)$, $K_i^+(c)$, $K_i^-(c)$, $L_i^-(c)$.

Let $L_i(c)$ be the strip between $L_i^+(c)$ and $L_i^-(c)$ for $i = 1, 2$ and $K_i(c)$ be the strip between $K_i^+(c)$ and $K_i^-(c)$ for $i = 1, 2$.

Provide the straight line through o and points d^+, d^- of support of C by $L_1^+(c)$ and $L_1^-(c)$, respectively. Denote by f^+, f^- its intersections with F^+, F^-, respectively. Denote by k^+, k^- its intersections with $K_i^+(c), K_i^-(c)$, respectively. Denote by h^+, h^- its intersections with $\mu F^+, \mu F^-$. Since h^+, d^+, k^+, $f+$, f^-, k^-, d^-, h^- are in this order on this straight line, we obtain $|h^+h^-|/|f^+f^-| \le |d^+d^-|/|k^+k^-|$. Hence by Lemma we obtain that

$$\mathrm{width}(L_i(c))/\mathrm{width}(K_i(c)) \le \mu$$

for $i = 1$. Analogously, we conclude this for $i = 2$.

If c is sufficiently close to a, then

$$\mathrm{width}(L_1(c))/\mathrm{width}(K_1(c)) > \mathrm{width}(L_2(c))/\mathrm{width}(K_2(c)).$$

If c is sufficiently close to b, then we have the opposite inequality. Moreover, observe that when c moves on $\widehat{ab}$ from a to b, then the strips $L_i(c)$ and $K_i(c)$, where $i = 1, 2$, change continuously. Consequently, there is at least one position c_0 of c for which

$$\mathrm{width}(L_1(c_0))/\mathrm{width}(K_1(c_0)) = \mathrm{width}(L_2(c_0))/\mathrm{width}(K_2(c_0)).$$

Therefore the thesis of our proposition is true for the parallelograms $P' = K_1(c_0) \cap K_2(c_0)$. Still the homothetic parallelogram $\mu' P = L_1(c_0) \cap L_2(c_0)$, where $\mu' \le \mu$, is circumscribed about C. $\square$

Corollary. *Denote the Banach–Mazur distance between $C \in \mathcal{M}^2$ and P by μ. Assume that $P \subset C \subset \mu P$ for a particular affine image $P \in \mathcal{M}^2$ of P_4. Then the parallelogram P is inscribed in C and μP is circumscribed about C.*

The author does not know if Proposition holds true in higher dimensions for the parallelotope or the cross-polytope in place of the parallelogram.

3. Banach–Mazur Distance Between the Parallelogram and the Affine-Regular Hexagon

Theorem 1. *We have $\delta_{BM}(P_4, P_6) = \frac{3}{2}$.*

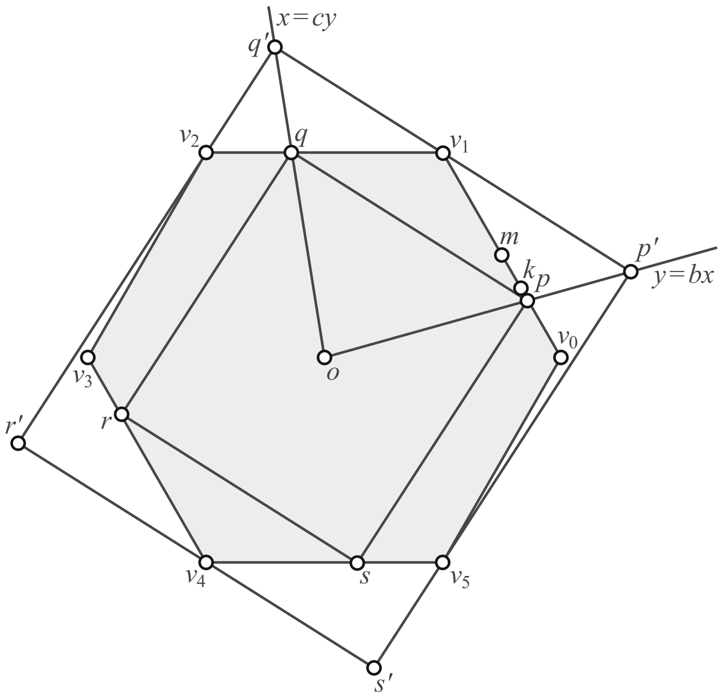

FIGURE 1. The positions of $P \in \mathcal{P}$ and $h(a)P$ with respect to the hexagon

Proof. The parallelogram with the four vertices at $(\pm 1, 0)$ and $(0, \pm\frac{1}{2}\sqrt{3})$ is contained in P_6 and its homothetic image with ratio $\frac{3}{2}$ contains P_6. Consequently, $\delta_{BM}(P_4, P_6) \le \frac{3}{2}$.

Having in mind Proposition, in order to show that $\delta_{BM}(P_4, P_6) \ge \frac{3}{2}$, it is sufficient to consider only any parallelogram $P = a(P_4)$ inscribed in P_6 such that a positive homothetic copy λP of P is circumscribed about P_6, and to show that this homothety ratio λ is at least $\frac{3}{2}$.

Denote by $\mathcal{P}$ the class of all such parallelograms P.

Consider a parallelogram $P = pqrs$ from $\mathcal{P}$. Some two consecutive vertices of P must be in two consecutive sides of P_6. The reason is that in the opposite case no positive homothetic image of our P is circumscribed about P_6 in contradiction to $P \in \mathcal{P}$. In order to fix attention, thanks to the symmetries of P_6, we do not make our considerations narrower assuming that $p \in v_0v_1$ and $q \in v_1v_2$. For the same reason, we may additionally assume that $p \in v_0m$, where m denotes the middle of the side v_0v_1.

Take the line $y = bx$ passing through p. It is easy to show that p has the form $(\frac{\sqrt{3}}{b+\sqrt{3}}, \frac{\sqrt{3}b}{b+\sqrt{3}})$, where b belongs to the interval $[0, \frac{\sqrt{3}}{3}]$. Here $b = 0$ generates v_0, while $b = \frac{\sqrt{3}}{3}$ generates m.

Our q is the intersection of the segment v_1v_2 with a straight line $x = cy$, where $c \in [-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}]$. So q has the form $(\frac{\sqrt{3}}{2}c, \frac{\sqrt{3}}{2})$. An easy calculation shows that the directional coefficient of the straight line containing pq is $\sigma = \frac{b-\sqrt{3}}{2-bc-\sqrt{3}c}$ and that the directional coefficient of the straight line containing sp is $\varsigma = \frac{3b+\sqrt{3}}{2+bc+\sqrt{3}c}$.

Denote by S the strip between the straight lines containing pq and rs, and by S^+ the narrowest strip parallel to S which contains P_6. Denote by T the strip between the straight lines containing qr and sp, and by T^+ the narrowest strip parallel to T which contains P_6.

We see that $P = S \cap T$. Of course, $S^+ \cap T^+$ is the parallelogram with sides parallel to the sides of P which is circumscribed about P_6. Since we are looking only for parallelograms $P \in \mathcal{P}$, the parallelogram $S^+ \cap T^+$ should be a positive homothetic copy of P.

Now, our task is to describe such parallelograms P.

We find the first coordinate of the intersection of the straight line through pq with the straight line$y = \sqrt{3}x$. It is $x_0 = \frac{\sqrt{3}}{2} \cdot \frac{1-c\sigma}{\sqrt{3}-\sigma}$. Next we evaluate the ratio of the first coordinate of v_1 to x_0 which is $\frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}-\sigma}{1-c\sigma}$.

By Lemma this ratio equals to $\mathrm{width}(S^+)/\mathrm{width}(S)$. By the substitution of σ we get $\mathrm{width}(S^+)/\mathrm{width}(S) = \frac{\sqrt{3}}{3} \cdot \frac{3\sqrt{3}-\sqrt{3}bc-3c-b}{2-2bc}$. In a similar way we obtain that $\mathrm{width}(T^+)/\mathrm{width}(T) = \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}+\varsigma}{1-c\varsigma} = \frac{\sqrt{3}}{3} \cdot \frac{3\sqrt{3}+3b+3c+\sqrt{3}bc}{2-2bc}$.

Solving the equation $\mathrm{width}(S^+)/\mathrm{width}(S) = \mathrm{width}(T^+)/\mathrm{width}(T)$, we conclude that only for $c = \frac{-2b}{\sqrt{3}b+3}$ its both sides are equal. We see that $q \in nv_2$, where n is the midpoint of v_1v_2. Since our P is a function of b, we denote it by $P(b)$. Substituting $c = \frac{-2b}{\sqrt{3}b+3}$ into $\mathrm{width}(S^+)/\mathrm{width}(S)$ we obtain that the common value of $w(S^+)/w(S)$ and $\mathrm{width}(T^+)/\mathrm{width}(T)$ is $h(b) = \frac{b^2+4\sqrt{3}b+9}{4b^2+2\sqrt{3}b+6}$, where $h(b)$ stands in place of $h(P(b))$. This ends our task.

Every $P(b)$ is inscribed in P_6 and every $h(b)P(b)$ is circumscribed about P_6 (see Fig. 1). The vertices of $h(b)P(b)$ being the images of vertices p, q, r, s of $P(b)$ are denoted by p', q', t', u', respectively. Of course, the straight line containing the side $p'q'$ of $h(b)P(b)$ supports P_6 at v_1.

We are considering here only the situation when the straight line containing the side $s'p'$ of $h(b)P(b)$ supports P_6 at v_5. This holds true if and only if the directional coefficient ς of the straight line containing $s'p'$ is at most the directional coefficient of the straight line containing v_5v_0, so when it is at most $\sqrt{3}$. Recall that $\varsigma = \frac{3b+\sqrt{3}}{2+bc+\sqrt{3}c}$, which after substituting $c = \frac{-2b}{\sqrt{3}b+3}$ gives $\varsigma = \frac{3\sqrt{3}b+3}{2(\sqrt{3}-b)}$. Solving $\frac{3\sqrt{3}b+3}{2(\sqrt{3}-b)} = \sqrt{3}$ we obtain $b = \frac{\sqrt{3}}{5}$.

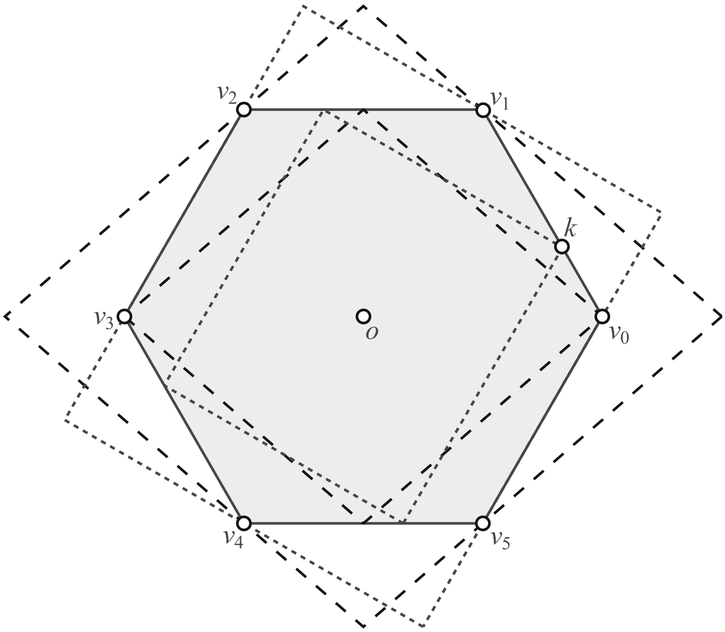

Figure 2. Two best positions of $P \in \mathcal{P}$ and $h(a)P$ with respect to the hexagon

So the straight line containing the side $h(b)ur$ of $h(b)P(b)$ supports P_6 at v_5 if and only if b is in the interval $[0, \frac{\sqrt{3}}{5}]$. In other words, if and only if $r \in v_0k$, where k is the point of intersection of $y = \frac{\sqrt{3}}{5}x$ with v_0v_1.

The derivative of the function $h(b)$ is $\frac{-7\sqrt{3}b^2-30b+3\sqrt{3}}{4(b^2+\sqrt{3}b+6)^2}$. An evaluation shows that this derivative is 0 if and only if $b = \frac{1}{14}(-10\sqrt{3} \pm \sqrt{384})$.

Only $b = \frac{1}{14}(-10\sqrt{3} + \sqrt{384}) \approx 0.1625$ in the interval $[0, \frac{\sqrt{3}}{5}]$. The value of $h(b)$ for this b is approximately 1.5224, so over $\frac{3}{2}$. Since $h(0) = \frac{3}{2}$ and $h(\frac{\sqrt{3}}{5}) = \frac{3}{2}$, we conclude that the global minimum of $h(b)$ in the interval $[0, \frac{\sqrt{3}}{5}]$ is $\frac{3}{2}$ and that it is attained only for $b = 0$ and $b = \frac{\sqrt{3}}{5}$. In Fig. 2 we see the two pairs $P(b)$, $\frac{3}{2}P(b)$ for these two values of b.

It remains to explain what happens for $b \in [\frac{\sqrt{3}}{5}, \frac{\sqrt{3}}{3}]$, so when $p \in km$. Observe that then the boundary of the smallest positive homothetic copy of P containing P_6 touches it at points v_1, v_3, v_4, v_0. Hence this situation is symmetric to the preceding one with respect to the straight line through o perpendicular to v_5v_0. Considering b only in the interval $[0, \frac{\sqrt{3}}{3}]$ is sufficient since we have in mind rotations of $P(b)$ by $60°$ and $120°$ and the axial symmetries with respect to the lines containing v_0v_3, v_1v_4, v_2v_5. $\square$

Let us add that when b changes from 0 to $\frac{\sqrt{3}}{5}$ in the paragraph before the last of the proof, the point p changes from $v_0 = (0,1)$ to $k = (\frac{5}{6}, \frac{\sqrt{3}}{6})$ beating $\frac{1}{3}$ of the unit. Simultaneously q changes from $(0, \frac{\sqrt{3}}{2})$ to $(-\frac{1}{6}, \frac{\sqrt{3}}{2})$ beating $\frac{1}{6}$ of the unit.

Remark. The only two positions of $P \subset P_6$ such that $P_6 \subset \frac{3}{2}P$ (besides their rotations by 60° and 120° and axial symmetries with respect to the lines containing v_0v_3, v_1v_4, v_2v_5) are the parallelograms with vertices $(1,0)$, $(0, \frac{\sqrt{3}}{2})$, $(-1,0)$, $(0, -\frac{\sqrt{3}}{2})$ and with vertices $(\frac{5}{6}, \frac{\sqrt{3}}{6})$, $(-\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2})$, $(-\frac{5}{6}, -\frac{\sqrt{3}}{6})$, $(\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2})$. Again see Fig. 2.

4. Banach–Mazur Distance Between the Parallelogram and the Affine-Regular Even-Gons

In this section we consider the BM-distances from P_4 to the regular even-gons with more than six vertices. In Theorem 2 we find these distances to P_{8j} and P_{8j+4}, and we present the estimates from above from P_4 to P_{8j+2} and P_{8j+6}. Next we conjecture that the values of these two upper estimates are just the BM-distances from P_4 to P_{8j+2} and P_{8j+6}.

Theorem 2. *We have*

(I) $\delta_{BM}(P_4, P_{8j}) = \sqrt{2}$,

(II) $\delta_{BM}(P_4, P_{8j+2}) \leq \frac{1}{2} \sec \frac{2j}{8j+2}\pi + \cos \frac{2j}{8j+2}\pi$,

(III) $\delta_{BM}(P_4, P_{8j+4}) = \sqrt{2} \cos \frac{1}{8j+4}\pi$,

(IV) $\delta_{BM}(P_4, P_{8j+6}) \leq \sin \frac{2j+2}{8j+6}\pi \cdot \csc \frac{4j+2}{8j+6}\pi + \cos \frac{2j+2}{8j+6}\pi$.

Proof. The inequalities showing that the left sides are at most the right sides result from positions of the inscribed parallelograms whose vertices are the intersections of the coordinate axes with the boundaries of P_{8j}, P_{8j+2}, P_{8j+4}, P_{8j+6}, respectively. Evaluating the ratio of the smallest homothetic copy of such an inscribed parallelogram which contains our polygon from amongst $P_{8j}, P_{8j+2}, P_{8j+4}, P_{8j+6}$, we find each value at the right sides of (I)–(IV).

Let us show the opposite inequalities for (I) and (III).

Ad (I). To prove that $\delta_{BM}(P_4, P_{8j}) \geq \sqrt{2}$, we have to show that for any parallelogram $P \in \mathcal{M}^2$ contained in P_{8j} such that $P_{8j} \subset \lambda P$, where λ is positive, the inequality $\lambda \geq \sqrt{2}$ holds true. By Proposition, in order to show that $\delta_{BM}(P_4, P_{8j}) \geq \sqrt{2}$, it is sufficient to take into account only any $P = a(P_4)$ inscribed in P_{8j} such that a positive homothetic copy λP is circumscribed about P_{8j}, and to show that $\lambda \geq \sqrt{2}$. Just we may disregard the other $P \subset P_{8j}$. This is realized in the following Parts (α) and (β).

(α) *If a parallelogram $P \in \mathcal{M}^2$ is inscribed in P_{8j} and its homothetic image is circumscribed, then P is a square.*

Take into account a parallelogram $P \in \mathcal{M}^2$ inscribed in P_{8j}, which is not a square. Denote its successive vertices by a, b, c, d, when we go counterclockwise.

First let us explain that they must be in some $2j$-th sides of P_{8j}. In the opposite case, some of them, say a and b do not fulfill this. We may assume that $a \in v_0v_1$ and $b \notin v_{2j}v_{2j+1}$ (besides when $a = v_0$ or $b = v_{2j+1}$). Provide the straight line parallel to ab which supports P_{8j} at a vertex v_k with $1 < k < 2j$. Provide the straight line parallel to bc which supports P_{8j} at a vertex v_l with $2j < l < 4j$. Then the distances from v_k to ab and from v_l to bc are different. This means that no homothetic image of $abcd$ has a chance to be circumscribed about P_{8j}.

Hence consider the situation when succesive vertices of P are at every $2j$-th side of P_{8j}. Clearly, a, c are symmetric with respect to o, and o is in different distances from a and b. Say, let $a \in v_0v_1$ and $b \in v_{2j}v_{2j+1}$. From the symmetry we see that $|v_0a| = |v_{4j}c|$. Let $b^* \in v_{2j}v_{2j+1}$ fulfill $|v_oa| = |v_{2j}b^*|$ and let d^* be opposite to b^*. Then ab^*cd^* is a square inscribed in P_{8j} Clearly an enlarged homothetic image of it is circumscribed about P_{8j}. Thus the half-lines with origin at o through v_1 and v_{2j+1} intersect the sides ab^* and b^*c, respectively, at points w_1^+, w_{2j+1}^-, respectively, in equal distances from o. These half-lines intersect the sides ab and cb at points z_1^+, z_{2j+1}^-, respectively. Since $b \neq b^*$, then P is not a rhombus and thus $|oz_1^+| \neq |oz_{2j+1}^-|$. Since $|ov_1| = |ov_{2j+1}|$, we get $|z_1^+z_1^-|/|w_1^+w_1^-| \neq |z_{2j+1}^+z_{2j+1}^-|/|w_{2j+1}^+w_{2j+1}^-|$ where w_i^-, z_i^- denote the points symmetric to w_i^+, z_i^+. From Lemma we conclude that the strip between the straight lines through v_1 and v_{4j+1}, parallel to ab has a different width than the strip between the straight lines through v_{2j+1} and v_{6j+1}, parallel to bc. Hence no homothetic image of $abcd$ is circumscribed about P_{8j}.

Consequently, P must be a square.

(β) *If a square P is inscribed in P_{8j} and λP is circumscribed about P_{8j}, then $\lambda \geq \sqrt{2}$, and $\lambda = \sqrt{2}$ if and only if the vertices of $P$are at every $2j$-th vertex of P_{8j} or at the middle of every $2j$-th side of P_{8j}.*

Having in mind the axial symmetries of P_{8j}, we may limit our considerations to the squares P inscribed in P_{8j} whose one vertex denoted by p is in v_0m, where m is the midpoint of v_0v_1.

Denote by k the directional factor $\frac{\sin(\pi/4j)}{\cos(\pi/4j)-1}$ of the straight line containing the side v_0v_1. So the line has equation $y = k(x-1)+1$. Clearly, p is in the intersection of the segment v_0m with a ray $y = bx$, where $x \geq 0$ and $0 \leq b \leq \tan\frac{\pi}{8j}$. We omit an easy calculation showing that $p = (\frac{k}{k-b}, \frac{kb}{k-b})$. Since the diagonals of P are orthogonal, the successive vertex q of P is the intersection of the rotated by 90° ray $y = -\frac{1}{b}x$, where $x \geq 0$, with the side $v_{2j}v_{2j+1}$. We easily establish that $q = (\frac{-kb}{k-b}, \frac{k}{k-b})$.

We provide the straight line containing pq. Its equation is $y = \frac{b-1}{b+1}x + \frac{k(b^2+1)}{(k-b)(b+1)}$. Consider its intersection point u with the line $y = x$. An easy calculation shows that both coordinates of u are equal to $\frac{k}{2}\cdot\frac{b^2+1}{k-b}$. Consequently, the ratio of the first coordinate of v_j to the first coordinate of u equals to $h(b) = \frac{\sqrt{2}}{2} : \left(\frac{k}{2} \cdot \frac{b^2+1}{k-b}\right) = \sqrt{2} \cdot \frac{k-b}{k(b^2+1)}$. By Proposition, the homothetic square $h(b)P$ is circumscribed about P_{8j}.

Clearly, we consider the function $h(b)$ in the interval $[0, \tan\frac{\pi}{8j}]$. Observe that $h(0) = \sqrt{2} = h(\tan\frac{\pi}{8j})$. Our aim is to show that $h(b) \geq \sqrt{2}$ for every b from this interval. We find the derivative $h'(b) = \frac{\sqrt{2}}{k} \cdot \frac{b^2-2bk-1}{(b^2+1)^2}$. An easy calculation shows that $h'(b) = 0$ in the interval $[0, \tan\frac{\pi}{8j}]$ if and only if $b = k + \sqrt{k^2+1}$. We have $h(k+\sqrt{k^2+1}) = -\frac{\sqrt{2}}{k} \cdot \frac{\sqrt{k^2+1}}{(k+\sqrt{k^2+1})^2+1}$. Applying the fact that $k < 0$ we show that this value is over $\sqrt{2}$. Hence $h(b) \geq \sqrt{2}$ for every $b \in [0, \tan\frac{\pi}{8j}]$ with equality only for $b = 0$ and $b = \tan\frac{\pi}{8j}$.

Consequently, the thesis of Part (β) holds true.

We conclude that $\delta_{BM}(P_4, P_{8j}) \geq \sqrt{2}$, which confirms the required opposite inequality of (I).

Ad (III). The vertices $v_0, v_{2h+1}, v_{4h+2}, v_{6h+3}$ of P_{8j+4} are the vertices of P_4. Observe that every side of P_4 is parallel to every h-th side of P_{8j+4}. So the side v_0v_{2h+1} of P_4 is parallel to the side v_jv_{j+1} of P_{8j+4}, and so on. An evaluation shows that the ratio of the first coordinates of the points of the intersection of the line $y = (\tan\frac{\pi}{4})x$ with these two sides (the points are centers of these two sides) is $\cos\frac{1}{8j+4}\pi \cdot \sec\frac{1}{4}\pi$. Analogous is true for every side of P_4 and the corresponding parallel side of P_{8j+4}. Consequently, by Lemma the homothetic copy of P_4 with ratio $\cos\frac{1}{8j+4}\pi \cdot \sec\frac{1}{4}\pi$ contains P_{8j+4}. Consequently, $\delta_{BM}(P_4, P_{8j+4}) \leq \cos\frac{1}{8j+4}\pi \cdot \sec\frac{1}{4}\pi$. □

From Part (β) we see the only positions of P for which $\delta_{BM}(P_4, P_{8j}) = \sqrt{2}$ is realized. Observe that the only positions of P for which $\delta_{BM}(P_4, P_{8j+4})$ is realized are these with the vertices at every $(2j+1)$-th vertex of P_{8j+4}.

Generalizing (III) we observe that $\delta_{BM}(P_n, P_{hn}) = \cos\frac{1}{hn}\pi \cdot \sec\frac{1}{n}\pi$ for every even $n \geq 4$ and every odd $h \geq 3$.

In connection with (II) and (IV) we conjecture that $\delta_{BM}(P_4, P_{8j+2}) = \frac{1}{2}\sec\frac{2j}{8j+2}\pi + \cos\frac{2j}{8j+2}\pi$ and $\delta_{BM}(P_4, P_{8j+6}) = \sin\frac{2j+2}{8j+6}\pi \cdot \csc\frac{4j+2}{8j+6}\pi + \cos\frac{2j+2}{8j+6}\pi$.

We also do not know the BM-distances between P_4 and the regular odd-gons. Besides some special cases, the task of finding the extended BM-distances (see “Appendix”) $\delta_{BM}(P_m, P_n)$ seems to be very complicated.

5. Appendix

According to the recommendation of the referee we present the notion of the extended BM-distance and explain why it extends the original one. The *extended Banach–Mazur distance of convex bodies $C, D \subset E^d$ is the number*

$$\delta^e_{BM}(C, D) = \inf_{a, h_\lambda} \{\lambda;\, a(C) \subset D \subset h_\lambda(a(C))\},$$

where a is an affine transformation and h_λ is a homothety with a positive ratio λ. The superscript e is added here only for the purpose of the proof of the below Claim, which permits to omit this superscript. This notion is considered in many papers and books, e.g., in [2,4–6,9]. We present Claim since we cannot find this somehow obvious fact in the literature.

Claim. For every $C, D \in \mathcal{M}^d$ the original and the extended Banach–Mazur distance of C and D are equal.

Proof. We can assume that D is centered at o in the definition of $\delta^{\mathrm{e}}_{BM}(C, D)$.

Since every $\lambda(C)$ from the definition of $\delta_{BM}(C, D)$ is a homothetical image of C in the definition of $\delta^{\mathrm{e}}_{BM}(C, D)$, we get $\delta^{\mathrm{e}}_{BM}(C, D) \le \delta_{BM}(C, D)$.

Show the reverse inequality. In the definition of $\delta^{\mathrm{e}}_{BM}(C, D)$ the infimum is realized by an affine image $a'(C)$ of C and by a homothetical image $I' = h'_\lambda a'(C)$ of it. Denote by $a^o(C)$ the translate of $a'(C)$ and by I^o the translate of I', both with center o. By $D \in \mathcal{M}^d$ we get $-a'(C) \subset D$ and $D \subset -I'$. So $a^o(C) \subset \operatorname{conv}\big(a'(C) \cup (-a'(C))\big) \subset D$ and $D \subset I' \cap (-I') \subset I^o$. Thus the infimum of $\delta^{\mathrm{e}}_{BM}(C, D)$ is realized by $a^o(C)$ and I^o in parts of $a(C)$ and $h_\lambda(a(C))$. Sets $a^o(C)$ and I^o, as centered at o, play the part of $a(C)$ and $\lambda a(C)$ in the definition of $\delta_{BM}(C, D)$. Hence the infimum of λ in this definition is at most the infimum of λ in the one of $\delta^{\mathrm{e}}_{BM}(C, D)$. So $\delta_{BM}(C, D) \le \delta^{\mathrm{e}}_{BM}(C, D)$. □

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