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Majorization of the Temljakov Operators for the Bavrin Families in $\mathbb{C}^n$

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Abstract. The paper concerns holomorphic functions in complete bounded n-circular domains $\mathcal{G}$ of the space $\mathbb{C}^n$. The object of the present investigation is to solve majorization problem of Temljakov operator. This type of problem has been studied earlier in Liczberski and Żywień (Folia Sci Univ Tech Res 33:37–42, 1986), Liczberski (Bull Technol Sci Univ Łódź 20:29–37, 1988) and Leś-Bomba and Liczberski (Demonstratio Math 42(3):491–503, 2009). In this paper we considered the family $\mathcal{M}_\mathcal{G} \cap \mathcal{F}_{0,k}(\mathcal{G})$, i.e. the functions of the Bavrin family $\mathcal{M}_\mathcal{G}$, which are $(0, k)$-symmetrical.

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1. Introduction

A domain $\mathcal{G} \subset \mathbb{C}^n$, $n \geq 2$, is called complete n-circular, if $z\Lambda = (z_1\lambda_1, \ldots, z_n\lambda_n) \in \mathcal{G}$ for each $z = (z_1, \ldots, z_n) \in \mathcal{G}$ and every $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \overline{\mathcal{U}^n}$, where $\mathcal{U}$ is the disc $\{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$. In the paper we assume that $\mathcal{G}$ is a bounded complete n-circular domain. Let us consider the Minkowski function $\mu_\mathcal{G} \colon \mathbb{C}^n \to [0, \infty)$

$$\mu_\mathcal{G}(z) = inf \left\{ t > 0 \colon \frac{1}{t} z \in G \right\}, \quad z \in \mathbb{C}^n.$$

We shall use the continuity of $\mu_\mathcal{G}$ and the following facts as well:

(i) $\mathcal{G} = \{z \in \mathbb{C}^n \colon \mu_\mathcal{G}(z) < 1\}$,
(ii) $\partial\mathcal{G} = \{z \in \mathbb{C}^n : \mu_\mathcal{G}(z) = 1\}$.

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Moreover, if a domain $\mathcal{G}$ is additionally bounded and convex, we have $\mu_{\mathcal{G}}(\cdot) = ||\cdot||$ (see [9]).

Let $\mathcal{H}_{\mathcal{G}}$ denote a family of holomorphic functions $f\colon \mathcal{G} \to \mathbb{C}$ and let $\mathcal{L} : \mathcal{H}_{\mathcal{G}} \to \mathcal{H}_{\mathcal{G}}$ be the Temljakov linear operator [10], which is defined by

$$\mathcal{L}f(z) = f(z) + Df(z)(z), \quad z \in \mathcal{G}, \tag{1}$$

where $Df(z)(w)$ is the value of the Frechet's derivative $Df(z)$ of f at the point z on a vector w (here $Df(z)$ is the row vector $\left[\frac{\partial f(z)}{\partial z_1}, ..., \frac{\partial f(z)}{\partial z_n}\right]$ and w is a column vector).

It is also know (see [10]) that the inverse of the Temljakov operator has the following form

$$(\mathcal{L}^{-1}f)(z) = \int_0^1 f(tz)dt, \quad z \in G. \tag{2}$$

In order to show our results, we will use the following property of the Temljakov operator.

Lemma 1. *If* $u, v \in \mathcal{H}_{\mathcal{G}}$ *then*

$$\mathcal{L}\left(u(z)v(z)\right) = -u(z)v(z) + u(z)\mathcal{L}v(z) + v(z)\mathcal{L}u(z),\ z \in \mathcal{G}. \tag{3}$$

Proof.

$$\begin{aligned}\mathcal{L}\left(u(z)v(z)\right) &= u(z)v(z) + D[u(z)v(z)](z)\\ &= u(z)v(z) + u(z)D[v(z)](z) + v(z)D[u(z)](z).\end{aligned}$$

After adding and subtracting the product $u(z)v(z)$ in the above equality we obtain

$$\begin{aligned}\mathcal{L}\left(u(z)v(z)\right) &= u(z)[v(z) + Dv(z)(z)]\\ &\quad +v(z[u(z) + Du(z)(z)] - u(z)v(z)\\ &= -u(z)v(z) + u(z)\mathcal{L}v(z) + v(z)\mathcal{L}u(z).\end{aligned}$$

□

We will consider some subfamilies $X_{\mathcal{G}}$ of functions $f \in \mathcal{H}_{\mathcal{G}}(1)$, where $\mathcal{H}_{\mathcal{G}}(1) = \{f \in \mathcal{H}_{\mathcal{G}} : f(0) = 1\}$. The below subfamilies $\mathcal{X}_{\mathcal{G}}$ are defined by the family $\mathcal{C}_{\mathcal{G}}$,

$$\mathcal{C}_{\mathcal{G}} = \{f \in \mathcal{H}_{\mathcal{G}}(1) : Ref(z) > 0, z \in \mathcal{G}\}.$$

We say that a function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{M}_{\mathcal{G}}$, $\mathcal{N}_{\mathcal{G}}$, $\mathcal{R}_{\mathcal{G}}$ (see [1]) if there exists a function $h \in \mathcal{C}_{\mathcal{G}}$ such that

$$\begin{aligned}\mathcal{L}f(z) &= f(z)h(z), \quad z \in \mathcal{G},\\ \mathcal{L}\mathcal{L}f(z) &= \mathcal{L}f(z)h(z), \quad z \in \mathcal{G}\\ \mathcal{L}f(z) &= \mathcal{L}\varphi(z)h(z), \quad \varphi \in \mathcal{N}_{\mathcal{G}}, \quad z \in \mathcal{G},\end{aligned}$$

respectively.

In the case $n = 2$ Bavrin ([1]) gave the following geometrical interpretation for functions from $\mathcal{M}_{\mathcal{G}}$. A function f belongs to $\mathcal{M}_G$ if and only if

(i) the function $z_1 f(z_1, z_2)$ is univalent starlike in the intersection of the domain $\mathcal{G}$ by every analitic plane $z_2 = \alpha z_1$, $\alpha \in \mathbb{C}$. In other words the function $z_1 f(z_1, \alpha z_1)$ of one variable is univalent starlike in the disc, which is the projection of the intersection $\mathcal{G} \cap \{z_2 = \alpha z_1\}$ onto the plane $z_2 = 0$,

(ii) the function $z_2 f(0, z_2)$ is univalent starlike in the intersection $\mathcal{G} \cap \{z_1 = 0\}$.

In connection with this interpretation we say that the family $\mathcal{M}_{\mathcal{G}}$ corresponds to the class $\mathcal{S}^*$ of normalize univalent starlike functions $F : \mathcal{U} \to \mathbb{C}$. In the same way we can say that the family $\mathcal{N}_{\mathcal{G}}$ ($\mathcal{R}_{\mathcal{G}}$) corresponds to the class $\mathcal{S}^c$ ($\mathcal{S}^{cc}$) (see [4]) of normalized univalent convex (close-to-convex) functions.

In the papers [3,5] the notion of $\mathcal{G}-$balance of linear functionals $A : \mathbb{C}^n \to \mathbb{C}$ was defined as follows

$$\mu_{\mathcal{G}}(A) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|A(w)|}{\mu_{\mathcal{G}}(w)} = \sup_{v \in \partial\mathcal{G}} |A(v)| = \sup_{u \in \mathcal{G}} |A(u)| .$$

Moreover, if the domain $\mathcal{G}$ is also convex then $\mu_{\mathcal{G}}(A)$ is a norm of the linear functional A.

Therefore $\mu_{\mathcal{G}}(\widehat{I})$ for the linear functional $\widehat{I} : \mathbb{C}^n \to C$ defined by

$$\widehat{I}\,(z) = \sum_{j=1}^{n} z_j, \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

means the same as $\Delta = \Delta(\mathcal{G})$-characteristic of domain $\mathcal{G}$ which Bavrin defined in [1] as follows

$$\Delta = \sup_{z=(z_1,\ldots,z_n) \in \mathcal{G}} \left| \sum_{j=1}^{n} z_j \right| .$$

In the sequel $I : \mathbb{C}^n \to \mathbb{C}$ is a linear operator defined by

$$I\,(z) = \frac{1}{\mu_{\mathcal{G}}(\widehat{I})} \widehat{I}\,(z)\,, \quad z \in \mathbb{C}^n.$$

We can see that $\mu_{\mathcal{G}}\,(I) = 1$ and we have

$$|I\,(z)\,| \le \mu_{\mathcal{G}}\,(I)\,\mu_{\mathcal{G}}\,(z) < 1, \quad z \in \mathcal{G}.$$

Let $k \ge 2$ be an arbitrarily fixed integer, $\varepsilon = \varepsilon_k = \exp \frac{2\pi i}{k}$ and a set $\mathcal{D} \subset \mathbb{C}^n$ be $k-$symmetric ($\varepsilon\mathcal{D} = \mathcal{D}$). For $j = 0, 1, \ldots, k-1$ we define the spaces $\mathcal{F}_{j,k} = \mathcal{F}_{j,k}(\mathcal{D})$ of functions (j, k)-symmetrical, i.e., all functions $f : \mathcal{D} \to \mathbb{C}$ such that

$$f\,(\varepsilon z) = \varepsilon^j f\,(z)\,, \quad z \in \mathcal{D}.$$

A very useful result concerning with (j, k)-symmetrical functions is the following [7]:

For every function $f\colon \mathcal{D} \to \mathbb{C}$ there exists exactly one sequence of functions $f_{j,k} \in \mathcal{F}_{j,k}$, $j = 0, 1, \ldots, k-1$, such that

$$\begin{aligned} f &= \sum_{j=0}^{k-1} f_{j,k}, \\ f_{j,k}(z) &= \frac{1}{k}\sum_{l=0}^{k-1} \varepsilon^{-jl} f\left(\varepsilon^l z\right), \quad z \in \mathcal{D}. \end{aligned} \tag{4}$$

By the uniqueness of the partition (4) the functions $f_{j,k}$ will be called further (j,k) −symmetrical components of the function f. Moreover, note that n-circular domain is k-symmetric.

2. The Majorization Problem

Let $f, F \in \mathcal{H}_\mathcal{G}$ and $r \in [0,1]$. If

$$|f(z)| \le |F(z)|, \quad z \in r\mathcal{G}, \tag{5}$$

we say that the function F majorizes the function f in the set $r\mathcal{G}$.

The second author (see [6]) has proved that if in a complete bounded two-circular domain $\mathcal{G} \subset \mathbb{C}^2$ a function $F \in \mathcal{M}_\mathcal{G}$ majorizes a function $f \in \mathcal{H}_\mathcal{G}$, then $\mathcal{L}F$ majorizes $\mathcal{L}f$ in $r\mathcal{G}$, $r = r(\mathcal{M}_\mathcal{G}) = 2 - \sqrt{3}$.

Moreover, the number $r(\mathcal{M}_\mathcal{G})$ cannot be increased by taking $\mathcal{G}$, to be the cone in $\mathbb{C}^2$

$$A(2;1) = \{z \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}.$$

Moreover, in paper [5] an analogous result optimal in case of any complete bounded n-circular domain $G \subset \mathbb{C}^n$ for the superclass $\mathcal{R}_G$ of the class $\mathcal{M}_G$ was given.

The main theorem is preceded by lemma.

Lemma 2. *If the function $F \in \mathcal{H}_\mathcal{G}(1)$ belongs to the family $\mathcal{M}_\mathcal{G} \cap \mathcal{F}_{0,k}(\mathcal{G})$, then for each fixed point $z \in \mathcal{G}\backslash\{0\}$, the function $G_z : \mathcal{U} \to \mathbb{C}$*

$$G_z(\xi) = \xi F\left(\xi \frac{z}{\mu_\mathcal{G}(z)}\right), \quad \xi \in \mathcal{U} \tag{6}$$

belongs to the family $S^* \cap \mathcal{F}_{1,k}(\mathcal{U})$ of the $(1,k)$-symmetric univalent starlike functions with normalization $G_z(0) = 0, (G_z)'(0) = 1$.

Proof. Proof of the relation $G_z \in S^*$ we can find in [1]. The $(1,k)$ symmetry of G_z follows from the following equalities

$$G_z(\varepsilon\xi) = \varepsilon\xi F\left(\varepsilon\xi \frac{z}{\mu_\mathcal{G}(z)}\right) = \varepsilon\xi F\left(\xi \frac{z}{\mu_\mathcal{G}(z)}\right) = \varepsilon G_z(\xi).$$

□

Theorem 1. *Let $\mathcal{G} \subset \mathbb{C}^n$, $n \geq 2$ be a bounded complete n-circular domain. If a function $f \in \mathcal{H}_{\mathcal{G}}$ is majorized in $\mathcal{G}$ by a function $F \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}$, then*

$$|\mathcal{L}f(z)| \leq T(r)|\mathcal{L}F(z)|, \quad \mu_{\mathcal{G}}(z) = r \in [0,1), \tag{7}$$

where

$$T(r) = \begin{cases} 1 & \text{for } r \in [0, r_k], \\ \frac{(1-r^k)^2(1-r^2)^2+4r^2(1+r^k)^2}{4r(1-r^2)(1-r^{2k})} & \text{for } r \in [r_k, 1). \end{cases} \tag{8}$$

and r_k is the unique solution in $(0,1)$ of the equation

$$r^{k+2} - 2r^{k+1} - r^k - 2r + 1 = 0. \tag{9}$$

The function T in (7) cannot be replaced by any function with values $T(r)$ smaller than the values of T defined by (8).

Proof. Let $f \in \mathcal{H}_{\mathcal{G}}$, $F \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}$. Thus by (5) we have

$$f(z) = \omega(z)F(z), \quad z \in \mathcal{G} \tag{10}$$

where $\omega \in S_{\mathcal{G}} \cup \{1\}$ and $S_{\mathcal{G}} = \{\omega \in \mathcal{H}_{\mathcal{G}} : \omega(\mathcal{G}) \subset \mathcal{U}\}$.

Indeed, since $F(z)\mathcal{L}F(z) \neq 0$ for $F \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}$, $z \in \mathcal{G}$ (see [1]), we have in view of (5) that

$$\left|\frac{f(z)}{F(z)}\right| \leq 1, \quad z \in \mathcal{G}.$$

Consequently, the function $\omega(z) = \frac{f(z)}{F(z)}$, $z \in \mathcal{G}$ is holomorphic in $\mathcal{G}$ and $|\omega(z)| < 1$ for $z \in \mathcal{G}$ or $\omega(z) \equiv 1$ in $\mathcal{G}$. Now, we will found the upper bound of the quotient $\left|\frac{\mathcal{L}f(z)}{\mathcal{L}F(z)}\right|$, $z \in \mathcal{G}$. If $\mu_{\mathcal{G}}(z) = r \in [0,1)$, then (10) and (3) give

$$\begin{aligned} \left|\frac{\mathcal{L}f(z)}{\mathcal{L}F(z)}\right| &= \left|\frac{\mathcal{L}[\omega(z)F(z)]}{\mathcal{L}F(z)}\right| = \left|\frac{D\omega(z)(z)F(z)}{\mathcal{L}F(z)} + \omega(z)\right| \\ &\leq |D\omega(z)(z)|\left|\frac{F(z)}{\mathcal{L}F(z)}\right| + |\omega(z)|. \end{aligned}$$

Let us recall that for $\omega \in S_{\mathcal{G}} \cup \{1\}$ we have (see [1])

$$|D\omega(z)(z)| \leq \frac{r}{1-r^2}(1-|\omega(z)|^2), \quad \mu_G(z) = r \in [0,1).$$

Now we go to the estimate of the expression $\left|\frac{\mathcal{F}(z)}{\mathcal{L}F(z)}\right|$. We will use Lemma 2. It is known (see [11]) that for the function $G \in S^* \cap \mathcal{F}_{1,k}(\mathcal{U})$ there holds the bound

$$\left|\frac{G(\xi)}{\xi G'(\xi)}\right| \leq \frac{1+|\xi|^k}{1-|\xi|^k}, \quad 0 \leq |\xi| < 1, \quad k \geq 2.$$

Taking into account the above $\xi = \mu_{\mathcal{G}}(z) = r \in [0,1)$ we have by (6)

$$\left|\frac{F(z)}{\mathcal{L}F(z)}\right| \leq \frac{1+r^k}{1-r^k}.$$

As a result we have

$$\left|\frac{\mathcal{L}f(z)}{\mathcal{L}F(z)}\right| \le \frac{1+r^k}{1-r^k}\frac{r}{1-r^2}(1-|\omega(z)|^2)+|\omega(z)|\,, \quad r\in[0,1).$$

Let us consider the right-hand side part of this inequality as a square function of the variable x:

$$y_r(x) = -\frac{1+r^k}{1-r^k}\frac{r}{1-r^2}x^2 + x + \frac{1+r^k}{1-r^k}\frac{r}{1-r^2}, \quad x=|\omega(z)|\,.$$

Then its maximum in interval [0,1] is equal to

$$\frac{(1-r^k)^2(1-r^2)^2+4r^2(1+r^k)^2}{4r(1-r^2)(1-r^{2k})} \quad for \quad r\in(r_k,1)$$

and it is 1 for $r\in[0,r_k]$.

Indeed, the maximum ordinate y_v is attained for the abscissa $x_v\in(0,1)$, in opposite case the maximum $y_v=y_v(1)=1$. Setting

$$x_v = x_v(r) = \frac{(1-r^k)(1-r^2)}{2r(1+r^k)}, \quad r\in(0,1] \tag{11}$$

we have

$$x_v(r)<1$$

if

$$r^{k+2}-2r^{k+1}-r^k-r^2-2r+1<0, \quad r\in(0,1].$$

Now, we show that the polynomial

$$q(r)=r^{k+2}-2r^{k+1}-r^k-r^2-2r+1$$

has exactly one root in the interval $(0,1)$. Finally, it is sufficient to note that

$$q(0)q(1)<0$$

and for $r\in[0,1)$

$$\begin{aligned} q'(r) &= (k+2)r^{k+1}-2(k+1)r^k-kr^{k-1}-2r-2 \\ &= kr^{k-1}(r^2-1)+2r(r^k-1)-2(k+1)r^k-2<0. \end{aligned}$$

Therefore in the interval $(0,1)$ the function $q(r)$ has exactly one root r_k.

For x_v given by (11) we have

$$y_v = \frac{(1-r^k)^2(1-r^2)^2+4r^2(1+r^k)^2}{4r(1-r^2)(1-r^{2k})} \quad for \quad r\in[r_k,1].$$

Hence we obtain (8).

In order to prove the second part of the theorem, let us assume that $r\in[r_k,1)$, the point $\overset{\circ}{z}\in\mathcal{G}$, $\mu_{\mathcal{G}}(\overset{\circ}{z})=r$ and the function f is of the form

$$f(z)=\omega(z)F(z),$$

where

$$F(z) = \frac{1}{[1 + I^k(z)]^{\frac{2}{k}}} \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k},$$

$I^k(z)$ means the product of k identical factors $I(z)$ (see [2]) and

$$\omega(z) = \frac{\alpha + I(z)}{1 + \alpha I(z)}, \quad z \in \mathcal{G},$$
$$\alpha = \frac{(1 - r^k)(1 - r^2) - 2r^2(1 + r^k)}{r[2(1 + r^k) - (1 - r^k)(1 - r^2)]}.$$

We set the α parameter from the condition

$$\frac{\alpha + I(z)}{1 + \alpha I(z)} = \frac{(1 - r^k)(1 - r^2)}{2r(1 + r^k)}, \quad z = (r, 0, \ldots, 0), \quad \alpha \in (0, 1).$$

Thus $F \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}$, f is majorized by F in $\mathcal{G}$ and for the point $\overset{\circ}{z}$ we have equality in (7). However, by putting $f = F$ for $r \in [0, r_k]$, where $F \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}$, we have $\mathcal{L}f = \mathcal{L}F$ and equality in (7) holds for points $z \in \mathcal{G}$ such that $\mu_{\mathcal{G}}(z) = r \in [0, r_k]$. This completes the proof. □

Corollary 1. *Let $n \geq 2$ and $\mathcal{G}$ be a bounded complete n-circular domain of $\mathbb{C}^n$. If a function $F \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}$ majorizes a function $f \in \mathcal{H}_{\mathcal{G}}$ in $\mathcal{G}$, then the function $\mathcal{L}F$ majorizes the function $\mathcal{L}f$ in the domain $r_k\mathcal{G}$, where r_k is the unique solution in $(0, 1)$ of the equation (9). The constant r_k cannot be replaced by any greater number r.*

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