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Results in Mathematics

A Lower Density Operator for the Borel Algebra

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Abstract. We prove that the existence of a Borel lower density operator (a Borel lifting) with respect to the σ-ideal of countable sets, for an uncountable Polish space, is equivalent to **CH**. One of the implications is known (due to K. Musiał) and the remaining implication is derived from a general abstract result dealing with the negation of **GCH**. We observe that there is no lower density Borel operator with respect to the σ-ideal of countable sets, whose range is of bounded level in the Borel hierarchy.

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1. A General Negative Result on Lower Densities

Let S be a σ-algebra of subsets of a nonempty set X and let $J \subseteq S$ be a σ-ideal. We write $A \sim B$ whenever the symmetric difference $A \triangle B$ is in J. This is an equivalence relation on S and its quotient space is denoted by S/J. A mapping $\Phi\colon S \to S$ is called a *lower density operator* (respectively, a *lifting*) with respect to J if it satisfies the following conditions (1)–(4) (respectively, (1)–(5)):

(1) $\Phi(X) = X$ and $\Phi(\emptyset) = \emptyset$,
(2) $A \sim B \implies \Phi(A) = \Phi(B)$ for every $A, B \in S$,
(3) $A \sim \Phi(A)$ for every $A \in S$,
(4) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for every $A, B \in S$,
(5) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ for every $A, B \in S$.

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The problem of the existence of liftings together with their various applications were widely discussed in the monograph [5] and in the later survey [14]. Lower density operators play an important role in constructions of density like topologies; see [4,11,15]. Several solutions of interesting problems on liftings require some set-theoretic tools. Our paper also use such methods.

We will start from a general negative result dealing with the negation of generalized continuum hypothesis (**GCH**). Then we will apply it to obtain a theorem stating the existence of a lower density operator on a Borel algebra in an uncountable Polish space, with respect to the σ-ideal of countable sets.

Let κ and λ be uncountable cardinals with $\kappa \le \lambda$. Let $|X| = \lambda$. A family F of subsets of X is called κ-additive if $\bigcup G \in F$ whenever $G \subseteq F$ and $|G| < \kappa$. Note that an ω_1-additive algebra is simply a σ-algebra. An algebra of subsets of $X \times X$ is called a cross-type algebra if it contains all sets of the form $\{x\} \times X$ and $X \times \{x\}$ for $x \in X$. Of course, all singletons $\{(x, y)\}$, with $x, y \in X$, belong to a cross-type algebra. Let J_κ denote the ideal of all subsets of $X \times X$ of cardinality $< \kappa$. Note that J_κ is contained in every κ-additive cross-type algebra of subsets of $X \times X$.

Theorem 1. *Assume that κ and λ are infinite cardinals such that $\kappa^+ < \lambda$. Fix a set X with $|X| = \lambda$. Then for every κ^+-additive cross-type algebra S of subsets of $X \times X$, there is no lower density operator $\Phi\colon S \to S$ with respect to the σ-ideal J_{κ^+}. In particular, this is true if $\lambda := 2^\kappa$ and we assume that $\kappa^+ < 2^\kappa$ (a part of $\neg$**GCH**).*

Proof. Enumerate X as $\{x_\alpha\colon \alpha < \lambda\}$. Suppose that there exist κ^+-additive cross-type algebra S of subsets of $X \times X$ and a lower density operator $\Phi\colon S \to S$ with respect to J_{κ^+}. Let $Q_\alpha := \Phi(P_\alpha)$ where $P_\alpha := \{x_\alpha\} \times X$ for $\alpha < \lambda$. Note that if $\alpha \ne \beta$ then $Q_\alpha \cap Q_\beta = \Phi(P_\alpha \cap P_\beta) = \emptyset$ by (4) and (1). Let $\pi_2\colon X \times X \to X$ be given by $\pi_2(x, y) := y$. □

Claim. *There is $x \in X$ such that $|\{\beta < \lambda\colon x \in \pi_2[Q_\beta]\}| \ge \kappa^+$.*

Proof of Claim. Suppose that $|\{\beta < \lambda\colon x \in \pi_2[Q_\beta]\}| < \kappa^+$ for each $x \in X$. Let

$$L_\alpha := \{\beta < \lambda\colon x_\alpha \in \pi_2[Q_\beta]\} \quad \text{for } \alpha < \lambda.$$

Then $|\bigcup_{\alpha<\kappa^+} L_\alpha| \le \kappa^+$ by our supposition. Since $\kappa^+ < \lambda$, the set $\lambda \setminus \bigcup_{\alpha<\kappa^+} L_\alpha$ is nonempty (of cardinality λ). Take $\xi \in \lambda \setminus \bigcup_{\alpha<\kappa^+} L_\alpha$. Then

$$\{x_\alpha\colon \alpha < \kappa^+\} \subseteq X \setminus \pi_2[Q_\xi] = \pi_2[P_\xi] \setminus \pi_2[Q_\xi] \subseteq \pi_2[P_\xi \setminus Q_\xi]$$

which gives a contradiction since $|\pi_2[P_\xi \setminus Q_\xi]| \le |P_\xi \setminus \Phi(P_\xi)| < \kappa^+$ by (3).

Take $x \in X$ as in the Claim. Consider $P := X \times \{x\}$. Then $|P \cap P_\alpha| = 1$ and $\Phi(P) \cap Q_\alpha = \Phi(P) \cap \Phi(P_\alpha) = \Phi(P \cap P_\alpha) = \emptyset$ for each $\alpha < \lambda$, by (4), (2) and (1). Therefore $\Phi(P) \cap \bigcup_{\alpha<\lambda} Q_\alpha = \emptyset$. On the other hand, $|P \cap \bigcup_{\alpha<\lambda} Q_\alpha| \ge$

κ^+ by the choice of x. Thus $P \backslash \Phi(P) \notin J_{\kappa^+}$ which yields a contradiction with (3). □

2. A Theorem on the Existence of Borel Liftings

If S is the σ-algebra of Borel sets in a given Hausdorff space, then the respective operator Φ satisfying conditions (1)–(5) is called a Borel lifting. Note that von Neumann and Stone [9] proved the existence of a lifting for a Borel measure space on $[0, 1]$ under the assumption of the continuum hypothesis (**CH**). A simple proof of the same result was then given by Musiał [8]. This was later generalized by Mokobodzki [6] and Fremlin [3] who showed that, subject to **CH**, any σ-finite measure space with the measure algebra of cardinality $\leq \omega_2$ has a lifting. On the other hand, Shelah [12] proved that $2^\omega = \omega_2$ is consistent with the nonexistence of Borel lifting for the Lebesgue measure algebra. Later in [2], it was shown that $2^\omega = \omega_2$ is consistent with the existence of Borel lifting for the Lebesgue measure algebra.

We focus on Borel liftings in the following case. We consider the σ-algebra $\mathcal{B}(X)$ of Borel subsets of an uncountable Polish space X and the σ-ideal $[X]^{\leq\omega}$ of all countable subsets of X. Since any two uncountable Borel subsets of Polish spaces are Borel isomorphic [13, Theorem 3.3.13], if we seek a lifting from $\mathcal{B}(X)$ into $\mathcal{B}(X)$ with respect to $[X]^{\leq\omega}$, it does not matter which Polish space is considered.

Theorem 2. *For an uncountable Polish space X, the following conditions are equivalent:*

(i) **CH***;*
(ii) there exists a lifting $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ with respect to $[X]^{\leq\omega}$;
(iii) there exists a lower density operator $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ with respect to $[X]^{\leq\omega}$.

Proof. Implication (i) $\Longrightarrow$ (ii) follows from [8, Theorem 1] where it is shown that, for any σ-algebra S and any σ-ideal $J \subseteq S$, if $|S/J| \leq \omega_1$, there exists a lifting from S to S, with respect to J.

Implication (ii) $\Longrightarrow$ (iii) is obvious.

To prove (iii) $\Longrightarrow$ (i) assume $\neg$**CH**. We work with $\mathbb{R} \times \mathbb{R}$ as a Polish space. It suffices to apply Theorem 1 where $\kappa := \omega$ and $\lambda := 2^\omega = |\mathbb{R}|$. Then J_{ω_1} consists of all countable subsets of the plane and the role of a cross-type σ-algebra is played by $\mathcal{B}(\mathbb{R} \times \mathbb{R})$. □

Note that implication (iii) $\Longrightarrow$ (ii) follows from the final part of [11] or from [4, Theorem 2.8]. In fact, the existence of a lower density operator implies the existence of a lifting in a general case.

Theorem 2 answers a question posed by Jacek Hejduk during his invited talk given on the Conference on Real Function Theory in Stará Lesná in September 2016. He asked about the existence of a lower density operator on $\mathcal{B}(\mathbb{R})$ with respect to $[\mathbb{R}]^{\leq\omega}$.

3. Nonexistence of a Range of Bounded Borel Level

It can happen that the values of a lower density operator $\Phi\colon S \to S$ with respect to $J \subseteq S$ are located in a proper subfamily of S. Denote by $\mathcal{L}$ the σ-algebra $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbb{R}$. Recall a canonical lower density operator $\Phi\colon \mathcal{L} \to \mathcal{L}$ with respect to the σ-ideal of null sets. Namely, Φ is given by

$$\Phi(A) := \left\{x \in \mathbb{R}\colon \lim_{h\to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1\right\}$$

for $A \in \mathcal{L}$ where λ stands for Lebesgue measure on $\mathbb{R}$ (see [10,15]). It is shown in [15] that the values of Φ hit into the Borel class $\mathbf{\Pi}^0_3$ (that is, $F_{\sigma\delta}$ in the classic notation; cf. [13, 3.6]). In [1], an exact Borel complexity of sets $\Phi(A)$ was studied where, instead of $\mathbb{R}$, the Cantor space with the respective measure is considered.

In the above context, we return to lower density operators $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ which exist under **CH** by Theorem 2. We may ask whether the range of Φ can be contained in $\mathbf{\Sigma}^0_\alpha$ for some $\alpha < \omega_1$ (we then say that this range is of bounded Borel level). We will show that the answer is negative. Pick $\mathbb{R}$ as a Polish space and let $\mathcal{B} := \mathcal{B}(\mathbb{R})$.

Proposition 3. *There is no lower density operator $\Phi\colon \mathcal{B} \to \mathcal{B}$ with respect to $[\mathbb{R}]^{\leq\omega}$, whose range is of bounded Borel level.*

Proof. Suppose that there exists Φ with the range $\Phi[\mathcal{B}] \subseteq \mathbf{\Sigma}^0_\alpha$ for some $\alpha < \omega_1$. We may assume that $\alpha \geq 3$. Fix $A \subseteq \mathbb{R}$ with $A \in \mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$ (cf. [13, Corollary 3.6.8]). We will show that $\Phi(A) \notin \mathbf{\Sigma}^0_\alpha$ which yields a contradiction. Suppose that $\Phi(A) \in \mathbf{\Sigma}^0_\alpha$. Since $A \triangle \Phi(A)$ is countable, we have $A = (\Phi(A) \cap B^c) \cup C$ for some countable sets $B, C \subseteq \mathbb{R}$. Then $B^c \in \mathbf{\Pi}^0_2 \subseteq \mathbf{\Sigma}^0_\alpha$ and $C \in \mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_\alpha$; see [13, Proposition 3.6.1]. Consequently, $A \in \mathbf{\Sigma}^0_\alpha$ which is impossible. □

Finally, let us mention another negative result obtained in the recent paper [7]. Namely, it is proved that there is no reasonably definable selector that chooses representatives for the equivalence relation on the Borel sets of having countable symmetric difference.

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