Trigonometric Approximation of Functions from $L_{2\pi}^{p(x)}$

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Abstract. We generalize and improve the results of A. Guven, D. Israfilov, Xh. Z. Krasniqi and T. N. Shakh-Emirov. We consider the general methods of summability of Fourier series of functions from $L_{2\pi}^{p(x)}$ with $p(x) \geq 1$. For estimate of the error of approximation of functions by the matrix means we use a modulus of continuity constructed by the Steklov functions of the increments of considered functions without of absolute values.

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1. Introduction

Let $p = p(x)$ will be a measurable 2π -periodic function, $p_- = \inf \{p(x) : x \in \mathbb{R}\}$, $p^+ = \sup \{p(x) : x \in \mathbb{R}\}$, $1 \leq p_- \leq p \leq p^+ < \infty$ and $L_{2\pi}^{p(x)}$ will be the space of all measurable 2π -periodic functions f such that $\int_{-\pi}^{\pi} |f(x)|^{p(x)} dx < \infty$.

Putting

$$\|f\|_p = \|f(\cdot)\|_{p(\cdot)} = \inf \left\{ \alpha > 0 : \int_{-\pi}^{\pi} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}$$

we turn $L_{2\pi}^{p(x)}$ into a Banach space (see [8]). We write $\Pi_{2\pi}$ for the set of all 2π -periodic variable exponents $p = p(x) \geq 1$ satisfying the condition

$$|p(x) - p(y)| \ln \frac{2\pi}{|x - y|} = O(1), \quad (x, y \in [-\pi, \pi]).$$

For $f \in L_{2\pi}^1$ we will consider the trigonometric Fourier series

$$S[f](x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_{\nu}(f) \cos \nu x + b_{\nu}(f) \sin \nu x)$$

with the partial sums $S_k[f]$.

We will also need some notations on methods of summability of the series $S[f]$ and modulus of continuity of f in the space $L_{2\pi}^p$.

If $A := (a_{n,k})_{0 \leq n,k < \infty}$ be an infinite matrix of real numbers such that

$$a_{n,k} \geq 0 \text{ when } k, n = 0, 1, 2, \dots, \lim_{n \rightarrow \infty} a_{n,k} = 0 \text{ and } \sum_{k=0}^{\infty} a_{n,k} = 1$$

or $A_0 := (a_{n,k})_{0 \leq k \leq n < \infty}$, where

$$a_{n,k} = 0 \quad \text{when } k > n,$$

then

$$T_{n,A}[f](x) := \sum_{k=0}^{\infty} a_{n,k} S_k[f](x) \quad (n = 0, 1, 2, \dots)$$

or

$$T_{n,A_0}[f](x) := \sum_{k=0}^n a_{n,k} S_k[f](x) \quad (n = 0, 1, 2, \dots),$$

respectively.

Let $f \in L_{2\pi}^p$ with $p = p(x) \in \Pi_{2\pi}$. For $n = 0, 1, 2, \dots$ we denote the best approximation of f by

$$E_n(f)_p = \inf_{T_n} \|f - T_n\|_p,$$

where the infimum is taken over all trigonometric polynomials $T_n(x) = \sum_{k=-n}^n c_k e^{ikx}$ of the degrees at most n but the role of modulus of continuity of $f \in L_{2\pi}^p$ in the case of a variable exponent $p = p(x)$ play the function

$$\Omega f(\delta)_p = \sup_{0 < h \leq \delta} \left\| f(\cdot) - \frac{1}{h} \int_{-h}^h f(\cdot + t) dt \right\|_{p(\cdot)}.$$

It follows from the results of [9] that if $p = p(x) \in \Pi_{2\pi}$, then the function $\Omega f(\cdot)_p$ is continuous on $[0; \infty)$ and $\lim_{\delta \rightarrow 0} \Omega f(\delta)_p = 0$. It also follows from the above definition that $\Omega f(\delta)_p$ is a non-decreasing function of δ . We will call $\Omega f(\cdot)_p$ the modulus of continuity of a function $f \in L_{2\pi}^p$. With such modulus of continuity we can define the following class of functions:

$$\text{Lip}_p(\omega, M) = \{f \in L_{2\pi}^p : \Omega f(\delta)_p \leq M\omega(\delta), \delta > 0\},$$

where ω is a function of modulus of continuity type on the interval $[0, 2\pi]$, i.e. a nondecreasing continuous function having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$ and M is some positive constant.

It was proved in work [10, Theorem 6.1] that if the variable exponent $p = p(x) \in \Pi_{2\pi}$ and $f \in L_{2\pi}^p$, then the following Jackson-type inequality holds:

$$E_n(f)_p = O(1) \Omega f \left(\frac{1}{n+1} \right)_p.$$

In this paper we generalize and improve the results of A. Guven, D. Israfilov, Xh. Z. Krasniqi and T. N. Shakh-Emirov. We will consider the spaces $L_{2\pi}^{p(x)}$ with $p(x) \geq 1$ and as a measure of approximation we will use the modulus of continuity constructed by the Steklov functions without the modulus of the increments of functions.

2. Main Results

At the begin we prove the general result.

Theorem 1. *Let $f \in L_{2\pi}^p$ with $p = p(x) \in \Pi_{2\pi}$. If the conditions*

$$\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| = O \left(\frac{1}{n+1} \right) \quad (1)$$

for some $\beta \geq 0$ and

$$\sum_{k=0}^{\infty} (k+1) a_{n,k} = O(n+1) \quad (2)$$

hold, then

$$\|T_{n,A}[f] - f\|_p = O \left(\Omega f \left(\frac{1}{n+1} \right)_p + \sum_{k=0}^n a_{n,k} \Omega f \left(\frac{1}{k+1} \right)_p \right).$$

We observe that all of the lower triangular matrices satisfy (2). In this case we have:

Theorem 2. *Let $f \in L_{2\pi}^p$ with $p = p(x) \in \Pi_{2\pi}$. If the conditions*

$$\sum_{k=0}^{n-1} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| = O(a_{n,n}) \quad (3)$$

for some $\beta \geq 0$ and

$$(n+1) a_{n,n} = O(1) \quad (4)$$

hold, then

$$\|T_{n,A_0}[f] - f\|_p = O \left(\sum_{k=0}^n a_{n,k} \Omega f \left(\frac{1}{k+1} \right)_p \right).$$

Next we will consider some special cases and we will approximate of $f \in Lip_p(\omega, M)$ with $p = p(x) \in \Pi_{2\pi}$.

Theorem 3. Let $f \in Lip_p(\omega, M)$ with $p = p(x) \in \Pi_{2\pi}$. If the conditions (1) for some $\beta > 0$ and (2) hold, then

$$\|T_{n,A}[f] - f\|_p = O\left(\omega\left(\frac{1}{n+1}\right)\right).$$

In the similar way we obtain, by Theorem 2 the following

Theorem 4. Let $f \in Lip_p(\omega, M)$ with $p = p(x) \in \Pi_{2\pi}$. If the conditions (3) for some $\beta > 0$ and (4) hold, then

$$\|T_{n,A_0}[f] - f\|_p = O\left(\omega\left(\frac{1}{n+1}\right)\right).$$

Finally, we have some examples and remarks.

Example 1. One can easily verify that $a_{n,k} = e^{-n} \sum_{j=k}^{\infty} \frac{n^j}{(j+1)!}$, where $n, k = 0, 1, 2, \dots$, satisfies the conditions (1) for any $\beta \geq 0$ and (2).

Example 2. We can verify that $a_{n,k} = \frac{(k+1)^\beta - k^\beta}{(n+1)^\beta}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, where $n, k = 0, 1, 2, \dots$, satisfies the conditions (3) for any $\beta > 1$ and (4).

Remark 1. If $((k+1)^{-\beta} a_{n,k}) \in HBVS$ for some $\beta \geq 0$, where

$$HBVS := \left\{ (c_k) \subset [0, \infty) : \sum_{k=0}^{m-1} |c_k - c_{k+1}| = O(c_m) \text{ for any } m = 1, 2, 3, \dots, n \right\},$$

then the condition (3) holds.

Remark 2. Let $f \in Lip_p(\omega, M)$ with $\omega(\delta) = \delta^\alpha$, where $\alpha \in (0, 1]$ and $p = p(x) \in \Pi_{2\pi}$. From Theorem 4 the results of [1–5, 7] and [11, Theorem 5 (v)], [6, Theorem 5 (vi)] follow at once in the more general and improved forms. More precisely, in the mentioned papers [2–4] there is considered the following modulus of continuity

$$\Omega_p f(\delta) = \sup_{0 < h \leq \delta} \left\| \frac{1}{h} \int_{-h}^h |f(\cdot + t) - f(\cdot)| dt \right\|_{p(\cdot)}$$

greater than $\Omega f(\delta)_p$ and therefore the class $Lip_p(\omega, M)$ constructed by modulus $\Omega f(\delta)_p$ is wider than such one constructed by $\Omega_p f(\delta)$. Moreover, in these papers and also in [7] there is the assumption $p(x) \geq p_- > 1$, since for $p_- = 1$ the quantity $\Omega_p f(\delta)$ may not tend to 0 when $\delta \rightarrow 0$ for some $f \in L_{2\pi}^p$. In the papers [1, 5], [11, Theorem 5 (v)], [6, Theorem 5 (vi)] where $p(x) = \text{const}$. there is used yet bigger the integral modulus of continuity. We can note that in all above cited papers the simultaneous assumptions $p_- = 1$ and $\alpha = 1$ are impossible. Furthermore, in Theorem 4 there is considered very general class of matrices defined of the means $T_{n,A}[f]$ which special cases occur in the above cited papers.

3. Lemmas

We start with two lemmas from the papers of I. I. Sharapudinov. The first one from [10, Lemma 2.3] will be formulated without a proof but the second one similar to that from the paper [9] but in more general form will be proved.

Lemma 1 (see [10, Lemma 2.3]). *Let $p = p(x)$ be a measurable 2π -periodic function, such that $1 \leq p_- \leq p \leq p^+ < \infty$ and g be a function of two variables, 2π -periodic and measurable on $[-\pi, \pi] \times [-\pi, \pi]$. Then*

$$\left\| \int_{-\pi}^{\pi} g(\cdot, t) dt \right\|_p \leq 2 \int_{-\pi}^{\pi} \|g(\cdot, t)\|_p dt.$$

Lemma 2 (cf. [9, Lemma 3.1]). *Let $f \in L_{2\pi}^p$ with $p = p(x) \in \Pi_{2\pi}$. Then*

$$\left\| f_{\frac{1}{2\lambda}}(\cdot + \tau) \right\|_p = O(1) \|f(\cdot)\|_p$$

for every real τ , where $f_{\frac{1}{2\lambda}}(\tau) = \lambda \int_{-\frac{1}{2\lambda} + \tau}^{\frac{1}{2\lambda} + \tau} f(t) dt$ with $\lambda > 1$.

Proof. Let $h = \frac{1}{[\lambda]}$,

$$x_k = (kh - 1)\pi, \quad k = 0, \pm 1, \pm 2, \dots, \quad (5)$$

$$s_k = \min \{p(x) : x_{k-1} \leq x \leq x_{k+2}\}, \quad k = 0, \pm 1, \pm 2, \dots, \quad (6)$$

$$p_t(x) = s_k \quad (x_k - t \leq x < x_{k+1} - t), \quad k = 0, \pm 1, \pm 2, \dots, \quad (7)$$

whence $p_t(x) = p_0(x + t)$ is a 2π -periodic step function such that,

$$s_{k-(m+1)} = p_t(x) \leq p(x)$$

for $m\pi h \leq t \leq (m+2)\pi h$, since $x_{k-(m+2)} \leq x \leq x_{k-(m-1)}$ for such t .

Let $\|f(\cdot)\|_p \leq 1$ and $\tau \in \mathbb{R}$. There exists an integer m such that

$$m\pi h \leq \tau \leq (m+2)\pi h.$$

It is clear that

$$\begin{aligned} J &= \int_{-\pi}^{\pi} \left| f_{\frac{1}{2\lambda}}(x + \tau) \right|^{p(x)} dx = \int_{-\pi}^{\pi} \left| \lambda \int_{-\frac{1}{2\lambda} + x + \tau}^{\frac{1}{2\lambda} + x + \tau} f(t) dt \right|^{p(x)} dx \\ &= \int_{-\pi - (m+1)\pi h}^{\pi - (m+1)\pi h} \left| \lambda \int_{-\frac{1}{2\lambda} + x + \tau}^{\frac{1}{2\lambda} + x + \tau} f(t) dt \right|^{p(x)} dx \\ &= \sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}}^{x_{k-m}} \left| \lambda \int_{-\frac{1}{2\lambda} + x + \tau}^{\frac{1}{2\lambda} + x + \tau} f(t) dt \right|^{p(x)} dx \\ &= \sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}}^{x_{k-m}} \left| \lambda \int_{-\frac{1}{2\lambda} + x + \tau}^{\frac{1}{2\lambda} + x + \tau} f(t) dt \right|^{p(x) - s_{k-(m+1)} + s_{k-(m+1)}} dx. \end{aligned}$$

Further, similarly as in [9, Theorem 2.1],

$$\begin{aligned} \left| \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} f(t) dt \right| &\leq \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} |f(t)| dt \leq \int_{-\pi+x+\tau}^{\pi+x+\tau} |f(t)| dt = \int_{-\pi}^{\pi} |f(t)| dt \\ &= \|f(\cdot)\|_1 \leq (2\pi + 1) \|f(\cdot)\|_p \leq (2\pi + 1) \end{aligned}$$

and for $x_{k-(m+1)} \leq x \leq x_{k-m}$

$$p(x) - s_{k-(m+1)} = O\left(\frac{1}{\ln \lambda}\right),$$

whence

$$\lambda^{p(x)-s_{k-(m+1)}} = O\left(\lambda^{1/\ln \lambda}\right) = O(1).$$

Next, by the Jensen inequality,

$$\begin{aligned} J &= O(1) \sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}}^{x_{k-m}} \left| \lambda \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} f(t) dt \right|^{s_{k-(m+1)}} dx \\ &= O(1) \sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}}^{x_{k-m}} \left[\lambda \int_{-\frac{1}{2\lambda}+x+\tau}^{\frac{1}{2\lambda}+x+\tau} |f(t)|^{s_{k-(m+1)}} dt \right] dx \\ &= O(1) \sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}}^{x_{k-m}} \left[\lambda \int_{-\frac{1}{2\lambda}+\tau}^{\frac{1}{2\lambda}+\tau} |f(t+x)|^{s_{k-(m+1)}} dt \right] dx \\ &= O(1) \lambda \int_{-\frac{1}{2\lambda}+\tau}^{\frac{1}{2\lambda}+\tau} \left[\sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}}^{x_{k-m}} |f(t+x)|^{s_{k-(m+1)}} dx \right] dt \\ &= O(1) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \left[\sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}}^{x_{k-m}} |f(x-t)|^{s_{k-(m+1)}} dx \right] dt \\ &= O(1) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \left[\sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}-t}^{x_{k-m}-t} |f(x)|^{s_{k-(m+1)}} dx \right] dt \\ &= O(1) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \left[\sum_{k=0}^{2[\lambda]-1} \int_{x_{k-(m+1)}-t}^{x_{k-m}-t} |f(x)|^{p_t(x)} dx \right] dt \\ &= O(1) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \left[\int_{-\pi-(m+1)\pi h-t}^{\pi-(m+1)\pi h-t} |f(x)|^{p_t(x)} dx \right] dt \\ &= O(1) \lambda \int_{-\frac{1}{2\lambda}-\tau}^{\frac{1}{2\lambda}-\tau} \left[\int_{-\pi}^{\pi} |f(x)|^{p_t(x)} dx \right] dt = O(1), \end{aligned}$$

since similarly as in [9, p. 143] $\int_{-\pi}^{\pi} |f(x)|^{p_t(x)} dx = O(1)$. Thus our result follows. $\square$

Next we present some estimate of the kernel.

Lemma 3. *If $\beta \geq 0$ and $0 < t \leq \pi$, then*

$$\left| \sum_{k=0}^n (k+1)^\beta \frac{\sin \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} \right| \leq \pi^2 \frac{(n+1)^\beta}{t^2}.$$

Proof. Using the Abel transformation

$$\begin{aligned} & \left| \sum_{k=0}^n (k+1)^\beta \frac{\sin \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} \right| \\ & \leq \left| \sum_{k=0}^{n-1} [(k+1)^\beta - (k+2)^\beta] \sum_{l=0}^k \frac{\sin \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} + (n+1)^\beta \sum_{l=0}^n \frac{\sin \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} \right|. \end{aligned}$$

Since $\sum_{l=0}^n \frac{\sin \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} = \frac{\sin^2 \frac{(n+1)t}{2}}{2 \sin^2 \frac{t}{2}}$, the above expression does not exceed

$$\begin{aligned} & \sum_{k=0}^{n-1} |(k+1)^\beta - (k+2)^\beta| \frac{\sin^2 \frac{(k+1)t}{2}}{2 \sin^2 \frac{t}{2}} + (n+1)^\beta \frac{\sin^2 \frac{(n+1)t}{2}}{2 \sin^2 \frac{t}{2}} \\ & \leq \frac{\pi^2}{2t^2} \left[\sum_{k=0}^{n-1} |(k+1)^\beta - (k+2)^\beta| + (n+1)^\beta \right] \\ & \leq \frac{\pi^2}{2t^2} [2(n+1)^\beta - 1] \leq \frac{\pi^2}{t^2} (n+1)^\beta, \end{aligned}$$

whence the desired estimate follows. $\square$

Application of Lemma 3 gives next more general estimate.

Lemma 4. *If (1) for some $\beta \geq 0$ and (2) hold, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} a_{n,k} \sum_{\nu=0}^k l_{\nu} \cos \nu t \right| dt = O(1), \text{ where } l_{\nu} = \begin{cases} 1, & \text{when } \nu = 0, \\ \frac{\pi}{4 \sin \frac{\pi}{8}}, & \text{when } \nu > 0. \end{cases}$$

Proof. Since

$$1 + \frac{\pi}{4 \sin \frac{\pi}{8}} \sum_{\nu=1}^k \cos \nu t = 1 + \frac{\pi}{4 \sin \frac{\pi}{8}} \left(\frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} - \frac{1}{2} \right)$$

therefore

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} a_{n,k} \left\{ 1 - \frac{\pi}{8 \sin \frac{\pi}{8}} + \frac{\pi}{4 \sin \frac{\pi}{8}} \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right\} \right| dt \\ & \leq \left| 1 - \frac{\pi}{8 \sin \frac{\pi}{8}} \right| + \frac{1}{8 \sin \frac{\pi}{8}} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} a_{n,k} \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt. \end{aligned}$$

Using the Abel transformation, Lemma 3 and (1) we get

$$\begin{aligned} & \left| \sum_{k=0}^{\infty} \frac{a_{n,k}}{(k+1)^{\beta}} (k+1)^{\beta} \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right| \\ &= \left| \sum_{k=0}^{\infty} \left[\frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right] \sum_{l=0}^k (l+1)^{\beta} \frac{\sin \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} \right| \\ &\leq \frac{\pi^2}{t^2} \sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| = \frac{O(1)}{t^2(n+1)}. \end{aligned}$$

Thus

$$\begin{aligned} & \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} a_{n,k} \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt \\ &= \left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi} \right) \left| \sum_{k=0}^{\infty} a_{n,k} \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt \\ &\leq \int_0^{\frac{\pi}{n+1}} \sum_{k=0}^{\infty} a_{n,k} \frac{2k+1}{2} dt + \int_{\frac{\pi}{n+1}}^{\pi} \frac{O(1)}{t^2(n+1)} dt \end{aligned}$$

and by the assumption (2)

$$\begin{aligned} & \int_{-\pi}^{\pi} \left| \sum_{k=0}^n a_{n,k} \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt \\ &\leq \frac{\pi}{n+1} O(n+1) + \frac{O(1)}{n+1} \left(\frac{\pi}{n+1} \right)^{-1} = O(1), \end{aligned}$$

whence our estimate follows. $\square$

We will yet to need the following approximation result.

Lemma 5. Let $f \in L_{2\pi}^p$ with $p = p(x) \in \Pi_{2\pi}$ and let t_n be a trigonometric polynomial of the degree at most n , such that $\|f - t_n\|_p = O(1) \Omega f(\frac{1}{n+1})_p$. If (1) for some $\beta \geq 0$ and (2) hold, then

$$\left\| \sum_{k=0}^{\infty} a_{n,k} S_k[f - t_n] \right\|_p = O \left(\Omega f \left(\frac{1}{n+1} \right)_p \right).$$

Proof. First of all, under the notation $f_h(t) = \frac{1}{2h} \int_{-h}^h f(y+t) dy$, we will prove that

$$a_0(f) = a_0(f_h), \quad a_{\nu}(f) = \frac{\nu h}{\sin \nu h} a_{\nu}(f_h), \quad (\nu = 1, 2, 3, \dots)$$

and

$$b_{\nu}(f) = \frac{\nu h}{\sin \nu h} b_{\nu}(f_h), \quad (\nu = 1, 2, 3, \dots).$$

Really, it is clear that $a_0(f_h) = a_0(f)$ and for $\nu = 1, 2, 3, \dots$,

$$\begin{aligned}
 \frac{a_\nu}{b_\nu}(f_h) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2h} \int_{-h}^h f(y+t) dy \frac{\cos}{\sin} \nu t dt \\
 &= \frac{1}{2\pi} \frac{1}{2h} \int_{-h}^h \left(\int_{-\pi}^{\pi} f(y+t) \frac{\cos}{\sin} \nu t dt \right) dy \\
 &= \frac{1}{2\pi} \frac{1}{2h} \int_{-h}^h \left(\int_{-\pi}^{\pi} f(t) \frac{\cos}{\sin} \nu(t-y) dt \right) dy \\
 &= \frac{1}{2\pi} \frac{1}{2h} \int_{-h}^h \left(\int_{-\pi}^{\pi} f(t) \left(\frac{\cos}{\sin} \nu t \cos \nu y \pm \frac{\sin}{\cos} \nu t \sin \nu y \right) dt \right) dy \\
 &= \frac{1}{2\pi} \frac{1}{2h} \int_{-h}^h \left(\cos \nu y \int_{-\pi}^{\pi} f(t) \frac{\cos}{\sin} \nu t dt \pm \sin \nu y \int_{-\pi}^{\pi} f(t) \frac{\sin}{\cos} \nu t dt \right) dy \\
 &= \frac{a_\nu}{b_\nu}(f) \frac{1}{2h} \int_{-h}^h \cos \nu t dt \pm \frac{b_\nu}{a_\nu}(f) \frac{1}{2h} \int_{-h}^h \sin \nu t dt = \frac{a_\nu}{b_\nu}(f) \frac{\sin \nu h}{\nu h}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 S_k[f](x) &= \frac{a_0(f)}{2} + \sum_{\nu=1}^k (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x) \\
 &= \frac{a_0(f_h)}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} a_\nu(f_h) \cos \nu x + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} b_\nu(f_h) \sin \nu x \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_h(t) dt + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \frac{1}{\pi} \int_{-\pi}^{\pi} f_h(t) \cos \nu t dt \cos \nu x \\
 &\quad + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \frac{1}{\pi} \int_{-\pi}^{\pi} f_h(t) \sin \nu t dt \sin \nu x \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_h(t) dt + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \frac{1}{\pi} \int_{-\pi}^{\pi} f_h(t) \cos \nu(t-x) dt \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_h(t) dt + \frac{1}{\pi} \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \int_{-\pi}^{\pi} f_h(t+x) \cos \nu t dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_h(t+x) \left(\frac{1}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \cos \nu t \right) dt.
 \end{aligned}$$

Consequently, denoting $t_{n,h}(t) = \frac{1}{2h} \int_{-h}^h t_n(y+t) dy$, we obtain

$$\begin{aligned} S_k[f - t_n](x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} [f - t_n]_h(t+x) \left(\frac{1}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \cos \nu t \right) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (f_h(t+x) - t_{n,h}(t+x)) \left(\frac{1}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \cos \nu t \right) dt \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{k=0}^{\infty} a_{n,k} S_k[f - t_n](x) &= \sum_{k=0}^{\infty} a_{n,k} \frac{1}{\pi} \int_{-\pi}^{\pi} (f_h(t+x) - t_{n,h}(t+x)) \left(\frac{1}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \cos \nu t \right) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (f_h(t+x) - t_{n,h}(t+x)) \sum_{k=0}^{\infty} a_{n,k} \left(\frac{1}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \cos \nu t \right) dt. \end{aligned}$$

Hence, by Lemma 1 and Lemma 2 with $0 < h < \frac{1}{2}$ and $|t| \leq \pi$

$$\begin{aligned} &\left\| \sum_{k=0}^{\infty} a_{n,k} S_k[f - t_n] \right\|_p \\ &\leq \frac{2}{\pi} \int_{-\pi}^{\pi} \|f_h(t+\cdot) - t_{n,h}(t+\cdot)\|_p \left| \sum_{k=0}^{\infty} a_{n,k} \left(\frac{1}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \cos \nu t \right) \right| dt \\ &= O(1) \frac{1}{\pi} \int_{-\pi}^{\pi} \|f - t_n\|_p \left| \sum_{k=0}^{\infty} a_{n,k} \left(\frac{1}{2} + \sum_{\nu=1}^k \frac{\nu h}{\sin \nu h} \cos \nu t \right) \right| dt \end{aligned}$$

and further, taking $h = \frac{\pi}{8\nu} < \frac{1}{2}$ for $\nu = 1, 2, \dots$, we have

$$\begin{aligned} &\left\| \sum_{k=0}^{\infty} a_{n,k} S_k[f - t_n] \right\|_p \\ &= O(1) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} a_{n,k} \left(1 + \frac{\pi}{4 \sin \frac{\pi}{8}} \sum_{\nu=1}^k \cos \nu t \right) \right| dt \|f - t_n\|_p. \end{aligned}$$

Since, by Lemma 4

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} a_{n,k} \left(1 + \frac{\pi}{4 \sin \frac{\pi}{8}} \sum_{\nu=1}^k \cos \nu t \right) \right| dt = O(1)$$

therefore, by the assumptions,

$$\left\| \sum_{k=0}^{\infty} a_{n,k} S_k [f - t_n] \right\|_p = O(1) \|f - t_n\|_p = O(1) \Omega f \left(\frac{1}{n+1} \right)_p$$

and our proof is ended. $\square$

4. Proofs of Main Results

4.1. Proof of Theorem 1

If t_n is a such polynomial that $\|f - t_n\|_p = O \left(\Omega f \left(\frac{1}{n+1} \right)_p \right)$, then

$$\begin{aligned} & \|T_{n,A} [f] - f\|_p \\ &= \left\| T_{n,A} [f] - \sum_{k=0}^n a_{n,k} t_k - \sum_{k=n+1}^{\infty} a_{n,k} t_n + \sum_{k=0}^n a_{n,k} t_k + \sum_{k=n+1}^{\infty} a_{n,k} t_n - f \right\|_p \\ &\leq \left\| T_{n,A} [f] - \sum_{k=0}^n a_{n,k} t_k - \sum_{k=n+1}^{\infty} a_{n,k} t_n \right\|_p + \left\| \sum_{k=0}^n a_{n,k} t_k + \sum_{k=n+1}^{\infty} a_{n,k} t_n - f \right\|_p \\ &= \left\| \sum_{k=0}^n a_{n,k} \{S_k [f] - t_k\} + \sum_{k=n+1}^{\infty} a_{n,k} \{S_k [f] - t_n\} \right\|_p \\ &\quad + \left\| \sum_{k=0}^n a_{n,k} (f - t_k) + \sum_{k=n+1}^{\infty} a_{n,k} (f - t_n) \right\|_p \\ &\leq \left\| \sum_{k=0}^{\infty} a_{n,k} S_k [f - t_n] \right\|_p + \left\| \sum_{k=0}^n a_{n,k} (f - t_k) \right\|_p + \left\| \sum_{k=n+1}^{\infty} a_{n,k} (f - t_n) \right\|_p \\ &= \left\| \sum_{k=0}^{\infty} a_{n,k} S_k [f - t_n] \right\|_p + O(1) \sum_{k=0}^n a_{n,k} \Omega f \left(\frac{1}{k+1} \right)_p \\ &\quad + \sum_{k=n+1}^{\infty} a_{n,k} O \left(\Omega f \left(\frac{1}{n+1} \right)_p \right), \end{aligned}$$

since

$$S_k [f - t_n] (x) = \begin{cases} S_k f (x) - t_k (x), & \text{for } k \leq n. \\ S_k f (x) - t_n (x), & \text{for } k \geq n. \end{cases}$$

We note that by Lemma 5

$$\left\| \sum_{k=0}^{\infty} a_{n,k} S_k [f - t_n] \right\|_p = O \left(\Omega f \left(\frac{1}{n+1} \right)_p \right).$$

Thus our result follows. $\square$

4.2. Proof of Theorem 2

We can note that the assumptions on entries of A_0 yield

$$\begin{aligned} & \sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \\ &= \sum_{k=0}^{n-1} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| + a_{n,n} \\ &= O(a_{n,n}) + a_{n,n} = O(a_{n,n}) = O\left(\frac{1}{n+1}\right), \end{aligned}$$

and

$$\sum_{k=0}^{\infty} (k+1)a_{n,k} = \sum_{k=0}^n (k+1)a_{n,k} \leq (n+1) \sum_{k=0}^n a_{n,k} = n+1.$$

Thus the result follows by Theorem 1, since using the monotonicity of $\Omega f(\delta)_p$ with respect to $\delta > 0$

$$\sum_{k=0}^n a_{n,k} \Omega f\left(\frac{1}{k+1}\right)_p \geq \Omega f\left(\frac{1}{n+1}\right)_p \sum_{k=0}^n a_{n,k} = \Omega f\left(\frac{1}{n+1}\right)_p.$$

This ends the proof of Theorem 2. $\square$

4.3. Proof of Theorem 3

The subadditivity of ω implies $\omega(n\delta) \leq n\omega(\delta)$, whence $\omega(\lambda\delta) \leq (\lambda+1)\omega(\delta)$ and therefore $\frac{\omega(\delta_2)}{\delta_2} \leq 2\frac{\omega(\delta_1)}{\delta_1}$ since $\omega(\delta_2) = \omega\left(\frac{\delta_1}{\delta_1}\delta_2\right) \leq \left(\frac{\delta_2}{\delta_1}+1\right)\omega(\delta_1) = \left(\frac{\delta_2}{\delta_1} + \frac{\delta_1}{\delta_1}\right)\omega(\delta_1) \leq \left(\frac{\delta_2}{\delta_1} + \frac{\delta_2}{\delta_1}\right)\omega(\delta_1) = 2\frac{\delta_2}{\delta_1}\omega(\delta_1)$, where $n \in \mathbb{N}_0$, $\lambda \geq 0$ and $0 < \delta_1 \leq \delta_2$. Hence, by (1) with $\beta > 0$,

$$\begin{aligned} \sum_{k=0}^n a_{n,k} \omega\left(\frac{1}{k+1}\right) &= \sum_{k=0}^n \frac{a_{n,k}}{k+1} \frac{\omega\left(\frac{1}{k+1}\right)}{\frac{1}{k+1}} \\ &\leq 2(n+1) \omega\left(\frac{1}{n+1}\right) \sum_{k=0}^n \frac{a_{n,k}}{k+1} \\ &= 2(n+1) \omega\left(\frac{1}{n+1}\right) \sum_{k=0}^{\infty} (k+1)^{\beta-1} \frac{a_{n,k}}{(k+1)^{\beta}} \\ &\leq 2(n+1) \omega\left(\frac{1}{n+1}\right) \sum_{k=0}^{\infty} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \sum_{l=0}^k (l+1)^{\beta-1} \\ &= O(n+1) \omega\left(\frac{1}{n+1}\right) \sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \\ &= (n+1) \omega\left(\frac{1}{n+1}\right) O\left(\frac{1}{n+1}\right) = O\left(\omega\left(\frac{1}{n+1}\right)\right). \end{aligned}$$

Thus, by Theorem 1, our result follows. $\square$

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References

- [1] Chandra, P.: Trigonometric approximation of functions in L_p -norm. J. Math. Anal. Appl. **275**, 13–26 (2002)
- [2] Guven, A.: Trigonometric approximation by matrix transformation in $L^{p(x)}$ spaces. Anal. Appl. (Singap.) **10**(1), 47–65 (2012)
- [3] Guven, A., Israfilov, D.: Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$. J. Math. Inequal. **4**(2), 285–299 (2010)
- [4] Krasniqi, X.Z.: Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$. Palest. J. Math. **6**(1), 84–93 (2017)
- [5] Leindler, L.: Trigonometric approximation in L_p -norm. J. Math. Anal. Appl. **302**(1), 129–136 (2005)
- [6] Mohapatra, R.N., Szal, B.: On trigonometric approximation of functions in the L^q norm. Demonstr. Math. **51**(1), 17–26 (2018)
- [7] Shakh-Emirov, T.N.: Approximation properties of some types of linear means in space $L_{2\pi}^{p(x)}$. Izv. Sarat. Inst. New ser. Ser. Mat. Mech. Inf. **1**(2), 108–112 (2013)
- [8] Sharapudinov, I.I.: Topology of the space $L^{p(t)}([0, 1])$. Math. Notes Acad. Sci. USSR **26**, 796–806 (1979)
- [9] Sharapudinov, I.I.: Several questions of approximation theory in spaces $L^{p(x)}$. Anal. Math. **33**(2), 135–153 (2007)
- [10] Sharapudinov, I.I.: Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials. Izv. RAN. Ser. Mat. **77**(2), 197–224 (2013)
- [11] Szal, B.: Trigonometric approximation by Nörlund type means in L^p -norm. Comment. Math. Univ. Carol. **50**(4), 575–589 (2009)

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