A Common q -Analogue of Two Supercongruences

Victor J. W. Guo and Wadim Zudilin

Abstract. We give a q -congruence whose specializations $q = -1$ and $q = 1$ correspond to supercongruences (B.2) and (H.2) on Van Hamme's list (in: *p*-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Applied Mathematics, vol 192. Dekker, New York, pp 223–236, 1997):

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) A_k \equiv p(-1)^{(p-1)/2} \pmod{p^3} \quad \text{and}$$

$$\sum_{k=0}^{(p-1)/2} A_k \equiv a(p) \pmod{p^2},$$

where $p > 2$ is prime,

$$A_k = \prod_{j=0}^{k-1} \left(\frac{1/2 + j}{1 + j} \right)^3 = \frac{1}{2^{6k}} \binom{2k}{k}^3 \quad \text{for } k = 0, 1, 2, \dots,$$

and $a(p)$ is the p th coefficient of the modular form $q \prod_{j=1}^{\infty} (1 - q^{4j})^6$ (of weight 3). We complement our result with a general common q -congruence for related hypergeometric sums.

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1. Introduction

The formula of Bauer [1] from 1859,

$$\sum_{k=0}^{\infty} (-1)^k (4k+1) A_k = \frac{2}{\pi}, \quad \text{where } A_k = \frac{1}{2^{6k}} \binom{2k}{k}^3 \quad \text{for } k = 0, 1, 2, \dots, \quad (1.1)$$

is one of traditional targets for different methods of proofs of hypergeometric identities. Its special status is probably linked to the fact that it belongs to a family of series for $1/\pi$ of Ramanujan type, after Ramanujan [21] brought to life in 1914 a long list of similar looking equalities for the constant but with a faster convergence. Identity (1.1) is a particular instance of ${}_4F_3$ hypergeometric summation (known to Ramanujan) but there are several proofs of it, including the original one [1] of Bauer, that do not require any knowledge of hypergeometric functions. One notable—computer—proof of (1.1) was given in 1994 by Ekhad and Zeilberger [2] using the Wilf–Zeilberger (WZ) method of creative telescoping.

It was observed in 1997 by Van Hamme [28] that many Ramanujan's and Ramanujan-like evaluations have nice p -adic analogues; for example, the congruence

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) A_k \equiv p(-1)^{(p-1)/2} \pmod{p^3} \quad (1.2)$$

(tagged (B.2) on Van Hamme's list) is valid for any prime $p > 2$ and corresponds to the equality (1.1). The congruence (1.2) was first proved by Mortenson [19] using a ${}_6F_5$ hypergeometric transformation; it later received another proof by one of these authors [29] via the WZ method [in fact, using the very same 'WZ certificate' as in [2] for (1.1)]. Notice that (1.2) is an example of *supercongruence* meaning that it holds modulo a power of p greater than 1.

Another entry on Van Hamme's 1997 list [28], tagged (H.2), is the congruence

$$\sum_{k=0}^{(p-1)/2} A_k \equiv \begin{cases} -\Gamma_p(1/4)^4 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (1.3)$$

again for any $p > 2$ prime, and $\Gamma_p(x)$ is the p -adic Gamma function. Van Hamme not only observed but also proved (1.3) in [28], and it was later generalized by Sun [23, 24, Theorem 2.5], Guo and Zeng [12, Corollary 1.2], Long and Ramakrishna [17], Liu [15, 16, Theorem 1.5] in different ways. For example, Long and Ramakrishna [17, Theorem 3] gave the following generalization of (1.3):

$$\sum_{k=0}^{(p-1)/2} A_k \equiv \begin{cases} -\Gamma_p(1/4)^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{p^2}{16} \Gamma_p(1/4)^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (1.4)$$

Recently, these authors [14, Theorem 2] proved that, for any positive odd integer n , modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^2; q^2)_k^2 (q^4; q^4)_k} q^{2k} \equiv \begin{cases} (q^2; q^4)_{(n-1)/4}^2 q^{(n-1)/2} & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}. \end{cases} \quad (1.5)$$

Here and in what follows, $\Phi_n(q)$ denotes the n th *cyclotomic polynomial*; the q -shifted factorial is given by $(a; q)_0 = 1$ and $(a; q)_n = (1-a)(1-aq) \dots (1-aq^{n-1})$ for $n \geq 1$ or $n = \infty$, while $[n] = [n]_q = 1 + q + \dots + q^{n-1}$ stands for the q -integer. Van Hamme [27, Theorem 3] also proved that

$$\binom{-1/2}{(p-1)/4} \equiv -\frac{\Gamma_p(1/4)^2}{\Gamma_p(1/2)} \pmod{p^2};$$

in view of $\Gamma_p(1/2)^2 = -1$ for $p \equiv 1 \pmod{4}$, by letting $q \rightarrow 1$ in (1.5) for $n = p$ we immediately obtain (1.3).

One feature of (1.3) (not highlighted in [28]) is its connection with the coefficients

$$a(p) = \begin{cases} 2(a^2 - b^2) & \text{if } p = a^2 + b^2, a \text{ odd}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (1.6)$$

of CM modular form $q \prod_{j=1}^{\infty} (1 - q^{4j})^6$ of weight 3, namely, the congruence

$$a(p) \equiv -\Gamma_p(1/4)^4 \pmod{p^2} \quad \text{for primes } p \equiv 1 \pmod{4}.$$

This served as a main motivation in [14] for not only establishing (1.5) but also speculating on possible q -deformation of modular forms.

For some other recent progress on q -analogues of supercongruences, the reader is referred to [4, 5, 7–11, 13, 20, 22, 26, 29]. In particular, the authors [13] introduced and executed a new method of creative microscoping to prove (and reprove) many q -analogues of classical supercongruences and also raised some problems on q -congruences. Using this method, the first author [6] gave a refinement of (1.5) modulo $\Phi_n(q)^3$ for $n \equiv 3 \pmod{4}$, in other words, a q -analogue of (1.4) for $p \equiv 3 \pmod{4}$.

A goal of this note is to present the following new q -analogue of Van Hamme's supercongruence (1.3).

Theorem 1.1. *Let n be a positive odd integer. Then*

$$\begin{aligned} & \sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \\ & \equiv \frac{[n]_{q^2}(q^3; q^4)_{(n-1)/2}}{(q^5; q^4)_{(n-1)/2}} q^{(1-n)/2} \begin{cases} (\bmod \Phi_n(q)^2 \Phi_n(-q)^3) & \text{if } n \equiv 1 \pmod{4}, \\ (\bmod \Phi_n(q)^3 \Phi_n(-q)^3) & \text{if } n \equiv 3 \pmod{4}. \end{cases} \end{aligned} \quad (1.7)$$

Note that $\Phi_n(q)\Phi_n(-q) = \Phi_n(q^2)$ for odd indices n .

The $n \equiv 3 \pmod{4}$ case of Theorem 1.1 confirms a conjecture of these authors [13, Conjecture 4.13], which states that, for $n \equiv 3 \pmod{4}$,

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})(q^2; q^4)_k^3}{(1+q)(q^4; q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2 \Phi_n(-q)}.$$

It is not difficult to verify that

$$\frac{(3/4)_{(p-1)/2}}{(5/4)_{(p-1)/2}} \equiv -\frac{p}{16} \Gamma_p(1/4)^4 \pmod{p^2}$$

for $p \equiv 3 \pmod{4}$, where $(a)_n = a(a+1)\dots(a+n-1)$ denotes the rising factorial (also known as Pochhammer's symbol). Therefore, the q -congruence (1.7) reduces to (1.4) for $p \equiv 3 \pmod{4}$ when $n = p$ and $q \rightarrow 1$, and it reduces to (1.3) for $p \equiv 1 \pmod{4}$ when $n = p$ and $q \rightarrow 1$. Moreover, letting $n = p$ and $q \rightarrow -1$ in (1.7), we immediately get (1.2). Thus, Theorem 1.1 presents a common q -analogue of supercongruences (1.2) and (1.3). We point out that other different q -analogues of (1.2) have been given in [7, 8].

Recently, Mao and Pan [18] (see also Sun [25, Theorem 1.3]) proved that, if $p \equiv 1 \pmod{4}$ is a prime, then

$$\sum_{k=0}^{(p+1)/2} \frac{(-1/2)_k^3}{k!^3} \equiv 0 \pmod{p^2}. \quad (1.8)$$

In this note, we prove the following q -analogue of (1.8).

Theorem 1.2. *Let $n > 1$ be an odd integer. Then*

$$\begin{aligned} & \sum_{k=0}^{(n+1)/2} \frac{(1+q^{4k-1})(q^{-2}; q^4)_k^3}{(1+q)(q^4; q^4)_k^3} q^{7k} \\ & \equiv \frac{[n]_{q^2}(q; q^4)_{(n-1)/2}}{(q^7; q^4)_{(n-1)/2}} q^{(n-3)/2} \begin{cases} (\bmod \Phi_n(q)^3 \Phi_n(-q)^3) & \text{if } n \equiv 1 \pmod{4}, \\ (\bmod \Phi_n(q)^2 \Phi_n(-q)^3) & \text{if } n \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

The $n \equiv 1 \pmod{4}$ case of Theorem 1.2 also confirms a conjecture of the first author and Schlosser [11, Conjecture 10.2].

For n prime, letting $q \rightarrow 1$ in Theorem 1.2 we obtain the following generalization of (1.8).

Corollary 1.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p+1)/2} \frac{(-1/2)_k^3}{k!^3} \equiv p \frac{(1/4)_{(p-1)/2}}{(7/4)_{(p-1)/2}} \begin{cases} (\bmod p^3) & \text{if } p \equiv 1 \pmod{4}, \\ (\bmod p^2) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

On the other hand, for n prime and $q \rightarrow -1$ in Theorem 1.2, we are led to the following result:

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k-1) \frac{(-1/2)_k^3}{k!^3} \equiv p(-1)^{(p+1)/2} \pmod{p^3}. \quad (1.9)$$

It should be mentioned that a different q -analogue of (1.9) was given in [13, Theorem 4.9] with $r = -1$, $d = 2$ and $a = 1$ (see also [11, Section 5]).

Moreover, for the summation formula

$$\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k^3}{k!^3} = 12 \frac{\Gamma(3/4)^4}{\pi^3},$$

we have the following q -analogue.

Theorem 1.4. *We have*

$$\sum_{k=0}^{\infty} \frac{(1 + q^{4k-1})(q^{-2}; q^4)_k^3}{(1 + q^{-1})(q^4; q^4)_k^3} q^{7k} = \frac{(q^2; q^4)_{\infty} (q^5; q^4)_{\infty}^2 (q^6; q^4)_{\infty}}{(q^3; q^4)_{\infty} (q^4; q^4)_{\infty}^2 (q^7; q^4)_{\infty}}.$$

Both Theorems 1.1 and 1.2 are particular cases of a more general result, which we state and prove in the next section, while Theorem 1.4 follows from a classical q -identity.

2. A Family of q -Congruences from the q -Dixon Sum

In this section we establish the following family of one-parameter q -congruences.

Theorem 2.1. *Let $n \geq 1$ be an odd integer and ℓ an integer with $0 \leq \ell \leq (n-1)/2$. Then*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(1 + q^{4k-2\ell+1})(q^{2-4\ell}; q^4)_k^3}{(1 + q^{1-2\ell})(q^4; q^4)_k^3} q^{(6\ell+1)k} \\ & \equiv \frac{(1 - q^{2n})(q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1 - q^{2-4\ell})(q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} q^{(2\ell-1)((n-1)/2+\ell)} \begin{cases} (\bmod \Phi_n(q)^2 \Phi_n(-q)^3) \\ \text{if } n + 2\ell \equiv 1 \pmod{4}, \\ (\bmod \Phi_n(q)^3 \Phi_n(-q)^3) \\ \text{if } n + 2\ell \equiv 3 \pmod{4}. \end{cases} \end{aligned} \quad (2.1)$$

Note that the q -congruence (2.1) remains true when the sum is over k from 0 to $(n-1)/2 + \ell$, since $(q^{2-4\ell}; q^4)_k / (q^4; q^4)_k \equiv 0 \pmod{\Phi_n(q^2)}$ for $(n-1)/2 + \ell < k \leq n-1$. Furthermore, when $\ell = 0$ and $\ell = 1$ (hence $n \geq 3$) the theorem reduces to Theorems 1.1 and 1.2, respectively.

The following easily proved q -congruence (see [11, Lemma 3.1]) is necessary in our derivation of Theorem 2.1.

Lemma 2.2. *Let n be a positive odd integer. Then, for $0 \leq k \leq (n-1)/2$, we have*

$$\frac{(aq; q^2)_{(n-1)/2-k}}{(q^2/a; q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k} \frac{(aq; q^2)_k}{(q^2/a; q^2)_k} q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

Like the proofs given in [13], we start with the following generalization of (1.7) with an extra parameter a .

Theorem 2.3. *Let $n > 1$ be an odd integer and $0 \leq \ell \leq (n-1)/2$. Then*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(1+q^{4k-2\ell+1})(aq^{2-4\ell}; q^4)_k (q^{2-4\ell}/a; q^4)_k (q^{2-4\ell}; q^4)_k}{(1+q^{1-2\ell})(aq^4; q^4)_k (q^4/a; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k} \\ & \equiv \frac{(1-q^{2n})(q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1-q^{2-4\ell})(q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} \\ & \quad \times q^{(2\ell-1)((n-1)/2+\ell)} \begin{cases} (\bmod \Phi_n(-q)(1-aq^{2n})(a-q^{2n})) \\ \text{if } n+2\ell \equiv 1 \pmod{4}, \\ (\bmod \Phi_n(q^2)(1-aq^{2n})(a-q^{2n})) \\ \text{if } n+2\ell \equiv 3 \pmod{4}. \end{cases} \end{aligned} \quad (2.2)$$

Proof. Performing the parameter substitutions $q \mapsto q^4$, $a \mapsto q^{2-4\ell}$, $b \mapsto bq^{2-4\ell}$ and $c \mapsto cq^{2-4\ell}$ in the q -Dixon sum [3, Appendix (II.13)], we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(1+q^{4k-2\ell+1})(q^{2-4\ell}; q^4)_k (bq^{2-4\ell}; q^4)_k (cq^{2-4\ell}; q^4)_k}{(1+q^{1-2\ell})(q^4/b; q^4)_k (q^4/c; q^4)_k (q^4; q^4)_k} \left(\frac{q^{6\ell+1}}{bc} \right)^k \\ & = \frac{(q^{6-4\ell}; q^4)_{\infty} (q^{2\ell+3}/b; q^4)_{\infty} (q^{2\ell+3}/c; q^4)_{\infty} (q^{4\ell+2}/bc; q^4)_{\infty}}{(q^4/b; q^4)_{\infty} (q^4/c; q^4)_{\infty} (q^{5-2\ell}; q^4)_{\infty} (q^{6\ell+1}/bc; q^4)_{\infty}}. \end{aligned} \quad (2.3)$$

Since n is odd, putting $b = q^{-2n}$ and $c = q^{2n}$ in (2.3) we see that the left-hand side terminates and is equal to

$$\begin{aligned} & \sum_{k=0}^{(n-1)/2+\ell} \frac{(1+q^{4k-2\ell+1})(q^{2-4\ell-2n}; q^4)_k (q^{2-4\ell+2n}; q^4)_k (q^{2-4\ell}; q^4)_k}{(1+q^{1-2\ell})(q^{4-2n}; q^4)_k (q^{4+2n}; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k} \\ & = \sum_{k=0}^{n-1} \frac{(1+q^{4k-2\ell+1})(q^{2-4\ell-2n}; q^4)_k (q^{2-4\ell+2n}; q^4)_k (q^{2-4\ell}; q^4)_k}{(1+q^{1-2\ell})(q^{4-2n}; q^4)_k (q^{4+2n}; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k}, \end{aligned}$$

while the right-hand side becomes

$$\begin{aligned} & \frac{(q^{2\ell-2n+3}; q^4)_{(n-1)/2+\ell} (q^{6-4\ell}; q^4)_{(n-1)/2+\ell}}{(q^{4-2n}; q^4)_{(n-1)/2+\ell} (q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} \\ & = \frac{(1-q^{2n})(q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1-q^{2-4\ell})(q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} q^{(2\ell-1)((n-1)/2+\ell)}. \end{aligned}$$

This proves that the q -congruence (2.2) holds modulo $1-aq^{2n}$ or $a-q^{2n}$.

On the other hand, by Lemma 2.2, for $0 \leq k \leq (n-1)/2 + \ell$, modulo $\Phi_n(q)$ we have

$$\begin{aligned}
& \frac{(aq^{1-2\ell}; q^2)_{(n-1)/2+\ell-k}}{(q^2/a; q^2)_{(n-1)/2+\ell-k}} \\
&= \frac{(aq^{1-2\ell}; q^2)_\ell (aq; q^2)_{(n-1)/2-k}}{(q^{n+1-2k}/a; q^2)_\ell (q^2/a; q^2)_{(n-1)/2-k}} \\
&\equiv (-a)^{(n-1)/2-2k} \frac{(aq^{1-2\ell}; q^2)_\ell (aq; q^2)_k}{(q^{n+1-2k}/a; q^2)_\ell (q^2/a; q^2)_k} q^{(n-1)^2/4+k} \\
&= (-a)^{(n-1)/2-2k} \frac{(aq^{1-2\ell}; q^2)_k (aq^{2k-2\ell+1}; q^2)_\ell}{(q^{n+1-2k}/a; q^2)_\ell (q^2/a; q^2)_k} q^{(n-1)^2/4+k} \\
&\equiv (-a)^{(n-1)/2+\ell-2k} \frac{(aq^{1-2\ell}; q^2)_k}{(q^2/a; q^2)_k} q^{(n-1)^2/4+k+(2k-\ell)\ell},
\end{aligned}$$

where we used $q^n \equiv 1 \pmod{\Phi_n(q)}$ in the last step. Using the above q -congruence we can easily check that, for odd $n > 1$ and $0 \leq k \leq (n-1)/2 + \ell$, sum of the k th and $((n-1)/2 + \ell - k)$ th summands on the left-hand side of (2.2) is congruent to 0 modulo $\Phi_n(-q)$ (or modulo $\Phi_n(q^2)$ if $n \equiv 3 - 2\ell \pmod{4}$). It follows that

$$\begin{aligned}
& \sum_{k=0}^{(n-1)/2+\ell} \frac{(1 + q^{4k-2\ell+1}) (aq^{2-4\ell}; q^4)_k (q^{2-4\ell}/a; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell}) (aq^4; q^4)_k (q^4/a; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k} \\
&\equiv 0 \begin{cases} \pmod{\Phi_n(-q)} & \text{if } n + 2\ell \equiv 1 \pmod{4}, \\ \pmod{\Phi_n(q^2)} & \text{if } n + 2\ell \equiv 3 \pmod{4}. \end{cases}
\end{aligned}$$

Clearly, the right-hand side of (2.1) is congruent to 0 modulo $\Phi_n(-q)$ if $n + 2\ell \equiv 1 \pmod{4}$ and modulo $\Phi_n(q^2)$ if $n + 2\ell \equiv 3 \pmod{4}$. Therefore, the q -congruence (2.2) holds modulo $\Phi_n(-q)$ if $n + 2\ell \equiv 1 \pmod{4}$ and modulo $\Phi_n(q^2)$ if $n + 2\ell \equiv 3 \pmod{4}$. Since the polynomials $1 - aq^{2n}$, $a - q^{2n}$ and $\Phi_n(-q)$ (or $\Phi_n(q^2)$) are pairwise coprime, we complete the proof of (2.2). $\square$

Proof of Theorem 2.1. We assume that $n > 1$, since the $n = 1$ case (making $\ell = 0$ only possible) is trivial. The limits of the denominators on both sides of (2.2) as $a \rightarrow 1$ are relatively prime to $\Phi_n(q^2)$, since k is in the range $0 \leq k \leq (n-1)/2 + \ell$. On the other hand, the limit of $(1 - aq^{2n})(a - q^{2n})$ as $a \rightarrow 1$ contains the factor $\Phi_n(q^2)^2$. $\square$

Proof of Theorem 1.4. Take $b = c = \ell = 1$ in Eq. (2.3). $\square$

3. Discussion

The method of creative microscoping used in our proofs indicates the origin of q -congruences from *infinite* q -hypergeometric identities; for example, the q -congruence (1.7) corresponds to the identity

$$\sum_{k=0}^{\infty} \frac{(1 + q^{4k+1}) (q^2; q^4)_k^3}{(1 + q) (q^4; q^4)_k^3} q^k = \frac{(q^2; q^4)_{\infty}^2 (q^3; q^4)_{\infty}^2}{(1 + q) (q; q^4)_{\infty}^2 (q^4; q^4)_{\infty}^2}, \quad (3.1)$$

which is just a particular instance of (2.3). Note that the limiting cases as $q \rightarrow -1$ and $q \rightarrow 1$ of (3.1) give the formulas (1.1) and

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} = \frac{\Gamma(1/4)^4}{4\pi^3} = \frac{8L(f, 1)}{\pi} = \frac{16L(f, 2)}{\pi^2} \quad (3.2)$$

where

$$f(\tau) = q \prod_{j=1}^{\infty} (1 - q^{4j})^6 = \sum_{n=1}^{\infty} a(n)q^n, \quad \text{with } q = \exp(2\pi i\tau),$$

is the CM modular form from the introduction and $L(f, s)$ denotes its L -function. This means that the q -identity (3.1) presents a common q -extension of evaluations (1.1) and (3.2)—the fact that makes it less surprising that the q -congruence (1.7) simultaneously extends (1.2) and (1.3).

The intermediate use of *parametric* q -hypergeometric identities in our proof of Theorem 2.1 based on the q -Dixon sum suggests that different q -congruences underlying (3.1) are possible. This is indeed the case when we analyze the formula (3.1) as the $a = 1$ specialization of

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(1 + q^{4k+1})(aq; q^2)_k (q/a; q^2)_k (-q; q^2)_k^2 (q^2; q^4)_k}{(1 + q)(q^2; q^2)_k^2 (-aq^2; q^2)_k (-q^2/a; q^2)_k (q^4; q^4)_k} q^k \\ &= \frac{(-q; q^2)_{\infty}^2 (aq^3; q^4)_{\infty}^2 (q^3/a; q^4)_{\infty}^2}{(1 + q)(-aq^2; q^2)_{\infty} (-q^2/a; q^2)_{\infty} (q^2; q^2)_{\infty}^2} \end{aligned} \quad (3.3)$$

which originates from a q -analogue of Watson's ${}_3F_2$ sum [3, Appendix (II.16)]. When we choose $a = q^n$ (or $a = q^{-n}$) in (3.3), for $n > 1$ odd, we get the sum terminating after $(n-1)/2$ terms on the left-hand side of (3.3), while the right-hand side vanishes if n is of the form $4m+3$ and it becomes equal to

$$\frac{(-q; q^2)_{\infty}^2 (q^{4m+4}; q^4)_{\infty}^2 (q^{2-4m}; q^4)_{\infty}^2}{(1 + q)(-q^{4m+3}; q^2)_{\infty} (-q^{1-4m}; q^2)_{\infty} (q^2, q^4; q^4)_{\infty}^2} = [4m+1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2}$$

if $n = 4m+1$. This means that modulo $(a - q^n)(1 - aq^n)$ we have

$$\begin{aligned} & \sum_{k=0}^N \frac{(1 + q^{4k+1})(aq; q^2)_k (q/a; q^2)_k (-q; q^2)_k^2 (q^2; q^4)_k}{(1 + q)(q^2; q^2)_k^2 (-aq^2; q^2)_k (-q^2/a; q^2)_k (q^4; q^4)_k} q^k \\ & \equiv \begin{cases} [4m+1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2} & \text{if } n = 4m+1, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \end{cases} \end{aligned}$$

for any $N \geq (n-1)/2$. The limiting $a \rightarrow 1$ case of the congruences can be shown to be

$$\sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv \begin{cases} [4m+1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2} & \text{if } n = 4m+1, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \end{cases} \quad (3.4)$$

modulo $\Phi_n(q)^2\Phi_n(-q)$. This is quite similar in spirit to (1.5), though still far from constructing q -analogues for the coefficients $a(p)$ in (1.6) of the modular form $f(\tau)$. The latter means that a hunt for q -rational functions, which equal the left-hand side of (1.5) or (3.4) modulo $\Phi_n(q)^2$ and specialize to $a(n)$ as $q \rightarrow 1$ (at least for n prime), is still on its way. Such q -rational functions are also expected to be self-reciprocal, that is, invariant under the involution $q \mapsto 1/q$, as all the left- and right-hand sides in (1.5), (1.7), (3.4) and also (2.1) are.

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