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Results in Mathematics

On Certain Sums of Arithmetic Functions Involving the GCD and LCM of Two Positive Integers

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Abstract. We obtain asymptotic formulas with remainder terms for the hyperbolic summations $\sum_{mn\le x} f((m,n))$ and $\sum_{mn\le x} f([m,n])$, where f belongs to certain classes of arithmetic functions, (m,n) and $[m,n]$ denoting the gcd and lcm of the integers m,n. In particular, we investigate the functions $f(n)=\tau(n), \log n, \omega(n)$ and $\Omega(n)$. We also define a common generalization of the latter three functions, and prove a corresponding result.

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1. Introduction

Let $F:\mathbb{N}^2\to\mathbb{C}$ be an arithmetic function of two variables. Several asymptotic results for sums $\sum F(m,n)$ with various bounds of summation are given in the literature. The usual 'rectangular' summations are of form $\sum_{m\le x,\, n\le y} F(m,n)$, in particular with $x=y$. The 'triangular' summations can be written as $\sum_{n\le x}\sum_{m\le n} F(m,n)$. Note that if the function F is symmetric in the variables, then

$$\sum_{m,n\le x} F(m,n) = 2\sum_{n\le x}\sum_{m\le n} F(m,n) - \sum_{n\le x} F(n,n).$$

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The 'hyperbolic' summations have the shape $\sum_{mn\le x} F(m,n)$, the sum being over the Dirichlet region $\{(m,n)\in \mathbb{N}^2 : mn\le x\}$. Hyperbolic summations have been less studied than rectangular and triangular summations and it is hyperbolic summations that are estimated in this paper.

We mention a few examples for functions F involving the greatest common divisor (gcd) and the least common multiple (lcm) of integers. If $F(m,n)=(m,n)$, the gcd of m and n, then

$$\sum_{m,n\le x} (m,n) = \frac{x^2}{\zeta(2)}\left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + O\left(x^{1+\theta+\varepsilon}\right), \quad (1.1)$$

holds for every $\varepsilon>0$, where ζ is the Riemann zeta function, ζ' is its derivative, γ is Euler's constant, and θ denotes the exponent appearing in Dirichlet's divisor problem. Furthermore,

$$\sum_{mn\le x} (m,n) = \frac{1}{4\zeta(2)} x(\log x)^2 + c_1 x\log x + c_2 x + O\left(x^{\beta}(\log x)^{\beta'}\right), \quad (1.2)$$

where c_1, c_2 are explicit constants, and $\beta = 547/832 \doteq 0.657451$, $\beta' = 26947/8320 \doteq 3.238822$. Estimate (1.1) (in the form of a triangular summation, involving Pillai's arithmetic function) was obtained by Chidambaraswamy and Sitaramachandrarao [4, Th. 3.1] using elementary arguments. Formula (1.2) was deduced applying analytic methods by Krätzel et al. [11, Th. 3.5].

If $F(m,n)=[m,n]$, the lcm of m and n, then we have

$$\sum_{m,n\le x} [m,n] = \frac{\zeta(3)}{4\zeta(2)} x^4 + O\left(x^3(\log x)^{2/3}(\log\log x)^{1/3}\right), \quad (1.3)$$

established by Bordellès [2, Th. 6.3] with a slightly weaker error term. The error in (1.3) comes from Liu's [13] improvement for the error term by Walfisz on $\sum_{n\le x}\varphi(n)$, where φ is Euler's function. Also see [8].

We also have

$$\sum_{m,n\le x} \frac{(m,n)}{[m,n]} = 3x + O\left((\log x)^2\right), \quad (1.4)$$

obtained by Hilberdink et al. [8, Th. 5.1].

For other related asymptotic results for functions involving the gcd and lcm of two (and several) integers see the papers [2,8,11,17,19] and their references. For summations of some other functions of two variables see [3,14,20].

We remark that for any arithmetic function $f:\mathbb{N}\to\mathbb{C}$ of one variable,

$$\sum_{mn\le x} f((m,n)) = \sum_{k\le x} G_f(k), \quad (1.5)$$

where $G_f(k)=\sum_{mn=k} f((m,n))$. Hence to find estimates for (1.5) is, in fact, a one variable summation problem. Formula (1.2) represents its special case when $f(k)=k$ ($k\in\mathbb{N}$). The sum of divisors function $f(k)=\sigma(k)$, giving an estimate similar to (1.2), with the same error term, was considered in paper

[11]. The case of the divisor function $f(k)=\tau(k)$ was discussed by Heyman [7], obtaining an asymptotic formula with error term $O(x^{1/2})$ by using elementary estimates. However, for every $k \in \mathbb{N}$,

$$\sum_{mn=k} \tau((m,n)) = \sum_{abc^2=k} 1 =: \tau(1,1,2;k),$$

which follows from the general arithmetic identities for $G_f(k)$ in Proposition 2.1, see in particular (2.2), and the summation of the divisor function $\tau(1,1,2;k)$ is well known in the literature. See, e.g., Krätzel [10, Ch. 6]. The best known related error term, to our knowledge, is $O(x^{63/178+\varepsilon})$, with $63/178 \doteq 0.353932$, given by Liu [12] using deep analytic methods.

We deduce the following result.

Theorem 1.1. *We have*

$$\sum_{mn\le x} \tau((m,n)) = \zeta(2)x\left(\log x + 2\gamma - 1 + 2\frac{\zeta'(2)}{\zeta(2)}\right) + \zeta^2(1/2)x^{1/2} + O(x^{63/178+\varepsilon}). \tag{1.6}$$

In an analogous manner to (1.5), let us define

$$\sum_{mn\le x} f([m,n]) = \sum_{k\le x} L_f(k), \tag{1.7}$$

where $L_f(k) = \sum_{mn=k} f([m,n])$.

We remark that if F is an arbitrary arithmetic function of two variables, then the one variable function

$$\widetilde{F}(k) = \sum_{mn=k} F(m,n)$$

is called the convolute of F. The function F of two variables is said to be multiplicative if $F(m_1m_2,n_1n_2) = F(m_1,n_1)F(m_2,n_2)$ provided that $(m_1n_1,m_2n_2) = 1$. If F is multiplicative, then $\widetilde{F}$ is also multiplicative. See Vaidyanathaswamy [21], Tóth [18, Sect. 6]. The functions G_f and L_f of above are special cases of this general concept. If f is multiplicative, then $G_f(k)$ and $L_f(k)$ are multiplicative as well.

In the present paper we deduce simple arithmetic representations of the functions $G_f(k)$ and $L_f(k)$ (Proposition 2.1), and establish new asymptotic estimates for sums of type (1.5) and (1.7). Namely, we give estimates for $\sum_{mn\le x} f((m,n))$ when f belongs to a wide class of functions (Theorem 2.2), and obtain better error terms in the case of a narrower class of functions (Theorem 2.4). In particular, we consider the functions $f(n) = \log n, \omega(n)$ and $\Omega(n)$ (Corollary 2.6). Actually, we define a common generalization of these three functions and prove a corresponding result (Corollary 2.5). We also point out the case of the function $f(n)=1/n$, the related result on $\sum_{mn\le x}(m,n)^{-1}$ (Corollary 2.3) being strongly connected with the sum

$\sum_{mn\le x}[m,n]$ (Theorem 2.7). Furthermore, we deduce estimates for the sums $\sum_{mn\le x} f([m,n])$ in the cases of $f(n)=\log n, \omega(n), \Omega(n)$ (Theorems 2.8, 2.9, 2.10) and $f(n)=\tau(n)$ (Theorem 2.11), respectively. Finally we obtain a formula for $\sum_{mn\le x}(m,n)[m,n]^{-1}$ (Theorem 2.12). The proofs are given in Sect. 3.

Throughout the paper we use the following notation: $\mathbb{N}=\{1,2,\ldots\}$; $\mathbb{P}=\{2,3,5,\ldots\}$ is the set of primes; $n=\prod_p p^{\nu_p(n)}$ is the prime power factorization of $n\in\mathbb{N}$, the product being over $p\in\mathbb{P}$, where all but a finite number of the exponents $\nu_p(n)$ are zero; $\tau(n)=\sum_{d\mid n} 1$ is the divisor function; $\mathbf{1}(n)=1$, $\operatorname{id}(n)=n$ ($n\in\mathbb{N}$); μ is the Möbius function; $\omega(n)=\#\{p:\nu_p(n)\ne 0\}$; $\Omega(n)=\sum_p \nu_p(n)$; $\kappa(n)=\prod_{\nu_p(n)\ne 0} p$ is the squarefree kernel of n; $*$ is the Dirichlet convolution of arithmetic functions; ζ is the Riemann zeta function, ζ' is its derivative, $\pi(x)=\sum_{p\le x} 1$; γ is Euler's constant.

2. Main Results

Useful arithmetic representations of the functions $G_f(n)=\sum_{ab=n} f((a,b))$ and $L_f(n)=\sum_{ab=n} f([a,b])$, already defined in the Introduction, are given by the next result.

Proposition 2.1. *Let f be an arbitrary arithmetic function. Then for every $n\in\mathbb{N}$,*

$$G_f(n)=\sum_{a^2b^2c=n} f(a)\mu(b)\tau(c) \tag{2.1}$$

$$=\sum_{a^2c=n} (f*\mu)(a)\tau(c) \tag{2.2}$$

$$=\sum_{a^2c=n} f(a)\, 2^{\omega(c)}, \tag{2.3}$$

and

$$L_f(n)=\sum_{a^2b^2c=n} f(n/a)\mu(b)\tau(c) \tag{2.4}$$

$$=\sum_{a^2c=n} f(ac)\, 2^{\omega(c)}. \tag{2.5}$$

If f is additive, then for every $n\in\mathbb{N}$,

$$L_f(n)=2(f*\mathbf{1})(n)-G_f(n). \tag{2.6}$$

If f is completely additive, then for every $n\in\mathbb{N}$,

$$L_f(n)=f(n)\tau(n)-G_f(n). \tag{2.7}$$

In terms of formal Dirichlet series, identities (2.1), (2.2) and (2.3) show that for every arithmetic function f,

$$\sum_{n=1}^{\infty} \frac{G_f(n)}{n^z} = \frac{\zeta^2(z)}{\zeta(2z)} \sum_{n=1}^{\infty} \frac{f(n)}{n^{2z}}.$$

See [11, Prop. 5.1] for a similar formula on the sum $\sum_{d_1\cdots d_k=n} g((d_1,\ldots,d_k))$, where $k \in \mathbb{N}$ and g is an arithmetic function.

Our first asymptotic formula applies to every function f satisfying a condition on its order of magnitude.

Theorem 2.2. *Let f be an arithmetic function such that $f(n) \ll n^{\beta}(\log n)^{\delta}$, as $n \to \infty$, for some fixed $\beta, \delta \in \mathbb{R}$ with $\beta < 1$. Then*

$$\sum_{mn\le x} f((m,n)) = x(C_f \log x + D_f) + R_f(x), \tag{2.8}$$

where the constants C_f and D_f are given by

$$C_f = \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{f(n)}{n^2},$$

$$D_f = \frac{1}{\zeta(2)} \left(C \sum_{n=1}^{\infty} \frac{f(n)}{n^2} - 2 \sum_{n=1}^{\infty} \frac{f(n)\log n}{n^2} \right),$$

with C defined by

$$C = 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}, \tag{2.9}$$

and the error term is

$$R_f(x) \ll \begin{cases} x^{(\beta+1)/2}(\log x)^{\delta+1}, & \text{if } 0<\beta<1 \text{ or } \beta=0, \delta\ne -1,\\ x^{1/2}\log\log x, & \text{if } \beta=0, \delta=-1,\\ x^{1/2}\lambda(x), & \text{if } \beta<0, \end{cases}$$

where

$$\lambda(x) := e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}, \tag{2.10}$$

with some constant $c > 0$.

The error term $R_f(x)$ can be improved assuming that the Riemann Hypothesis (RH) is true. For example, let $\varrho = 221/608 \doteq 0.363486$. If $\beta, \delta \in \mathbb{R}$ and $\beta < 2\varrho - 1 \doteq -0.273026$, then $R_f(x) \ll x^{\varrho+\varepsilon}$.

Theorem 2.2 applies, e.g., to the functions $f(n) = n^{\beta}$ (with $\beta < 1$, $\delta = 0$), $f(n) = (\log n)^{\delta}$ (with $\beta = 0$, $\delta \in \mathbb{R}$), $f(n) = \tau^k(n)$ ($k \in \mathbb{N}$, with $\beta = k\varepsilon$, $\varepsilon > 0$ arbitrary small, $\delta = 0$), $f(n) = \omega(n)$ or $\Omega(n)$ (with $0 < \beta = \varepsilon$ arbitrary small, $\delta = 0$). We point out the case of the function $f(n) = n^{-1}$.

Corollary 2.3. *We have*

$$\sum_{mn\le x} \frac{1}{(m,n)} = \frac{\zeta(3)}{\zeta(2)} x(\log x + D) + O(x^{1/2}\lambda(x)), \tag{2.11}$$

where

$$D = 2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)} + 2\frac{\zeta'(3)}{\zeta(3)},$$

and $\lambda(x)$ is defined by (2.10). If RH is true, then the error term is $O(x^{\varrho+\varepsilon})$, where ϱ is given in Theorem 2.2.

However, for some special functions asymptotic formulas with more terms or with better unconditional errors can be obtained. See, e.g. (1.6), namely the case $f(n) = \tau(n)$ and our next results.

Let f be a function such that $(\mu * f)(n) = 0$ for all $n \ne p^\nu$ (n is not a prime power), $(\mu * f)(p^\nu) = g(p)$ does not depend on ν and $g(p)$ is sufficiently small for the primes p. More exactly, we have the next result.

Theorem 2.4. *Let f be an arithmetic function such that there exists a subset Q of the set of primes $\mathbb{P}$ and there exists a subset S of $\mathbb{N}$ with $1 \in S$, satisfying the following properties:*

i) $(\mu * f)(n) = 0$ *for all* $n \ne p^\nu$*, where* $p \in Q$ *and* $\nu \in S$*,*

ii) $(\mu * f)(p^\nu) = g(p)$*, depending only on* p*, for all prime powers* p^ν *with* $p \in Q$*,* $\nu \in S$*.*

iii) $g(p) \ll (\log p)^\eta$*, as* $p \to \infty$*, where* $\eta \ge 0$ *is a fixed real number.*

Then for the error term in (2.8) we have $R_f(x) \ll x^{1/2}(\log x)^\eta$. Furthermore, the constants C_f and D_f can be given as

$$C_f = \sum_{p\in Q} g(p) \sum_{\nu\in S} \frac{1}{p^{2\nu}},$$

$$D_f = (2\gamma - 1)C_f - 2\sum_{p\in Q} g(p)\log p \sum_{\nu\in S} \frac{\nu}{p^{2\nu}}.$$

The prototype of functions f to which Theorem 2.4 applies is the function $f_{S,\eta}$ implicitly defined by

$$h_{S,\eta}(n) := (\mu * f_{S,\eta})(n) = \begin{cases} (\log p)^\eta, & \text{if } n = p^\nu \text{ a prime power with } \nu \in S, \\ 0, & \text{otherwise,} \end{cases} \tag{2.12}$$

where $1 \in S \subseteq \mathbb{N}$, $\eta \ge 0$ is real and $Q = \mathbb{P}$. It is possible to consider the corresponding generalization with $Q \subset \mathbb{P}$, as well. By Möbius inversion we obtain that for $n = \prod_p p^{\nu_p(n)} \in \mathbb{N}$,

$$f_{S,\eta}(n) = \sum_{d\mid n} h_{S,\eta}(d) = \sum_{p\mid n} (\log p)^\eta \#\{\nu : 1 \le \nu \le \nu_p(n), \nu \in S\},$$

where $f_{S,\eta}(1) = 0$ (empty sum).

Let $S = \mathbb{N}$. Then

$$f_{\mathbb{N},\eta}(n) := \sum_{p|n} \nu_p(n)(\log p)^\eta,$$

which gives for $\eta = 1$, $f_{\mathbb{N},1}(n) = \log n$, while $h_{\mathbb{N},1}(n) = \Lambda(n)$ is the von Mangoldt function. If $\eta = 0$, then $f_{\mathbb{N},0}(n) = \Omega(n)$.

Now let $S = \{1\}$. Then

$$f_{\{1\},\eta}(n) := \sum_{p|n} (\log p)^\eta,$$

and if $\eta = 0$, then $f_{\{1\},0}(n) = \omega(n)$. If $\eta = 1$, then $f_{\{1\},1}(n) = \log \kappa(n)$, where $\kappa(n) = \prod_{p|n} p$. Note that $\sum_{n \le x} h_{\{1\},1}(n) = \sum_{p \le x} \log p = \theta(x)$ is the Chebyshev theta function.

The functions $f_{S,\eta}(n)$ and $h_{S,\eta}(n)$ have not been studied in the literature, as far as we know.

According to (2.12), the conditions of Theorem 2.4 are satisfied and we deduce the next result.

Corollary 2.5. *If $1 \in S \subseteq \mathbb{N}$ and $\eta \ge 0$ is a real number, then*

$$\sum_{mn \le x} f_{S,\eta}((m,n)) = x(C_{f_{S,\eta}} \log x + D_{f_{S,\eta}}) + O(x^{1/2}(\log x)^\eta),$$

where the constants $C_{f_{S,\eta}}$ and $D_{f_{S,\eta}}$ are given by

$$C_{f_{S,\eta}} = \sum_p (\log p)^\eta \sum_{\nu \in S} \frac{1}{p^{2\nu}},$$

$$D_{f_{S,\eta}} = (2\gamma - 1) C_{f_{S,\eta}} - 2 \sum_p (\log p)^{\eta+1} \sum_{\nu \in S} \frac{\nu}{p^{2\nu}}.$$

In the special cases mentioned above we obtain the following results.

Corollary 2.6. *We have*

$$\sum_{mn \le x} \log(m,n) = x(C_{\log} \log x + D_{\log}) + O(x^{1/2} \log x), \tag{2.13}$$

$$\sum_{mn \le x} \log \kappa((m,n)) = x(C_{\log \kappa} \log x + D_{\log \kappa}) + O(x^{1/2} \log x), \tag{2.14}$$

$$\sum_{mn \le x} \omega((m,n)) = x(C_\omega \log x + D_\omega) + O(x^{1/2}), \tag{2.15}$$

$$\sum_{mn \le x} \Omega((m,n)) = x(C_\Omega \log x + D_\Omega) + O(x^{1/2}), \tag{2.16}$$

where

$$C_{\log} = -\frac{\zeta'(2)}{\zeta(2)} = \sum_p \frac{\log p}{p^2-1} \doteq 0.569960,$$

$$D_{\log} = -\frac{\zeta'(2)}{\zeta(2)}\left(2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)} + 2\frac{\zeta''(2)}{\zeta'(2)}\right), \tag{2.17}$$

$$C_{\log\kappa} = \sum_p \frac{\log p}{p^2} \doteq 0.493091, \quad D_{\log\kappa} = (2\gamma-1)\sum_p \frac{\log p}{p^2} - 2\sum_p \frac{(\log p)^2}{p^2},$$

$$C_{\omega} = \sum_p \frac{1}{p^2} \doteq 0.452247, \quad D_{\omega} = (2\gamma-1)\sum_p \frac{1}{p^2} - 2\sum_p \frac{\log p}{p^2}, \tag{2.18}$$

$$C_{\Omega} = \sum_p \frac{1}{p^2-1} \doteq 0.551693, \quad D_{\Omega} = (2\gamma-1)\sum_p \frac{1}{p^2-1} - 2\sum_p \frac{p^2\log p}{(p^2-1)^2}. \tag{2.19}$$

We deduce by (2.13) and (2.14) that $\prod_{mn\le x}(m,n) \sim x^{C_{\log} x}$ and $\prod_{mn\le x}\kappa((m,n)) \sim x^{C_{\log\kappa} x}$, as $x\to\infty$.

Now consider the functions given by $L_f(n) = \sum_{ab=n} f([a,b])$. If $f = \mathrm{id}$, then $L_{\mathrm{id}}(n) = \sum_{ab=n}[a,b] = n\sum_{ab=n}(a,b)^{-1}$. The next result follows from Corollary 2.3 by partial summation. It may be compared to estimates (1.1), (1.2) and (1.3).

Theorem 2.7. *We have*

$$\sum_{mn\le x}[m,n] = \frac{\zeta(3)}{2\zeta(2)}x^2(\log x + E) + O(x^{3/2}\lambda(x)),$$

where $\lambda(x)$ *is defined by* (2.10), *and*

$$E = 2\gamma - \frac{1}{2} - 2\frac{\zeta'(2)}{\zeta(2)} + 2\frac{\zeta'(3)}{\zeta(3)}.$$

If RH is true, then the error term is $O(x^{1+\varrho+\varepsilon})$, *where* ϱ *is given in Theorem 2.2.*

If the function f is (completely) additive, then identities (2.6) and (2.7) can be used to deduce asymptotic estimates for $\sum_{n\le x} L_f(n)$.

Theorem 2.8. *We have*

$$\sum_{mn\le x}\log[m,n] = x(\log x)^2 + (2\gamma - 2 - C_{\log})x\log x$$
$$-(2\gamma - 2 + D_{\log})x + O(x^{1/2}\log x),$$

where $C_{\log}$ *and* $D_{\log}$ *are given by* (2.17).

As a consequence we deduce that $\prod_{mn\le x}[m,n] \sim x^{x\log x}$ as $x\to\infty$.

Theorem 2.9. *We have*

$$\sum_{mn\le x} \omega([m,n]) = 2x(\log x)(\log\log x) + (K_\omega - C_\omega)x\log x + O(x), \quad (2.20)$$

where C_ω *is given by* (2.18) *and*

$$K_\omega = 2\left(\gamma - 1 + \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right)\right). \quad (2.21)$$

Theorem 2.10. *We have*

$$\sum_{mn\le x} \Omega([m,n]) = 2x(\log x)(\log\log x) + (K_\Omega - C_\Omega)x\log x + O(x), \quad (2.22)$$

where C_Ω *is given by* (2.19) *and*

$$K_\Omega = 2\left(\gamma - 1 + \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p-1}\right)\right). \quad (2.23)$$

Next we consider the divisor function $f(n) = \tau(n)$.

Theorem 2.11. *We have*

$$\sum_{mn\le x} \tau([m,n]) = x(C_1(\log x)^3 + C_2(\log x)^2 + C_3\log x + C_4) + O(x^{1/2+\varepsilon}) \quad (2.24)$$

for every $\varepsilon > 0$, *where*

$$C_1 = \frac{1}{\pi^2}\prod_p \left(1 - \frac{1}{(p+1)^2}\right) \doteq 0.078613,$$

and the constants C_2, C_3, C_4 *can also be given explicitly.*

Estimate (2.24) may be compared to (1.6) and to that for $\sum_{m,n\le x}\tau([m,n])$. See Tóth and Zhai [19, Th. 3.4].

Finally, we deduce the counterpart of formula (1.4) with hyperbolic summation.

Theorem 2.12. *We have*

$$\sum_{mn\le x} \frac{(m,n)}{[m,n]} = \frac{\zeta^2(3/2)}{\zeta(3)}x^{1/2} + O((\log x)^3). \quad (2.25)$$

3. Proofs

Proof of Proposition 2.1. Group the terms of the sum $G_f(n) = \sum_{ab=n} f((a,b))$ according to the values $(a,b) = d$, where $a = dc, b = de$ with $(c,e) = 1$. We obtain, using the property of the Möbius μ function,

$$\begin{aligned} G_f(n) &= \sum_{\substack{d^2ce=n \\ (c,e)=1}} f(d) = \sum_{d^2ce=n} f(d) \sum_{\delta\mid(c,e)} \mu(\delta) \\ &= \sum_{d^2\delta^2k\ell=n} f(d)\mu(\delta) = \sum_{d^2\delta^2t=n} f(d)\mu(\delta) \sum_{k\ell=t} 1 \\ &= \sum_{d^2\delta^2t=n} f(d)\mu(\delta)\tau(t), \end{aligned}$$

giving (2.1), which can be written as (2.2) and (2.3) by the definition of the Dirichlet convolution and the identity $\sum_{\delta^2t=k} \mu(\delta)\tau(t) = 2^{\omega(k)}$.

Alternatively, use the identity $f(n) = \sum_{d\mid n}(f*\mu)(d)$ to deduce that

$$\begin{aligned} G_f(n) &= \sum_{ab=n} \sum_{d\mid(a,b)} (f*\mu)(d) = \sum_{ab=n} \sum_{d\mid a, d\mid b} (f*\mu)(d) \\ &= \sum_{d^2ce=n} (f*\mu)(d) = \sum_{d^2t=n} (f*\mu)(d) \sum_{ce=t} 1 \\ &= \sum_{d^2t=n} (f*\mu)(d)\tau(t), \end{aligned}$$

giving (2.2).

For $L_f(n)$ use that $[a,b] = ab/(a,b)$ and apply the first method above to deduce (2.4) and (2.5).

If f is an additive function, then $f((a,b)) + f([a,b]) = f(a) + f(b)$ holds for every $a,b \in \mathbb{N}$. To see this, it is enough to consider the case when $a = p^r, b = p^s$ are powers of the same prime p. Now, $f(p^{\min(r,s)}) + f(p^{\max(r,s)}) = f(p^r) + f(p^s)$ trivially holds for every $r,s \geq 0$. Therefore,

$$L_f(n) = \sum_{ab=n} f([a,b]) = \sum_{ab=n} f(a) + \sum_{ab=n} f(b) - \sum_{ab=n} f((a,b)) \tag{3.1}$$

$$= 2(f*\mathbf{1})(n) - G_f(n),$$

which is (2.6).

Finally, to obtain (2.7), use that if f is completely additive, then in (3.1) one has

$$\sum_{ab=n} f(a) + \sum_{ab=n} f(b) = \sum_{ab=n} f(ab) = f(n)\tau(n).$$

□

For the proof of Theorem 2.2 we need the following Lemmas.

Lemma 3.1. *Let $s, \delta \in \mathbb{R}$ with $s > 1$. Then*

$$\sum_{n>x} \frac{(\log n)^\delta}{n^s} \ll \frac{(\log x)^\delta}{x^{s-1}}.$$

Proof. The function $t \mapsto t^{-s}(\log t)^\delta$ $(t > x)$ is decreasing for large x. By comparing the sum with the corresponding integral we have

$$\sum_{n>x} \frac{(\log n)^\delta}{n^s} \le \int_x^\infty \frac{(\log t)^\delta}{t^s}\, dt.$$

If $\delta < 0$, then trivially,

$$\int_x^\infty \frac{(\log t)^\delta}{t^s}\, dt \le (\log x)^\delta \int_x^\infty \frac{1}{t^s}\, dt \ll \frac{(\log x)^\delta}{x^{s-1}}.$$

If $\delta > 0$, then integrating by parts gives

$$\int_x^\infty \frac{(\log t)^\delta}{t^s}\, dt \ll \frac{(\log x)^\delta}{x^{s-1}} + \int_x^\infty \frac{(\log t)^{\delta-1}}{t^s}\, dt,$$

and repeated applications of the latter estimate, until the exponent of $\log t$ becomes negative, conclude the proof. □

Lemma 3.2. *Let $s, \delta \in \mathbb{R}$ with $s > 0$. Then*

$$\sum_{2\le n\le x} \frac{(\log n)^\delta}{n^s} \ll \begin{cases} x^{1-s}(\log x)^\delta, & \text{if } 0 < s < 1, \delta \in \mathbb{R}, \\ (\log x)^{\delta+1}, & \text{if } s = 1, \delta \ne -1, \\ \log\log x, & \text{if } s = 1, \delta = -1, \\ 1, & \text{if } s > 1. \end{cases}$$

Proof. Let $0 < s < 1$. If $\delta \ge 0$, then trivially

$$\sum_{n\le x} \frac{(\log n)^\delta}{n^s} \le (\log x)^\delta \sum_{n\le x} \frac{1}{n^s}$$

and by comparison of the sum with the corresponding integral we have

$$\sum_{n\le x} \frac{1}{n^s} \ll x^{1-s}. \tag{3.2}$$

If $\delta < 0$, then write

$$\sum_{2\le n\le x} \frac{(\log n)^\delta}{n^s} = \sum_{2\le n\le x^{1/2}} \frac{(\log n)^\delta}{n^s} + \sum_{x^{1/2}< n\le x} \frac{(\log n)^\delta}{n^s}$$
$$\ll \sum_{n\le x^{1/2}} \frac{1}{n^s} + (\log x^{1/2})^\delta \sum_{x^{1/2}< n\le x} \frac{1}{n^s},$$

which is, using again (3.2),

$$\ll x^{(1-s)/2} + (\log x)^\delta x^{1-s} \ll (\log x)^\delta x^{1-s},$$

where $1-s>0$. The case $s=1$ is well-known. If $s>1$, then the corresponding series is convergent. □

Proof of Theorem 2.2. We use identity (2.3) and the known estimate

$$\sum_{n\le x} 2^{\omega(n)} = \frac{x}{\zeta(2)}(\log x + C) + S(x), \tag{3.3}$$

where C is defined by (2.9) and $S(x) \ll x^{1/2}$ (see [6]), that can be improved to $S(x) \ll x^{1/2}\lambda(x)$ with $\lambda(x)$ given by (2.10) (see [15, Th. 3.1]).

We deduce by standard arguments that

$$\sum_{mn\le x} f((m,n)) = \sum_{d^2c\le x} f(d)2^{\omega(c)} = \sum_{d\le x^{1/2}} f(d) \sum_{c\le x/d^2} 2^{\omega(c)}$$

$$= \sum_{d\le x^{1/2}} f(d)\left(\frac{1}{\zeta(2)}\frac{x}{d^2}(\log\frac{x}{d^2}+C)+S\left(\frac{x}{d^2}\right)\right)$$

$$= \frac{x}{\zeta(2)}\left((\log x + C)\sum_{d\le x^{1/2}}\frac{f(d)}{d^2} - 2\sum_{d\le x^{1/2}}\frac{f(d)\log d}{d^2}\right) + \sum_{d\le x^{1/2}} f(d)S\left(\frac{x}{d^2}\right). \tag{3.4}$$

Here

$$\sum_{d\le x^{1/2}}\frac{f(d)}{d^2} = \sum_{d=1}^{\infty}\frac{f(d)}{d^2} + A_f(x),$$

where the series converges absolutely by the given assumption on f, and

$$A_f(x) = \sum_{d>x^{1/2}}\frac{|f(d)|}{d^2} \ll \sum_{d>x^{1/2}}\frac{(\log d)^{\delta}}{d^{2-\beta}} \ll x^{(\beta-1)/2}(\log x)^{\delta}$$

by using Lemma 3.1, where $2-\beta>1$, leading to the error $x(\log x)x^{(\beta-1)/2}(\log x)^{\delta} = x^{(\beta+1)/2}(\log x)^{\delta+1}$.

Furthermore,

$$\sum_{d\le x^{1/2}}\frac{f(d)\log d}{d^2} = \sum_{d=1}^{\infty}\frac{f(d)\log d}{d^2} + B_f(x),$$

where the series converges absolutely and

$$B_f(x) = \sum_{d>x^{1/2}}\frac{|f(d)|\log d}{d^2} \ll \sum_{d>x^{1/2}}\frac{(\log d)^{\delta+1}}{d^{2-\beta}} \ll x^{(\beta-1)/2}(\log x)^{\delta+1}$$

by using Lemma 3.1 again, giving the same error $x^{(\beta+1)/2}(\log x)^{\delta+1}$.

Finally, we estimate the last term in (3.4), namely the sum

$$T(x) := \sum_{d\le x^{1/2}} f(d)S\left(\frac{x}{d^2}\right). \tag{3.5}$$

We have by using that $S(x) \ll x^{1/2}$,

$$T(x) \ll x^{1/2} \sum_{d \le x^{1/2}} \frac{|f(d)|}{d} \ll x^{1/2} \sum_{d \le x^{1/2}} \frac{(\log d)^\delta}{d^{1-\beta}}$$

$$\ll \begin{cases} x^{(\beta+1)/2}(\log x)^\delta, & \text{if } 0 < \beta < 1, \\ x^{1/2}(\log x)^{\delta+1}, & \text{if } \beta = 0, \delta \neq -1, \\ x^{1/2} \log\log x, & \text{if } \beta = 0, \delta = -1, \end{cases}$$

by Lemma 3.2.

If $\beta < 0$, then we use that $S(x) \ll x^{1/2}\lambda(x)$ for $x \ge 3$ and $S(x) \ll 1$ for $1 \le x < 3$. Note that for $x \ge 3$, $\lambda(x) := e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}$ is decreasing, but $x^\varepsilon \lambda(x)$ is increasing for every $\varepsilon > 0$, if $c > 0$ is sufficiently small. We split the sum $T(x)$ defined by (3.5) in two sums: $T(x) = T_1(x) + T_2(x)$, according to $x/d^2 \ge 3$ and $x/d^2 < 3$, respectively. We deduce

$$\begin{aligned} T_1(x) &:= \sum_{\substack{d \le x^{1/2} \\ x/d^2 \ge 3}} f(d) S\left(\frac{x}{d^2}\right) \ll x^{1/2} \sum_{d \le (x/3)^{1/2}} \frac{(\log d)^\delta}{d^{1-\beta}} \lambda\left(\frac{x}{d^2}\right) \\ &= x^{1/2-\varepsilon} \sum_{d \le (x/3)^{1/2}} \frac{(\log d)^\delta}{d^{1-\beta-2\varepsilon}} \left(\frac{x}{d^2}\right)^\varepsilon \lambda\left(\frac{x}{d^2}\right) \\ &\ll x^{1/2-\varepsilon} x^\varepsilon \lambda(x) \sum_{d=1}^\infty \frac{(\log d)^\delta}{d^{1-\beta-2\varepsilon}} \ll x^{1/2}\lambda(x), \end{aligned}$$

by choosing $0 < \varepsilon < -\beta/2$. Furthermore, by Lemma 3.1 (with $s = -\beta > 0$),

$$T_2(x) := \sum_{\substack{d \le x^{1/2} \\ x/d^2 < 3}} f(d) S\left(\frac{x}{d^2}\right) \ll \sum_{d > (x/3)^{1/2}} d^\beta (\log d)^\delta \ll x^{(\beta+1)/2}(\log x)^\delta \ll x^{1/2}\lambda(x).$$

Baker [1] proved that under the Riemann Hypothesis for the error term $S(x)$ of estimate (3.3) one has $S(x) \ll x^{\frac{4}{11}+\varepsilon}$, whilst Kaczorowski and Wiertelak [9] remarked that a slight modification of the treatment in [22] yields $S(x) \ll x^{\frac{221}{608}+\varepsilon}$. This leads to the desired improvement of the error. □

Now we will prove Theorem 2.4. We need the following Lemmas.

Lemma 3.3. *If $\eta \ge 0$ and $s > 1$ are real numbers, then*

$$\sum_{p > x} \frac{(\log p)^\eta}{p^s} \ll \frac{(\log x)^{\eta-1}}{x^{s-1}}.$$

Proof. We have, by using Riemann–Stieltjes integration, integration by parts and the Chebyshev estimate $\pi(x) \ll x/\log x$,

$$\sum_{p>x} \frac{(\log p)^\eta}{p^s} = \int_x^\infty \frac{(\log t)^\eta}{t^s}\, d(\pi(t)) = \left[\frac{(\log t)^\eta}{t^s}\pi(t)\right]_{t=x}^{t=\infty} - \int_x^\infty \left(\frac{(\log t)^\eta}{t^s}\right)' \pi(t)\, dt$$
$$\ll \frac{(\log x)^{\eta-1}}{x^{s-1}} + \int_x^\infty \frac{(\log t)^{\eta-1}}{t^s}\, dt.$$

Integration by parts, again, gives

$$\int_x^\infty \frac{(\log t)^{\eta-1}}{t^s}\, dt \ll \frac{(\log x)^{\eta-1}}{x^{s-1}} + \int_x^\infty \frac{(\log t)^{\eta-2}}{t^s}\, dt,$$

and repeated applications of the latter estimate conclude the result. □

Lemma 3.4. *If $\eta \geq 0$ and $0 \leq s < 1$ are real numbers, then*

$$\sum_{p\leq x} \frac{(\log p)^\eta}{p^s} \ll \frac{(\log x)^{\eta-1}}{x^{s-1}}.$$

Proof. Similar to the previous proof, by using Riemann–Stieltjes integration,

$$\sum_{p\leq x} \frac{(\log p)^\eta}{p^s} = \int_2^x \frac{(\log t)^\eta}{t^s}\, d(\pi(t)) = \left[\frac{(\log t)^\eta}{t^s}\pi(t)\right]_{t=2}^{t=x} - \int_2^x \left(\frac{(\log t)^\eta}{t^s}\right)' \pi(t)\, dt$$
$$\ll \frac{(\log x)^{\eta-1}}{x^{s-1}} + \int_2^x \frac{(\log t)^{\eta-1}}{t^s}\, dt \ll \frac{(\log x)^{\eta-1}}{x^{s-1}}.$$

□

Proof of Theorem 2.4. Now we use identity (2.2) and the well-known estimate on $\tau(n)$,

$$\sum_{n\leq x} \tau(n) = x(\log x + C_1) + O(x^{\theta+\varepsilon}), \tag{3.6}$$

where $C_1 = 2\gamma - 1$ and $1/4 < \theta < 1/2$. Put $\theta_1 = \theta + \varepsilon$. We note that the final error term of our asymptotic formula will not depend on θ_1 and it will be enough to use $\theta_1 < 1/2$. We have

$$\sum_{mn\leq x} f((m,n)) = \sum_{d^2c\leq x} (\mu * f)(d)\tau(c) = \sum_{d\leq x^{1/2}} (\mu * f)(d) \sum_{c\leq x/d^2} \tau(c)$$
$$= \sum_{d\leq x^{1/2}} (\mu * f)(d) \left(\frac{x}{d^2}(\log \frac{x}{d^2} + C_1) + O((\frac{x}{d^2})^{\theta_1})\right)$$
$$= x\left((\log x + C_1) \sum_{d\leq x^{1/2}} \frac{(\mu * f)(d)}{d^2} - 2 \sum_{d\leq x^{1/2}} \frac{(\mu * f)(d)\log d}{d^2}\right)$$
$$+O(x^{\theta_1} \sum_{d\leq x^{1/2}} \frac{|(\mu * f)(d)|}{d^{2\theta_1}}).$$

Here

$$A := \sum_{d\le x^{1/2}} \frac{(\mu * f)(d)}{d^2} = \sum_{\substack{p^\nu \le x\\ p\in Q\\ \nu\in S}} \frac{g(p)}{p^{2\nu}} = \sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p) \sum_{\substack{1\le\nu\le m\\ \nu\in S}} \frac{1}{p^{2\nu}}$$

$$= \sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p) \left(H_S(p) - \sum_{\substack{\nu\ge m+1\\ \nu\in S}} \frac{1}{p^{2\nu}} \right), \tag{3.7}$$

where $m =: \lfloor \frac{\log x}{2\log p} \rfloor$, and for every prime p,

$$\frac{1}{p^2} \le H_S(p) := \sum_{\nu\in S}^{\infty} \frac{1}{p^{2\nu}} \le \sum_{\nu=1}^{\infty} \frac{1}{p^{2\nu}} = \frac{1}{p^2-1}, \tag{3.8}$$

using that $1 \in S$. Here

$$\sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p)H_S(p) = \sum_{p\in Q} g(p)H_S(p) - \sum_{\substack{p> x^{1/2}\\ p\in Q}} g(p)H_S(p), \tag{3.9}$$

where the series is absolutely convergent by the condition $g(p) \ll (\log p)^\eta$ and by (3.8), and the last sum is

$$\ll \sum_{p>x^{1/2}} \frac{(\log p)^\eta}{p^2-1} \ll \sum_{p>x^{1/2}} \frac{(\log p)^\eta}{p^2} \ll \frac{(\log x)^{\eta-1}}{x^{1/2}}$$

by Lemma 3.3. Also,

$$A_1 := \sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p) \sum_{\substack{\nu\ge m+1\\ \nu\in S}} \frac{1}{p^{2\nu}} \ll \sum_{p\le x^{1/2}} (\log p)^\eta \sum_{\nu\ge m+1} \frac{1}{p^{2\nu}} \tag{3.10}$$

$$= \sum_{p\le x^{1/2}} \frac{(\log p)^\eta}{p^{2m}(p^2-1)}. \tag{3.11}$$

By the definition of m we have $m > \frac{\log x}{2\log p} - 1$, hence $p^{2m} > \frac{x}{p^2}$. Thus the sum in (3.11) is

$$\le \frac{1}{x} \sum_{p\le x^{1/2}} \frac{p^2(\log p)^\eta}{p^2-1} \ll \frac{1}{x} \sum_{p\le x^{1/2}} (\log p)^\eta \le \frac{1}{x} (\log x)^\eta \pi(x^{1/2}), \tag{3.12}$$

hence

$$A_1 \ll \frac{(\log x)^{\eta-1}}{x^{1/2}}, \tag{3.13}$$

using $\eta \ge 0$ and the estimate $\pi(x^{1/2}) \ll \frac{x^{1/2}}{\log x}$.

We deduce by (3.7), (3.9), (3.10) and (3.13) that

$$A = \sum_{p\in Q} g(p)H_S(p) + O\left(\frac{(\log x)^{\eta-1}}{x^{1/2}}\right),$$

which leads to the error term $\ll x^{1/2}(\log x)^{\eta}$.

In a similar way,

$$B:=\sum_{d\le x^{1/2}} \frac{(\mu * f)(d)\log d}{d^2} = \sum_{\substack{p^\nu\le x\\ p\in Q\\ \nu\in S}} \frac{g(p)\log p^\nu}{p^{2\nu}} = \sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p)\log p \sum_{\substack{1\le\nu\le m\\ \nu\in S}} \frac{\nu}{p^{2\nu}}$$

$$= \sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p)\log p \left(K_S(p) - \sum_{\substack{\nu\ge m+1\\ \nu\in S}} \frac{\nu}{p^{2\nu}}\right),$$

where for every prime p,

$$\frac{1}{p^2} \le K_S(p) := \sum_{\substack{\nu=1\\ \nu\in S}}^{\infty} \frac{\nu}{p^{2\nu}} \le \sum_{\nu=1}^{\infty} \frac{\nu}{p^{2\nu}} = \frac{p^2}{(p^2-1)^2} \ll \frac{1}{p^2}, \tag{3.14}$$

since $1\in S$. We write

$$\sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p)(\log p)K_S(p) = \sum_{p\in Q} g(p)(\log p)K_S(p) - \sum_{\substack{p> x^{1/2}\\ p\in Q}} g(p)(\log p)K_S(p),$$

where the series is absolutely convergent by (3.14), and the last sum is

$$\ll \sum_{p>x^{1/2}} \frac{(\log p)^{\eta+1}}{p^2} \ll \frac{(\log x)^{\eta}}{x^{1/2}}$$

by Lemma 3.3. Also,

$$B_1 := \sum_{\substack{p\le x^{1/2}\\ p\in Q}} g(p)\log p \sum_{\substack{\nu\ge m+1\\ \nu\in S}} \frac{\nu}{p^{2\nu}} \ll \sum_{p\le x^{1/2}} (\log p)^{\eta+1} \sum_{\nu\ge m+1} \frac{\nu}{p^{2\nu}}, \tag{3.15}$$

where

$$\sum_{\nu\ge m+1} \frac{\nu}{p^{2\nu}} = \frac{p^2}{(p^2-1)^2}\left(\frac{m}{p^{2m+2}} - \frac{m+1}{p^{2m}}\right) \ll \frac{1}{p^2}\frac{m}{p^{2m}} \ll \frac{\log x}{x\log p},$$

using that $p^{2m} > \frac{x}{p^2}$ and $m \ll \frac{\log x}{\log p}$. This gives that for the sum B_1 in (3.15),

$$B_1 \le \frac{\log x}{x} \sum_{p\le x^{1/2}} (\log p)^{\eta} \ll \frac{(\log x)^{\eta}}{x^{1/2}},$$

see (3.12). Putting all of this together we obtain that

$$B = \sum_{p \in Q} g(p)(\log p) K_S(p) + O\left(\frac{(\log x)^\eta}{x^{1/2}}\right),$$

which leads to the error term $\ll x^{1/2}(\log x)^\eta$, the same as above.

Finally,

$$C: = x^{\theta_1} \sum_{d \le x^{1/2}} \frac{|(\mu * f)(d)|}{d^{2\theta_1}} = x^{\theta_1} \sum_{\substack{p^\nu \le x^{1/2} \\ p \in Q \\ \nu \in S}} \frac{g(p)}{p^{2\nu\theta_1}} \le x^{\theta_1} \sum_{p \le x^{1/2}} (\log p)^\eta \sum_{\nu=1}^{\infty} \frac{1}{p^{2\nu\theta_1}}$$

$$\ll x^{\theta_1} \sum_{p \le x^{1/2}} \frac{(\log p)^\eta}{p^{2\theta_1}} \ll x^{1/2}(\log x)^{\eta-1}$$

by Lemma 3.4. This finishes the proof. □

Proof of Theorem 2.7. We have

$$\sum_{ab \le x} [a,b] = \sum_{n \le x} n \sum_{ab=n} \frac{1}{(a,b)},$$

and partial summation applied to estimate (2.11) gives the result. □

Proof of Theorem 2.8. We have, by using identity (2.7),

$$\sum_{mn \le x} \log[m,n] = \sum_{n \le x} \tau(n) \log n - \sum_{mn \le x} \log(m,n).$$

We obtain by partial summation on (3.6) that

$$\sum_{n \le x} \tau(n) \log n = x((\log x)^2 + (2\gamma - 2)\log x + 2 - 2\gamma) + O(x^{\theta+\varepsilon}),$$

which can be combined with formula (2.13) on $\sum_{mn \le x} \log(m,n)$. □

Proof of Theorem 2.9. By identity (2.6) we have

$$\sum_{mn \le x} \omega([m,n]) = 2 \sum_{mn \le x} \omega(n) - \sum_{mn \le x} \omega((m,n)),$$

where

$$\sum_{mn \le x} \omega(n) = \sum_{n \le x} \omega(n) \sum_{m \le x/n} 1 = x \sum_{n \le x} \frac{\omega(n)}{n} + O\left(\sum_{n \le x} \omega(n)\right).$$

As well known,

$$\sum_{n \le x} \omega(n) = x \log\log x + Mx + O\left(\frac{x}{\log x}\right),$$

where

$$M = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) \doteq 0.261497$$

is the Mertens constant. By partial summation we have

$$\sum_{n \le x} \frac{\omega(n)}{n} = (\log x)(\log\log x) + (M-1)\log x + O(\log\log x).$$

Using also formula (2.15) on $\sum_{mn\le x} \omega((m,n))$ we deduce (2.20) with error term $O(x\log\log x)$. For a different approach observe that

$$\sum_{mn=k} \omega([m,n]) = \sum_{mn=k} \omega(k) = \omega(k)\tau(k),$$

since if $mn=k$, then the prime factors of $[m,n]$ coincide with the prime factors of k, so $\omega([m,n]) = \omega(k)$.

Hence

$$\sum_{mn\le x} \omega([m,n]) = \sum_{n\le x} \omega(n)\tau(n),$$

and the asymptotic formula for the latter sum, established by De Koninck and Mercier [5, Th. 9] using analytic arguments, gives the error $O(x)$. Note that in [5] the constant (2.21) is given in a different form. □

Proof of Theorem 2.10. By using that

$$\sum_{n\le x} \Omega(n) = x\log\log x + \left(M + \sum_p \frac{1}{p(p-1)}\right) x + O\left(\frac{x}{\log x}\right),$$

the first approach in the proof of Theorem 2.9 applies, and gives the error $O(x\log\log x)$. However, to obtain the better error term $O(x)$ we proceed as follows. The function $\Omega(n)$ is completely additive, hence by (2.7),

$$\sum_{mn\le x} \Omega([m,n]) = \sum_{n\le x} \Omega(n)\tau(n) - \sum_{mn\le x} \Omega((m,n)).$$

It follows from the general result by De Koninck and Mercier [5, Th. 8], applied to the function $\Omega(n)$ that

$$\sum_{n\le x} \Omega(n)\tau(n) = 2x(\log x)(\log\log x) + K_\Omega x\log x + O(x),$$

where the constant K_Ω is defined by (2.23). Now using also estimate (2.16) on $\sum_{mn\le x} \Omega((m,n))$ finishes the proof of (2.22). □

Proof of Theorem 2.11. We show that

$$h(n) := \sum_{ab=n} \tau([a,b]) = \sum_{dk=n} \psi(d)\tau^2(k), \tag{3.16}$$

where the function ψ is multiplicative and $\psi(p^\nu) = (-1)^{\nu-1}(\nu-1)$ for every prime power p^ν $(\nu \geq 0)$.

This can be done by multiplicativity and computing the values of both sides for prime powers. However, we present here a different approach, based on identity (2.5). The Dirichlet series of $h(n)$ is

$$H(z) := \sum_{n=1} \frac{h(n)}{n^z} = \sum_{d^2k=1}^{\infty} \frac{\tau(dk)2^{\omega(k)}}{d^{2z}k^z} = \sum_{k=1}^{\infty} \frac{2^{\omega(k)}}{k^z} \sum_{d=1}^{\infty} \frac{\tau(dk)}{d^{2z}}. \tag{3.17}$$

If f is any multiplicative function and $k = \prod_p p^{\nu_p(k)}$ a positive integer, then

$$\sum_{n=1}^{\infty} \frac{f(kn)}{n^z} = \prod_p \sum_{\nu=0}^{\infty} \frac{f(p^{\nu+\nu_p(k)})}{p^{\nu z}}.$$

If $f(n) = \tau(n)$, then this gives (see Titchmarsh [16, Sect. 1.4.2])

$$\sum_{n=1}^{\infty} \frac{\tau(kn)}{n^z} = \zeta^2(z) \prod_p \left(\nu_p(k) + 1 - \frac{\nu_p(k)}{p^z}\right). \tag{3.18}$$

By inserting (3.18) into (3.17) we deduce

$$H(z) = \zeta^2(2z) \sum_{k=0}^{\infty} \frac{2^{\omega(k)} h_z(k)}{k^z},$$

where h_z is the multiplicative function given by $h_z(k) = \prod_p \left(\nu_p(k) + 1 - \frac{\nu_p(k)}{p^{2z}}\right)$, depending on z. Therefore, by the Euler product formula,

$$\begin{aligned}
H(z) &= \zeta^2(2z) \prod_p \left(1 + \sum_{\nu=1}^{\infty} \frac{2}{p^{\nu z}} \left(\nu + 1 - \frac{\nu}{p^{2z}}\right)\right) \\
&= \zeta^2(2z) \prod_p \left(1 + 2(1 - \frac{1}{p^{2z}}) \sum_{\nu=1}^{\infty} \frac{\nu+1}{p^{\nu z}} + \frac{2}{p^{2z}} \sum_{\nu=1}^{\infty} \frac{1}{p^{\nu z}}\right) \\
&= \zeta^2(2z) \prod_p \left(1 + 2\left(1 - \frac{1}{p^{2z}}\right)\left(\left(1 - \frac{1}{p^z}\right)^{-2} - 1\right) + \frac{2}{p^{3z}}\left(1 - \frac{1}{p^z}\right)^{-1}\right) \\
&= \zeta^2(2z) \prod_p \left(1 + 2\left(1 + \frac{1}{p^z}\right)\left(1 - \frac{1}{p^z}\right)^{-1} - 2\left(1 - \frac{1}{p^{2z}}\right) + \frac{2}{p^{3z}}\left(1 - \frac{1}{p^z}\right)^{-1}\right) \\
&= \zeta(z)\zeta^2(2z) \prod_p \left(1 + \frac{1}{p^z}\right)\left(1 + \frac{2}{p^z}\right) \\
&= \zeta^2(z)\zeta(2z) \prod_p \left(1 + \frac{2}{p^z}\right),
\end{aligned}$$

which can be written as

$$H(z) = \frac{\zeta^4(z)}{\zeta(2z)} G(z),$$

where

$$G(z)=\prod_p\left(1-\frac{1}{(p^z+1)^2}\right)=\prod_p\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu-1}(\nu-1)}{p^{\nu z}}. \tag{3.19}$$

Here $\frac{\zeta^4(z)}{\zeta(2z)}=\sum_{n=1}^{\infty}\frac{\tau^2(n)}{n^z}$, as well known. This proves identity (3.16).

The infinite product in (3.19) is absolutely convergent for $\Re z>1/2$, Using Ramanujan's formula

$$\sum_{n\le x}\tau^2(n)=x(a(\log x)^3+b(\log x)+c\log x+d)+O(x^{1/2+\varepsilon}),$$

where $a=1/\pi^2$, the convolution method leads to asymptotic formula (2.24). The main coefficient is $(1/\pi^2)G(1)=(1/\pi^2)\prod_p\left(1-\frac{1}{(p+1)^2}\right)$. See the similar proof of [17, Th. **1**]. □

Proof of Theorem 2.12. We have

$$\sum_{ab\le x}\frac{(a,b)}{[a,b]}=\sum_{n\le x}\frac{1}{n}\sum_{ab=n}(a,b)^2. \tag{3.20}$$

Let $f(n)=n^2$. Then $(\mu*f)(n)=\phi_2(n)=n^2\prod_{p|n}(1-1/p^2)$ is the Jordan function of order 2. Here Theorems 2.2 and 2.4 cannot be applied. However, the estimate

$$\sum_{n\le x}\phi_2(n)=\frac{x^3}{3\zeta(3)}+O(x^2),$$

is well-known, and using identity (2.2) we deduce that

$$\begin{aligned}\sum_{ab\le x}(a,b)^2&=\sum_{d^2k\le x}\phi_2(d)\tau(k)=\sum_{k\le x}\tau(k)\sum_{d\le(x/k)^{1/2}}\phi_2(d)\\&=\frac{x^{3/2}}{3\zeta(3)}\sum_{k\le x}\frac{\tau(k)}{k^{3/2}}+O\left(x\sum_{k\le x}\frac{\tau(k)}{k}\right)\\&=\frac{\zeta^2(3/2)}{3\zeta(3)}x^{3/2}+O\left(x(\log x)^2\right).\end{aligned}$$

Now, taking into account (3.20), partial summation concludes formula (2.25). □

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