An Extension of Raşa's Conjecture to q -Monotone Functions

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Abstract. We extend an inequality involving the Bernstein basis polynomials and convex functions on $[0, 1]$. The inequality was originally conjectured by Raşa about thirty years ago, but was proved only recently. Our extension provides an inequality involving q -monotone functions, $q \in \mathbb{N}$. In particular, 1-monotone functions are nondecreasing functions, and 2-monotone functions are convex functions. In general, q -monotone functions on $[0, 1]$, for $q \geq 2$, possess a $(q - 2)$ nd derivative in $(0, 1)$, which is convex there. We also discuss some other linear positive approximation processes.

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1. Introduction

Given a function f defined on $[0, 1]$, the classical Bernstein polynomials associated with it are defined by,

$$(B_n f)(x) = \sum_{j=0}^n p_{n,j}(x) f\left(\frac{j}{n}\right), \quad x \in [0, 1],$$

where

$$p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad 0 \leq j \leq n.$$

For the sake of simplicity we need also the notation $p_{n,j}(x) = 0$, $j < 0$ and $j > n$.

Mrowiec et al. [11] proved the following theorem.

Theorem A. *If $f \in C[0, 1]$ is convex, then for all $n \in \mathbb{N}$,*

$$\sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y)] f\left(\frac{i+j}{2n}\right) \geq 0.$$

This inequality involving the Bernstein basis polynomials was stated by Ioan Raşa as an open problem about thirty years ago. During the Conference on Ulam's Type Stability (Rytro, Poland, 2014), Raşa [12] recalled his problem. Theorem A affirms the conjecture.

The proof given by Mrowiec et al. [11] makes heavy use of probability theory. As a tool they applied stochastic convex orderings (which they proved for binomial distributions) as well as the so-called concentration inequality. Recently [1], the first author gave an elementary proof of Theorem A, which has brought Komisarski and Rajba [8] to give a simple proof unifying that of Theorem A and of similar inequalities for the Favard–Mirakyan–Szász and Baskakov operators (see below), using probabilistic tools and the Hardy–Littlewood–Polya inequality (see also [9]).

Following the ideas of [1], Gavrea [6] extended Theorem A. He considered more general functions $a_{n,i}(x)$ than $p_{n,i}(x)$ and, more importantly, he replaced the function evaluations $f\left(\frac{i+j}{2n}\right)$ with $A_{((i+j)/(2n))}(f)$, where $\{A_t\}_{t \geq 0}$, is a set of positive linear functionals on the linear space of functions f satisfying certain conditions (see [6, (12) and Theorem 3.1]).

In [2], Abel and Raşa have proved a somewhat stronger inequality that implies Theorem A and shows the strong relation of Theorem A to the convexity preservation of the Bernstein polynomials. Recently, Gavrea [7] has followed [2] and studied Raşa's convexity problem for Baskakov–Mastroianni-type operators (see Sect. 4, below for definitions). The concrete examples discussed in [7] are restricted to Bernstein polynomials, the Favard–Mirakyan–Szász operators, also a slight generalization, by Schurer, and the Baskakov operators (which we discuss in Sect. 3, below).

Given $f \in C[0, 1]$, denote

$$\Delta_h^1 f(x) := \Delta_h f(x) := \begin{cases} f(x+h) - f(x), & x, x+h \in [0, 1], \\ 0, & \text{otherwise,} \end{cases}$$

and for $q \geq 1$,

$$\Delta_h^{q+1} f(x) := \Delta_h^q(\Delta_h f(x)).$$

Namely,

$$\Delta_h^q f(x) = \begin{cases} \sum_{i=0}^q (-1)^{q-i} \binom{q}{i} f(x+ih), & x, x+qh \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

A function f defined on $[0, 1]$ is called q -monotone there if $\Delta_h^q f(x) \geq 0$, for all $h \geq 0$. In particular a 1-monotone function is nondecreasing and a 2-monotone

one is convex. It is well known (see, e.g., [10]) that the Bernstein polynomials preserve q -monotonicity of all orders $q \geq 1$. In view of the above, it is, thus, natural to ask whether there is an analog of Theorem A for any other q . The purpose of this paper is to present this analogous result. Namely, we will show that

Theorem 1.1. *Let $q, n \in \mathbb{N}$. If $f \in C[0, 1]$ is a q -monotone function, then for all $x, y \in [0, 1]$,*

$$\begin{aligned} \operatorname{sgn}(x - y)^q & \sum_{\nu_1, \dots, \nu_q=0}^n \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j p_{n, \nu_i}(x) \right) \left(\prod_{i=j+1}^q p_{n, \nu_i}(y) \right) \\ & \times f\left(\frac{\nu_1 + \dots + \nu_q}{qn}\right) \geq 0. \end{aligned} \quad (1.1)$$

Here and in the sequel we follow the convention that an empty product is equal to 1.

Regrettably, the tools of stochastic processes and analysis that have been so elegantly used in [8, 9] for the convex case, are not available for $q > 2$.

Remark 1.2. Note that for $q = 1$, inequality (1.1) is simply rewriting the fact that if f is nondecreasing, so are the Bernstein polynomials associated with it. We don't know if for any of the other q 's, (1.1) is equivalent to the q -monotonicity preservation of the Bernstein polynomials.

Actually Theorem 1.1 will follow as a special case from Theorem 1.3 which we state after introducing some notation.

With a function f integrable on $[0, 1]$, and $\alpha \in [0, 1]$, we associate the polynomials

$$(B_n^\alpha f)(x) := \sum_{k=0}^n p_{n,k}(x) \int_0^1 f\left(\frac{k + \alpha t}{n + \alpha}\right) dt, \quad x \in [0, 1]. \quad (1.2)$$

(See, e.g., [3, (3.7)].) For $\alpha = 0$, the polynomials (1.2) reduce to the Bernstein polynomials, and if $\alpha = 1$, then we obtain the Kantorovich polynomials associated with f . It is well known that if $f \in C[0, 1]$, then

$$\lim_{n \rightarrow \infty} (B_n^\alpha f)(x) = f(x), \quad \text{uniformly in } x \in [0, 1].$$

Our result is

Theorem 1.3. *Let $q, n \in \mathbb{N}$. If $f \in C[0, 1]$ is a q -monotone function, then for all $x, y \in [0, 1]$,*

$$\begin{aligned} \operatorname{sgn}(x - y)^q & \sum_{\nu_1, \dots, \nu_q=0}^n \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j p_{n, \nu_i}(x) \right) \left(\prod_{i=j+1}^q p_{n, \nu_i}(y) \right) \\ & \times \int_0^1 f\left(\frac{\nu_1 + \dots + \nu_q + \alpha t}{qn + \alpha}\right) dt \geq 0. \end{aligned}$$

We will prove Theorem 1.3 in Sect. 2. Then, in Sect. 3, we will discuss the analogues of Theorem 1.3 for the basis elements of the Favard-Mirakyan-Szász and of the Baskakov operators.

2. Proof of Theorem 1.3

We begin with some auxiliary formulas.

Given $n, m, q \in \mathbb{N}$, denote $\nu := (\nu_1, \dots, \nu_q) \in (\mathbb{N}_0)^q$ and $|\nu| := \nu_1 + \dots + \nu_q$. Then

$$\begin{aligned} \sum_{|\nu|=m} \prod_{i=1}^q p_{n,\nu_i}(x_i) &= \sum_{|\nu|=m} \prod_{i=1}^q \binom{n}{\nu_i} x_i^{\nu_i} (1-x_i)^{n-\nu_i} \\ &= \frac{1}{m!} \sum_{|\nu|=m} \binom{m}{\nu} \prod_{i=1}^q \left[\left(\frac{\partial}{\partial z} \right)^{\nu_i} (1+x_i z)^n \right] \Big|_{z=-1} \\ &= \frac{1}{m!} \left(\frac{\partial}{\partial z} \right)^m \prod_{i=1}^q (1+x_i z)^n \Big|_{z=-1}, \end{aligned} \quad (2.1)$$

where the last equality follows by Leibniz rule for the differentiation of products of functions. Note that the formula is valid also if $m > n$ as we recall that $p_{n,j}(x) = 0$, for $j > n$.

Hence, for $x, y \in [0, 1]$ and any $(qn+1)$ -tuple $(a_k)_{k=0}^{qn}$,

$$\begin{aligned} &\sum_{|\nu| \leq qn} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j p_{n,\nu_i}(x) \right) \left(\prod_{i=j+1}^q p_{n,\nu_i}(y) \right) a_{|\nu|} \\ &= \sum_{m=0}^{qn} a_m \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \sum_{|\nu|=m} \left(\prod_{i=1}^j p_{n,\nu_i}(x) \right) \left(\prod_{i=j+1}^q p_{n,\nu_i}(y) \right) \\ &= \sum_{m=0}^{qn} a_m \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \frac{1}{m!} \sum_{|\nu|=m} \binom{m}{\nu} \\ &\quad \times \left(\prod_{i=1}^j \left[\left(\frac{\partial}{\partial z} \right)^{\nu_i} (1+xz)^n \right] \right) \left(\prod_{i=j+1}^q \left[\left(\frac{\partial}{\partial z} \right)^{\nu_i} (1+yz)^n \right] \right) \Big|_{z=-1} \\ &= \sum_{m=0}^{qn} a_m \frac{1}{m!} \left[\left(\frac{\partial}{\partial z} \right)^m \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} (1+xz)^{nj} (1+yz)^{n(q-j)} \right] \Big|_{z=-1} \\ &= \sum_{m=0}^{qn} a_m \frac{1}{m!} \left[\left(\frac{\partial}{\partial z} \right)^m [(1+xz)^n - (1+yz)^n]^q \right] \Big|_{z=-1}. \end{aligned} \quad (2.2)$$

Now, define

$$g(z) := g_n(z; x, y) := \left(\frac{(1+xz)^n - (1+yz)^n}{z} \right)^q,$$

a polynomial in z of degree at most $(n-1)q$. Then,

$$g(z) = (x-y)^q \left(\sum_{m=0}^{n-1} (1+xz)^m (1+yz)^{n-1-m} \right)^q.$$

Hence,

$$\operatorname{sgn}(x-y)^q g^{(k)}(-1) \geq 0, \quad k = 0, \dots, qn - q. \quad (2.3)$$

The next proposition is the key result.

Proposition 2.1. *Let $x, y \in [0, 1]$. Then,*

$$\begin{aligned} & \sum_{\nu_1, \dots, \nu_q=0}^n a_{|\nu|} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j p_{n, \nu_i}(x) \right) \left(\prod_{i=j+1}^q p_{n, \nu_i}(y) \right) \\ &= \sum_{k=0}^{qn-q} (\Delta^q a_k) \frac{1}{k!} g^{(k)}(-1). \end{aligned} \quad (2.4)$$

Here Δ denotes the forward difference $\Delta^1 a_k := \Delta a_k := a_{k+1} - a_k$, and for $m \in \mathbb{N}$, we recursively define $\Delta^{m+1} a_k := \Delta(\Delta^m a_k)$.

Proof. By virtue of (2.2), we have

$$\begin{aligned} & \sum_{|\nu| \leq qn} a_{|\nu|} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j p_{n, \nu_i}(x) \right) \left(\prod_{i=j+1}^q p_{n, \nu_i}(y) \right) \\ &= \sum_{m=0}^{qn} a_m \frac{1}{m!} \frac{d^m}{dz^m} [z^q g(z)] \Big|_{z=-1} \\ &= \sum_{m=0}^{qn} a_m \frac{1}{m!} \sum_{i=0}^m \binom{m}{i} q \cdots (q-i+1) z^{q-i} g^{(m-i)}(z) \Big|_{z=-1} \\ &= \sum_{m=0}^{qn} a_m \sum_{i=0}^m (-1)^{q-i} \binom{q}{i} \frac{1}{(m-i)!} g^{(m-i)}(-1) \\ &= \sum_{m=0}^{qn-q} \frac{1}{m!} g^{(m)}(-1) \sum_{i=0}^q (-1)^q \binom{q}{i} a_{m+i} \\ &= \sum_{m=0}^{qn-q} \frac{1}{m!} g^{(m)}(-1) \Delta^q a_m. \end{aligned}$$

which proves (2.4) □.

Proof of Theorem 1.3. For $m = 0, 1, \dots, qn - q$, we put

$$a_m = \int_0^1 f\left(\frac{m + \alpha t}{qn + \alpha}\right) dt.$$

If $f \in C[0, 1]$ is a q -monotone function, then $\Delta^q a_m \geq 0$, for $m = 0, 1, \dots, qn - q$. Therefore, Theorem 1.3 readily follows by combining Proposition 2.1 and (2.3). $\square$

3. Other Classical Linear Positive Approximation Operators

Given a function f defined on $[0, \infty)$, we are going to discuss two classical linear approximation processes, the Favard–Mirakyan–Szász operators and the Baskakov operators associated with it.

The Favard–Mirakyan–Szász operators, associated with f defined on $[0, \infty)$, such that $|f(x)| \leq Ce^{Ax}$, $x \in [0, \infty)$, for some constants $C, A > 0$, are defined by,

$$(S_u f)(x) := e^{-ux} \sum_{k=0}^{\infty} \frac{(ux)^k}{k!} f\left(\frac{k}{u}\right), \quad x \in [0, \infty), \quad 0 < u < \infty.$$

If we denote

$$s_k(x) := e^{-x} \frac{x^k}{k!} = \frac{1}{k!} \frac{\partial^k}{\partial z^k} e^{zx} \Big|_{z=-1}, \quad k = 0, 1, \dots, \quad (3.1)$$

then the operators may be represented as,

$$(S_u f)(x) = \sum_{k=0}^{\infty} s_k(ux) f\left(\frac{k}{u}\right), \quad x \in [0, \infty), \quad 0 < u < \infty.$$

We may generalize the operators as we have done in Sect. 1 to include both the above operators (for $\alpha = 0$) and their Kantorovich polynomials' variant (for $\alpha = 1$). Namely, for f as above which is integrable in every compact subinterval of $[0, \infty)$, and $\alpha \in [0, 1]$, let

$$(S_u^\alpha f)(x) = \sum_{k=0}^{\infty} s_k(ux) \int_0^1 f\left(\frac{k + \alpha t}{u + \alpha}\right) dt, \quad x \in [0, \infty), \quad 0 < u < \infty.$$

It follows that, for a continuous f as above,

$$\lim_{u \rightarrow \infty} (S_u^\alpha f)(x) = f(x), \quad \text{uniformly in every compact subinterval of } [0, \infty).$$

The Baskakov operators, associated with f defined on $[0, \infty)$, such that $|f(x)| \leq C(1+x)^\mu$, $x \in [0, \infty)$, for some $C, \mu > 0$, are defined by

$$(V_n f)(x) := \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \quad n = 1, 2, \dots$$

If we denote

$$m_{n,k}(x) := \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} = \frac{1}{k!} \frac{\partial^k}{\partial z^k} (1-zx)^{-n} \Big|_{z=-1}, \quad k=0,1,\dots, \quad (3.2)$$

then we may write

$$(V_n f)(x) := \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \quad n=1,2,\dots$$

We generalize these operators in the above spirit, namely, for an appropriate f integrable in every compact subinterval of $[0, \infty)$, and $\alpha \in [0, 1]$, let

$$(V_n^\alpha f)(x) := \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 f\left(\frac{k+\alpha t}{n+\alpha}\right) dt, \quad x \in [0, \infty), \quad n=1,2,\dots$$

We have the following two results analogous to Theorem 1.3.

Theorem 3.1. Let $q \in \mathbb{N}$ and $0 < u < \infty$. If f defined on $[0, \infty)$ and such that $|f(x)| \leq Ce^{Ax}$, for some constants $C, A > 0$, is a q -monotone function there, then for all $x, y \in [0, \infty)$,

$$\begin{aligned} & \operatorname{sgn}(x-y)^q \sum_{\nu_1, \dots, \nu_q=0}^{\infty} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j s_{\nu_i}(x) \right) \left(\prod_{i=j+1}^q s_{\nu_i}(y) \right) \\ & \times \int_0^1 f\left(\frac{\nu_1 + \dots + \nu_q + \alpha t}{u + \alpha}\right) dt \geq 0. \end{aligned}$$

And

Theorem 3.2. Let $n, q \in \mathbb{N}$. If f defined on $[0, \infty)$ and such that $|f(x)| \leq C(1+x)^\gamma$, for some $C, \gamma > 0$, is a q -monotone function there, then for all $x, y \in [0, \infty)$,

$$\begin{aligned} & \operatorname{sgn}(y-x)^q \sum_{\nu_1, \dots, \nu_q=0}^{\infty} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j m_{n,\nu_i}(x) \right) \left(\prod_{i=j+1}^q m_{n,\nu_i}(y) \right) \\ & \times \int_0^1 f\left(\frac{\nu_1 + \dots + \nu_q + \alpha t}{n + \alpha}\right) dt \geq 0. \end{aligned}$$

Proof of Theorem 3.1. First, we observe that for $\nu := (\nu_1, \dots, \nu_q) \in (\mathbb{N}_0)^q$ and integer $m \geq 0$, we have

$$(x_1 + \dots + x_q)^m = \sum_{|\nu|=m} \binom{m}{\nu} \left(\prod_{i=1}^q x_i^{\nu_i} \right),$$

so that by (3.1),

$$\sum_{|\nu|=m} \left(\prod_{i=1}^q s_{\nu_i}(x_i) \right) = \sum_{|\nu|=m} \left(\prod_{i=1}^q \frac{x_i^{\nu_i}}{\nu_i!} e^{-x_i} \right)$$

$$\begin{aligned}
&= \frac{1}{m!} \sum_{|\nu|=m} \binom{m}{\nu} \left(\prod_{i=1}^q x^{\nu_i} \right) e^{-(x_1+\dots+x_q)} \\
&= \frac{1}{m!} (x_1 + \dots + x_q)^m e^{-(x_1+\dots+x_q)} \\
&= \frac{1}{m!} \frac{\partial^m}{\partial z^m} e^{z(x_1+\dots+x_q)} \Big|_{z=-1}.
\end{aligned}$$

Denote

$$F_\nu := \int_0^1 f\left(\frac{|\nu| + \alpha t}{u + \alpha}\right) dt,$$

and

$$F_m := \int_0^1 f\left(\frac{m + \alpha t}{u + \alpha}\right) dt.$$

Then, it follows that

$$\begin{aligned}
&\sum_{\nu_1, \dots, \nu_q=0}^{\infty} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j s_{\nu_i}(x) \right) \left(\prod_{i=j+1}^q s_{\nu_i}(y) \right) F_\nu \\
&= \sum_{m=0}^{\infty} F_m \frac{1}{m!} \frac{\partial^m}{\partial z^m} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} e^{z(jx + (q-j)y)} \Big|_{z=-1} \\
&= \sum_{m=0}^{\infty} F_m \frac{1}{m!} \frac{\partial^m}{\partial z^m} (e^{zx} - e^{zy})^q \Big|_{z=-1}. \tag{3.3}
\end{aligned}$$

Denote,

$$g(z) := g(z; x, y) := \left(\frac{e^{zx} - e^{zy}}{z} \right)^q.$$

Then, the last line in (3.3) may be rewritten, as we had in Sect. 2, as

$$\begin{aligned}
&\sum_{m=0}^{\infty} F_m \frac{1}{m!} \frac{\partial^m}{\partial z^m} (e^{zx} - e^{zy})^q \Big|_{z=-1} \\
&= \sum_{m=0}^{\infty} F_m \frac{1}{m!} \frac{d^m}{dz^m} (z^q g(z)) \Big|_{z=-1} \\
&= \sum_{m=0}^{\infty} (\Delta^q F_m) \frac{1}{m!} g^{(m)}(-1).
\end{aligned}$$

Expanding $g(z)$ into a Taylor series about $z = -1$, it readily follows that

$$\begin{aligned} g(z) &= \underbrace{\int_y^x \cdots \int_y^x}_{q \text{ times}} e^{z(u_1 + \cdots + u_q)} du_1 \cdots du_q \\ &= \sum_{k=0}^{\infty} \frac{(z+1)^k}{k!} \underbrace{\int_y^x \cdots \int_y^x}_{q \text{ times}} (u_1 + \cdots + u_q)^k e^{-(u_1 + \cdots + u_q)} du_1 \cdots du_q. \end{aligned}$$

Therefore, $\operatorname{sgn}(x-y)^q g^{(k)}(-1) \geq 0$, for all $k = 0, 1, \dots$, and the proof of Theorem 3.1 is complete. $\square$

Proof of Theorem 3.2. Let $n, m, q \in \mathbb{N}$, and denote $\nu := (\nu_1, \dots, \nu_q) \in (\mathbb{N}_0)^q$. By (3.2) it follows that

$$\begin{aligned} \sum_{|\nu|=m} \prod_{i=1}^q m_{n, \nu_i}(x_i) &= \frac{1}{m!} \sum_{|\nu|=m} \binom{m}{\nu} \prod_{i=1}^q \left(\frac{\partial}{\partial z} \right)^{\nu_i} (1 - x_i z)^{-n} \Big|_{z=-1} \\ &= \frac{1}{m!} \left(\frac{\partial}{\partial z} \right)^m \prod_{i=1}^q (1 - x_i z)^{-n} \Big|_{z=-1}. \end{aligned} \quad (3.4)$$

Hence, as is done in (2.2), for $0 \leq x, y < \infty$ and any sequence $(a_k)_{k=0}^{\infty}$, we have

$$\begin{aligned} &\sum_{|\nu| \geq 0} \sum_{j=0}^q (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^j m_{n, \nu_i}(x) \right) \left(\prod_{i=j+1}^q m_{n, \nu_i}(y) \right) a_{|\nu|} \\ &= \sum_{m=0}^{\infty} a_m \frac{1}{m!} \left[\left(\frac{\partial}{\partial z} \right)^m [(1 - xz)^{-n} - (1 - yz)^{-n}]^q \right] \Big|_{z=-1}. \end{aligned} \quad (3.5)$$

Now, define

$$g(z) := g_n(z; x, y) := \left(\frac{(1 - xz)^{-n} - (1 - yz)^{-n}}{z} \right)^q.$$

Then,

$$\begin{aligned} g(z) &= (y - x)^q \left(\frac{(1 - xz)^{-n} - (1 - yz)^{-n}}{(1 - xz) - (1 - yz)} \right)^q \\ &= (y - x)^q \left(\sum_{m=0}^{n-1} (1 - xz)^{-m} (1 - yz)^{-(n-1-m)} \right)^q. \end{aligned}$$

Note that $(\partial/\partial z)^j (1 - uz)^{-m} \Big|_{z=-1} = j! m_{m,j}(x) \geq 0$, for $j, m = 0, 1, 2, \dots$, and $0 \leq u < \infty$. Hence

$$\operatorname{sgn}(y - x)^q \cdot g^{(k)}(-1) \geq 0, \quad k = 0, \dots \quad (3.6)$$

Finally, we may rewrite the last line in (3.5), as we did in (3.3),

$$\begin{aligned} & \sum_{m=0}^{\infty} a_m \frac{1}{m!} \left[\left(\frac{\partial}{\partial z} \right)^m [(1-xz)^{-n} - (1-yz)^{-n}]^q \right] \Big|_{z=-1} \\ &= \sum_{m=0}^{\infty} a_m \frac{1}{m!} \frac{d^m}{dz^m} (z^q g(z)) \Big|_{z=-1} \\ &= \sum_{m=0}^{\infty} \Delta^q a_m \frac{1}{m!} g^{(m)}(-1), \end{aligned}$$

and Theorem 3.2 follows by taking $a_m := \int_0^1 f\left(\frac{m+\alpha t}{n+\alpha}\right) dt$. $\square$

4. Concluding Remarks

In recent years there has been a unified approach that includes both the Bernstein and Baskakov operators, yet, one has to keep in mind that the functions with which the operators are associated are defined on different intervals.

For $c \neq 0$, denote

$$\begin{aligned} p_{n,j}^{[c]}(x) &= \binom{-n/c}{j} (-cx)^j (1+cx)^{-n/c-j} \\ &= \frac{1}{j!} \left[\left(\frac{\partial}{\partial z} \right)^j (1-cxz)^{-n/c} \right] \Big|_{z=-1}, \end{aligned}$$

where we recall that

$$\binom{-n/c}{j} = \frac{(-1)^j (n/c)(n/c+1) \cdots (n/c+j-1)}{j!}.$$

Note that the Bernstein basis polynomials are obtained for $c = -1$ and the Baskakov basis elements are obtained for $c = 1$. It is also interesting that,

$$\lim_{c \rightarrow 0} p_{n,j}^{[c]}(x) = \frac{(nx)^j}{j!} e^{-nx}.$$

Define the operators, associated with an appropriate function f and for the right range of x , by

$$(L_n^{[c]} f)(x) := \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \int_0^1 f\left(\frac{k+\alpha t}{n+\alpha}\right) dt,$$

and one can prove a theorem analogous to Theorems 1.3 and 3.2, along the same lines.

We omit the details as we do not find much interest in the general case.

Another more general sequence of operators is due originally to Baskakov [5] and modified by Mastroianni (see [4]) Let $(\phi_n)_{n=1}^\infty$ be a sequence of real valued functions defined on $\mathbb{R}_+$, such that $\phi_n(0) = 1$, $n \geq 1$, and

$$(-1)^k \phi_n^{(k)}(x) \geq 0, \quad n \geq 1, \quad k \geq 0, \quad x \in \mathbb{R}_+.$$

Finally, we assume that for each pair $(n, k) \in \mathbb{N} \times \mathbb{N}_0$, there exist a number $p(n, k) \in \mathbb{N}$ and a positive function $\alpha_{n,k}(x)$, such that for any fixed k ,

$$\lim_{n \rightarrow \infty} \frac{p(n, k)}{n} = \lim_{n \rightarrow \infty} \frac{\alpha_{n,k}(x)}{n^k} = 1,$$

for which

$$\phi_n^{(i+k)}(x) = (-1)^k \alpha_{n,k}(x) \phi_{p(n,k)}^{(i)}(x), \quad i \in \mathbb{N}_0.$$

Then, the general Baskakov–Mastroianni operators associated with a function f defined on $[0, \infty)$, are defined by

$$(L_n^\alpha f)(x) := \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \phi_n^{(k)}(x) \int_0^1 f\left(\frac{k + \alpha t}{n + \alpha}\right) dt, \quad 0 \leq x < \infty,$$

provided that the series converges for all $x \in [0, \infty)$. Again, the original definition was with $\alpha = 0$, and the Kantorovich variant, namely, $\alpha = 1$ appears in [4].

The Bernstein polynomials are obtained by taking $\phi_n(x) = (1 - x)^n$. In this case $p(n, k) = n - k$ and $\alpha_{n,k}(x) = k! \binom{n}{k}$. In the case $\phi_n(x) = e^{-nx}$, we have $p(n, k) = n$ and $\alpha_{n,k}(x) = n^k$, so that the Baskakov–Mastroianni operators reduce to the Favard–Mirakyan–Szász operators, with running index n instead of a real u , and in the case $\phi_n(x) = (1 + x)^{-n}$, we have $p(n, k) = n + k$ and $\alpha_{n,k}(x) = k! \binom{n+k-1}{k}$, so that the Baskakov–Mastroianni operators reduce to the Baskakov operators.

Again, one may obtain an analog of (2.2), (3.3) and (3.5), we omit the details as we do not know of any significant examples, other than the above three, where one may proceed and have something interesting.

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