Ohlin-Type Theorem for Convex Set-Valued Maps

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Abstract. A counterpart of the Ohlin theorem for convex set-valued maps is proved. An application of this result to obtain some inclusions related to convex set-valued maps in an alternative unified way is presented. In particular counterparts of the Jensen integral and discrete inequalities, the converse Jensen inequality and the Hermite–Hadamard inequalities are obtained.

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1. Introduction

In Ohlin [12] proved the following interesting and very useful result on convex functions in a probabilistic context (as usual, $\mathbb{E}[X]$ denotes the expectation of the random variable X):

Lemma 1 [12]. *Let X_1, X_2 be two real valued random variables such that $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If the distribution functions F_{X_1}, F_{X_2} cross one time, i.e. there exists $t_0 \in \mathbb{R}$ such that*

$$F_{X_1}(t) \leq F_{X_2}(t) \quad \text{if } t < t_0 \text{ and } F_{X_1}(t) \geq F_{X_2}(t) \quad \text{if } t > t_0, \quad (1)$$

then

$$\mathbb{E}[f(X_1)] \leq \mathbb{E}[f(X_2)]$$

for every convex function $f : \mathbb{R} \rightarrow \mathbb{R}$.

For years the above Ohlin lemma was not well-known in the mathematical community. It has been rediscovered by Rajba [14], who found its various applications to the theory of functional inequalities. In [13, 15, 18], the Ohlin lemma is used, among others, to get a simple proof of the known Hermite–Hadamard inequalities, as well as to obtaining new Hermite–Hadamard type inequalities.

In this note we prove counterparts of the Ohlin theorem for convex set-valued maps. We present also applications of these results to obtain some inclusions connected with convex set-valued maps.

2. Preliminaries

Let $(Y, \|\cdot\|)$ be a separable Banach space, B be the closed unit ball in Y , $(\Omega, \mathcal{A}, P)$ be a probability space with a non-atomic measure P and $I \subset \mathbb{R}$ be an open interval. Denote by $n(Y)$ the family of all nonempty subsets of Y and by $cl(Y)$ the family of all closed nonempty subsets of Y . For a given set-valued map $G : \Omega \rightarrow n(Y)$ the integral $\int_{\Omega} G(\omega) dP$ is understood in the sense of Aumann, i.e. it is the set of integrals of all integrable (in the sense of Bochner) selections of the map G (cf. [1, 2]):

$$\int_{\Omega} G(\omega) dP = \left\{ \int_{\Omega} g(\omega) dP : g : \Omega \rightarrow Y \text{ is integrable and } g(\omega) \in G(\omega), \omega \in \Omega \right\}.$$

A set-valued map $G : \Omega \rightarrow n(Y)$ is called *integrable bounded* if there exists a non-negative integrable function $k : \Omega \rightarrow \mathbb{R}$ such that $G(\omega) \subset k(\omega)B$, for all $\omega \in \Omega$. In this case every measurable selection of G is integrable and, consequently, the Aumann integral of G is nonempty whenever G is measurable.

The following properties of the Aumann integral will be needed in our investigations:

Lemma 2 [[1], Theorems 8.6.3, 8.6.4]. *Let $G : \Omega \rightarrow cl(Y)$ be a measurable set-valued map. a) The closure of the integral of G is convex and*

$$\overline{\int_{\Omega} G(\omega) dP} = \overline{conv} \left(\int_{\Omega} G(\omega) dP \right).$$

b) If Y is finite dimensional, then the integral of G is convex. In particular, if $Y = \mathbb{R}$ and $G(\omega) = [g_1(\omega), g_2(\omega)]$, $\omega \in \Omega$, then

$$\int_{\Omega} G(\omega) dP = \left[\int_{\Omega} g_1(\omega) dP, \int_{\Omega} g_2(\omega) dP \right].$$

c) If G is integrable bounded, then

$$\overline{\int_{\Omega} G(\omega) dP} = \int_{\Omega} \overline{conv} G(\omega) dP.$$

Recall that a set-valued map $G : I \rightarrow n(Y)$ is called *convex* if

$$tG(x_1) + (1-t)G(x_2) \subset G(tx_1 + (1-t)x_2) \quad (2)$$

for all $x_1, x_2 \in I$ and $t \in [0, 1]$ (see e.g. [1, 3, 4, 8] and the references therein). Note that by (2), all values of G are convex subsets of Y .

The following lemma characterizes convex set-valued maps with values in $cl(\mathbb{R})$.

Lemma 3 [8]. *A set-valued map $G : I \rightarrow cl(\mathbb{R})$ is convex if and only if it has one of the following forms:*

- a) $G(x) = [g_1(x), g_2(x)]$, $x \in I$,
- b) $G(x) = [g_1(x), +\infty)$, $x \in I$,
- c) $G(x) = (-\infty, g_2(x)]$, $x \in I$,
- d) $G(x) = (-\infty, +\infty)$, $x \in I$,

where $g_1 : I \rightarrow \mathbb{R}$ is convex and $g_2 : I \rightarrow \mathbb{R}$ is concave.

Clearly, if $G : I \rightarrow cl(\mathbb{R})$ is convex and integrable bounded, then it is of the form a).

3. Ohlin-Type Result for Convex Set-Valued Maps

The following result is a counterpart the Ohlin lemma for convex set-valued maps.

Theorem 4. *Let $(Y, \|\cdot\|)$ be a separable Banach space, $(\Omega, \mathcal{A}, P)$ be a probability space with a non-atomic measure P and $I \subset \mathbb{R}$ be an open interval. Assume that $X_1, X_2 : \Omega \rightarrow I$ are integrable random variables such that $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If there exists $t_0 \in \mathbb{R}$ such that*

$$F_{X_1}(t) \leq F_{X_2}(t) \quad \text{if } t < t_0 \text{ and } F_{X_1}(t) \geq F_{X_2}(t) \quad \text{if } t > t_0,$$

then

$$\int_{\Omega} G(X_2(\omega)) dP \subset \int_{\Omega} G(X_1(\omega)) dP$$

for every convex integrable bounded set-valued map $G : I \rightarrow cl(Y)$.

Proof. The proof is divided into two steps. First, we assume that $Y = \mathbb{R}$. Then, by Lemma 3 and the assumption that G is integrable bounded, we obtain that G is of the form $G(x) = [g_1(x), g_2(x)]$, $x \in I$, where $g_1 : I \rightarrow \mathbb{R}$ is convex and $g_2 : I \rightarrow \mathbb{R}$ is concave. By the Ohlin lemma (Lemma 1), we have

$$\mathbb{E}[g_1(X_1)] \leq \mathbb{E}[g_1(X_2)] \text{ and } \mathbb{E}[g_2(X_1)] \geq \mathbb{E}[g_2(X_2)]. \quad (3)$$

Hence, using Lemma 2(b), we get

$$\begin{aligned} \int_{\Omega} G(X_2(\omega)) dP &= [\mathbb{E}[g_1(X_2)], \mathbb{E}[g_2(X_2)]] \subset [\mathbb{E}[g_1(X_1)], \mathbb{E}[g_2(X_1)]] \\ &= \int_{\Omega} G(X_1(\omega)) dP. \end{aligned}$$

Now, assume that Y is an arbitrary separable Banach space. Take a nonzero continuous linear functional $y^* \in Y^*$. Since the set-valued map $x \mapsto \overline{y^*(G(x))}$, $x \in I$, is convex and has closed values in $\mathbb{R}$, by the previous step,

$$\int_{\Omega} \overline{y^*(G(X_2(\omega)))} dP \subset \int_{\Omega} \overline{y^*(G(X_1(\omega)))} dP. \quad (4)$$

Take arbitrary $z \in \int_{\Omega} G(X_2(\omega)) dP$. By the definition of the Aumann integral, there exists an integrable selection $g \circ X_2$ of the set-valued map $G \circ X_2$ such that $z = \int_{\Omega} g(X_2(\omega)) dP$. Using (4), we obtain

$$y^*(z) = y^*\left(\int_{\Omega} g(X_2(\omega)) dP\right) = \int_{\Omega} y^*(g(X_2(\omega))) dP \in \int_{\Omega} \overline{y^*(G(X_1(\omega)))} dP. \quad (5)$$

Since G is integrable bounded and the values $y^*(G(X_1(\omega)))$ are convex, by Lemma 2(c), we get

$$\begin{aligned} \int_{\Omega} \overline{y^*(G(X_1(\omega)))} dP &= \int_{\Omega} \overline{\text{conv}} y^*(G(X_1(\omega))) dP = \overline{\int_{\Omega} y^*(G(X_1(\omega))) dP} \\ &= \overline{y^*\left(\int_{\Omega} G(X_1(\omega)) dP\right)} \subset \overline{y^*\left(\int_{\Omega} G(X_1(\omega)) dP\right)}. \end{aligned} \quad (6)$$

From (5) and (6),

$$y^*(z) \in \overline{\overline{y^*\left(\int_{\Omega} G(X_1(\omega)) dP\right)}}.$$

Since this condition holds for arbitrary $y^* \in Y^*$ and, by Lemma 2(a) the set $\overline{\int_{\Omega} G(X_1(\omega)) dP}$ is convex and closed, by the separation theorem (see [16], Corollary 2.5.11), we obtain

$$z \in \overline{\int_{\Omega} G(X_1(\omega)) dP}$$

and hence, using once more Lemma 2(c),

$$z \in \int_{\Omega} G(X_1(\omega)) dP.$$

Consequently,

$$\int_{\Omega} G(X_2(\omega)) dP \subset \int_{\Omega} G(X_1(\omega)) dP,$$

which finishes the proof. $\square$

Remark 5. In the above proof we use the Ohlin lemma (Lemma 1) to obtain the inequalities (3). Replacing in Theorem 4 the assumptions on X_1 and X_2 (the same as in Ohlin's lemma) by any weaker conditions sufficient for (3) (for instance necessary and sufficient conditions such as in the Levin–Stečkin theorem [7]; cf. also [11]), we can obtain more general result. However, it should

be emphasized that the assumptions in the Ohlin lemma are very convenient because they are simple and can be easily verified.

4. Applications

In this section, we present an application of the Ohlin-type lemma to obtain various inclusions related to convex set-valued maps in a simple and unified way. Some of these results (Corollaries 6–10) are known, but we present alternative proofs of them.

The first result is a counterpart of the classical integral Jensen inequality.

Corollary 6 (cf. [8]). *Let $G : I \rightarrow cl(Y)$ be integrable bounded set-valued map and $(\Omega, \mathcal{A}, P)$ be a probability space with a non-atomic measure P . Then G is convex if and only if*

$$\int_{\Omega} G(X(\omega)) dP \subset G\left(\int_{\Omega} X(\omega) dP\right), \quad (7)$$

for every integrable random variable $X : \Omega \rightarrow I$.

Proof. Assume first that $G : I \rightarrow cl(Y)$ is a convex integrable bounded set-valued map and $X : \Omega \rightarrow I$ is an integrable random variable. Take a random variable $X_1 : \Omega \rightarrow I$ with the distribution $\mu_{X_1} = \delta_{\mathbb{E}[X]}$. Then the distribution functions F_X, F_{X_1} satisfy condition (1) and $\mathbb{E}[X] = \mathbb{E}[X_1]$. Therefore, by Theorem 4,

$$\int_{\Omega} G(X(\omega)) dP \subset \int_{\Omega} G(X_1(\omega)) dP = G(\mathbb{E}[X]) = G\left(\int_{\Omega} X(\omega) dP\right).$$

Now, assume that G satisfies condition (7) with every integrable random variable $X : \Omega \rightarrow I$. Fix $x_1, x_2 \in I$ and $t \in (0, 1)$, and take a random variable $X : \Omega \rightarrow I$ with the distribution $\mu_X = t\delta_{x_1} + (1-t)\delta_{x_2}$. Then $\int_{\Omega} X(\omega) dP = tx_1 + (1-t)x_2$ and $\int_{\Omega} G(X(\omega)) dP = tG(x_1) + (1-t)G(x_2)$. Therefore by (7)

$$tG(x_1) + (1-t)G(x_2) \subset G(tx_1 + (1-t)x_2),$$

which proves that G is convex. $\square$

If in the above corollary we take a random variable X with the distribution $\mu_X = t_1\delta_{x_1} + \dots + t_n\delta_{x_n}$, where $x_1, \dots, x_n \in I$ and $t_1, \dots, t_n > 0$ are such that $t_1 + \dots + t_n = 1$, then we obtain a counterpart of the discrete Jensen inequality.

Corollary 7 (cf. [10]). *If a set-valued map $G : I \rightarrow cl(Y)$ is convex and integrable bounded, then*

$$\sum_{i=1}^n t_i G(x_i) \subset G\left(\sum_{i=1}^n t_i x_i\right),$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$ and $t_1, \dots, t_n > 0$ with $t_1 + \dots + t_n = 1$.

We have also the following converse Jensen inclusion for convex set-valued maps.

Corollary 8 (cf. [6]). *Let $m, M \in I$, $m < M$. If $G : I \rightarrow cl(Y)$ is convex and integrable bounded, then*

$$\frac{M - \bar{x}}{M - m}G(m) + \frac{\bar{x} - m}{M - m}G(M) \subset \sum_{i=1}^n t_i G(x_i), \quad (8)$$

for all $x_1, \dots, x_n \in [m, M]$, $t_1, \dots, t_n > 0$ with $t_1 + \dots + t_n = 1$ and $\bar{x} = t_1 x_1 + \dots + t_n x_n$.

Proof. Take random variables $X_1, X_2 : \Omega \rightarrow I$ with the distributions

$$\mu_{X_1} = \sum_{i=1}^n t_i \delta_{x_i} \text{ and } \mu_{X_2} = \frac{M - \bar{x}}{M - m} \delta_m + \frac{\bar{x} - m}{M - m} \delta_M.$$

Then the distribution functions F_X, F_Y satisfy condition (1) and

$$\mathbb{E}[X_1] = \sum_{i=1}^n t_i x_i = \bar{x} = \frac{M - \bar{x}}{M - m} m + \frac{\bar{x} - m}{M - m} M = \mathbb{E}[X_2].$$

Moreover

$$\int_{\Omega} G(X_1(\omega)) dP = \sum_{i=1}^n t_i G(x_i),$$

and

$$\int_{\Omega} G(X_2(\omega)) dP = \frac{M - \bar{x}}{M - m} G(m) + \frac{\bar{x} - m}{M - m} G(M).$$

Therefore, by Theorem 4, we obtain (8). $\square$

The next two corollaries are versions of the Hermite–Hadamard inequalities for convex set-valued maps.

Corollary 9 (cf. [9, 17]). *If $G : I \rightarrow cl(Y)$ is convex and integrable bounded, then*

$$\frac{G(a) + G(b)}{2} \subset \frac{1}{b - a} \int_a^b G(x) dx \subset G\left(\frac{a + b}{2}\right), \quad (9)$$

for all $a, b \in I$, $a < b$.

Proof. Let $X_1, X_2 : \Omega \rightarrow I$ be random variables with the distributions $\mu_{X_1} = \delta_{(a+b)/2}$, $\mu_{X_2} = \frac{1}{2}(\delta_a + \delta_b)$ and let $X_3 : \Omega \rightarrow I$ has the uniform distribution on $[a, b]$. Then the pairs X_1, X_3 and X_3, X_2 satisfy the assumptions of Theorem 4. Moreover,

$$\int_{\Omega} G(X_1(\omega)) dP = G\left(\frac{a + b}{2}\right), \int_{\Omega} G(X_2(\omega)) dP = \frac{G(a) + G(b)}{2}$$

and

$$\int_{\Omega} G(X_3(\omega)) dP = \frac{1}{b - a} \int_a^b G(x) dx.$$

Therefore, by Theorem 4, we obtain (9). $\square$

Corollary 10. (cf. [9]) *If $G : I \rightarrow cl(Y)$ is convex and integrable bounded, $[a, b] \subset I$ and μ is a Borel measure on $[a, b]$ with $\mu([a, b]) > 0$, then*

$$\frac{b - x_\mu}{b - a} G(a) + \frac{x_\mu - a}{b - a} G(b) \subset \frac{1}{\mu([a, b])} \int_a^b G(x) d\mu(x) \subset G(x_\mu), \quad (10)$$

where $x_\mu = \frac{1}{\mu([a, b])} \int_a^b x d\mu(x)$ is the barycenter of μ on $[a, b]$.

Proof. By the mean value theorem $x_\mu \in [a, b]$. Let $X_1, X_2, X_3 : \Omega \rightarrow [a, b]$ be random variables with the distributions

$$\mu_{X_1} = \delta_{x_\mu}, \quad \mu_{X_2} = \frac{b - x_\mu}{b - a} \delta_a + \frac{x_\mu - a}{b - a} \delta_b, \quad \mu_{X_3} = \frac{1}{\mu([a, b])} \mu.$$

Then the pairs X_1, X_3 and X_3, X_2 satisfy the assumptions of Theorem 4. Moreover,

$$\int_\Omega G(X_1(\omega)) dP = G(x_\mu), \quad \int_\Omega G(X_2(\omega)) dP = \frac{b - x_\mu}{b - a} G(a) + \frac{x_\mu - a}{b - a} G(b)$$

and

$$\int_\Omega G(X_3(\omega)) dP = \frac{1}{\mu([a, b])} \int_a^b G(x) d\mu(x).$$

Therefore, by Theorem 4, we obtain (10). $\square$

The next two corollaries are counterparts for convex set-valued maps of the following inequalities concerning convex functions $f : I \rightarrow \mathbb{R}$ (cf. [5, 15]):

$$\frac{f(c) + f(d)}{2} - f\left(\frac{c + d}{2}\right) \leq \frac{f(a) + f(b)}{2} - f\left(\frac{a + b}{2}\right)$$

for all $a, b, c, d \in I$ such that $a < c < d < b$,

and the Popoviciu inequality

$$\frac{2}{3} \left[f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right) \right] \leq \frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x + y + z}{3}\right),$$

for all $x, y, z \in I$.

Corollary 11. *If $G : I \rightarrow cl(Y)$ is convex and integrable bounded, then*

$$\frac{G(a) + G(b)}{2} + G\left(\frac{c + d}{2}\right) \subset \frac{G(c) + G(d)}{2} + G\left(\frac{a + b}{2}\right), \quad (11)$$

for all $a, b, c, d \in I$ such that $a < c < d < b$.

Proof. Let $X_1, X_2 : \Omega \rightarrow I$ be random variables with the distributions

$$\mu_{X_1} = \frac{1}{4} (\delta_c + \delta_d) + \frac{1}{2} \delta_{(a+b)/2}, \quad \mu_{X_2} = \frac{1}{4} (\delta_a + \delta_b) + \frac{1}{2} \delta_{(c+d)/2}.$$

Then the pair X_1, X_2 satisfies the assumptions of Theorem 4. Moreover,

$$\begin{aligned}\int_{\Omega} G(X_1(\omega)) dP &= \frac{G(c) + G(d)}{4} + \frac{1}{2} G\left(\frac{a+b}{2}\right), \\ \int_{\Omega} G(X_2(\omega)) dP &= \frac{G(a) + G(b)}{4} + \frac{1}{2} G\left(\frac{c+d}{2}\right).\end{aligned}$$

Therefore, by Theorem 4, we obtain (11). $\square$

Corollary 12. *If $G : I \rightarrow \mathbb{R}$ is convex and integrable bounded, then*

$$\begin{aligned}\frac{G(x) + G(y) + G(z)}{3} + G\left(\frac{x+y+z}{3}\right) &\subset \frac{2}{3} \left[G\left(\frac{x+y}{2}\right) + G\left(\frac{y+z}{2}\right) \right. \\ &\quad \left. + G\left(\frac{z+x}{2}\right) \right]\end{aligned}$$

for all $x, y, z \in I$.

Proof. Let $X_1, X_2 : \Omega \rightarrow I$ be random variables with the distributions

$$\begin{aligned}\mu_{X_1} &= \frac{1}{3} (\delta_{(x+y)/2} + \delta_{(y+z)/2} + \delta_{(z+x)/2}), \\ \mu_{X_2} &= \frac{1}{6} (\delta_x + \delta_y + \delta_z) + \frac{1}{2} \delta_{(x+y+z)/3}.\end{aligned}$$

Then the pair X_1, X_2 satisfies the assumptions of Theorem 4. Moreover,

$$\begin{aligned}\int_{\Omega} G(X_1(\omega)) dP &= \frac{1}{3} \left[G\left(\frac{x+y}{2}\right) + G\left(\frac{y+z}{2}\right) + G\left(\frac{z+x}{2}\right) \right], \\ \int_{\Omega} G(X_2(\omega)) dP &= \frac{G(x) + G(y) + G(z)}{6} + \frac{1}{2} G\left(\frac{x+y+z}{3}\right).\end{aligned}$$

Therefore, by Theorem 4, the corollary is proved. $\square$

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