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Results in Mathematics

The Baire Category of Subsequences and Permutations which preserve Limit Points

Marek Balcerzak and Paolo Leonetti

Abstract. Let $\mathcal{I}$ be a meager ideal on $\mathbf{N}$. We show that if x is a sequence with values in a separable metric space then the set of subsequences [resp. permutations] of x which preserve the set of $\mathcal{I}$-cluster points of x is topologically large if and only if every ordinary limit point of x is also an $\mathcal{I}$-cluster point of x. The analogue statement fails for all maximal ideals. This extends the main results in [Topology Appl. **263** (2019), 221–229]. As an application, if x is a sequence with values in a first countable compact space which is $\mathcal{I}$-convergent to ℓ, then the set of subsequences [resp. permutations] which are $\mathcal{I}$-convergent to ℓ is topologically large if and only if x is convergent to ℓ in the ordinary sense. Analogous results hold for $\mathcal{I}$-limit points, provided $\mathcal{I}$ is an analytic P-ideal.

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1. Introduction

A classical result of Buck [7] states that, if x is real sequence, then "almost every" subsequence of x has the same set of ordinary limit points of the original sequence x, in a measure sense. The aim of this note is to prove its topological analogue and non-analogue in the context of ideal convergence.

Let $\mathcal{I}$ be an *ideal* on the positive integers $\mathbf{N}$, that is, a family a subsets of $\mathbf{N}$ closed under subsets and finite unions. Unless otherwise stated, it is also assumed that $\mathcal{I}$ contains the ideal Fin of finite sets and it is different from the power set $\mathcal{P}(\mathbf{N})$. $\mathcal{I}$ is a P-ideal if it is σ-directed modulo finite sets, i.e., for every sequence (A_n) of sets in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n. We regard ideals as subsets of the Cantor space $\{0,1\}^{\mathbf{N}}$, hence we may speak about their topological complexities. In particular, an ideal can be F_σ, analytic, etc. Among the most important ideals, we find the family of asymptotic density zero sets

$$\mathcal{Z} := \{A \subseteq \mathbf{N} : \lim_{n\to\infty} \tfrac{1}{n}|A \cap [1,n]| = 0\}.$$

We refer to [15] for a recent survey on ideals and associated filters.

Let $x = (x_n)$ be a sequence taking values in a topological space X, which will be always assumed to be Hausdorff. Then $\ell \in X$ is an $\mathcal{I}$-*cluster point* of x if

$$\{n \in \mathbf{N} : x_n \in U\} \notin \mathcal{I}$$

for each neighborhood U of ℓ. The set of $\mathcal{I}$-cluster points of x is denoted by $\Gamma_x(\mathcal{I})$. Moreover, $\ell \in X$ is an $\mathcal{I}$-*limit point* of x if there exists a subsequence (x_{n_k}) such that

$$\lim_{k\to\infty} x_{n_k} = \ell \quad \text{and} \quad \{n_k : k \in \mathbf{N}\} \notin \mathcal{I}.$$

The set of $\mathcal{I}$-limit points is denoted by $\Lambda_x(\mathcal{I})$. Statistical cluster points and statistical limits points (that is, $\mathcal{Z}$-cluster points and $\mathcal{Z}$-limit points) of real sequences were introduced by Fridy in [13] and studied by many authors, see e.g. [8,10,14,19,28,29]. It is worth noting that ideal cluster points have been studied much before under a different name. Indeed, as it follows by [26, Theorem 4.2], they correspond to classical "cluster points" of a filter $\mathscr{F}$ (depending on x) on the underlying space, cf. [6, Definition 2, p.69]. Let also $\mathrm{L}_x := \Gamma_x(\mathrm{Fin})$ be the set of accumulation points of x, and note that $\mathrm{L}_x = \Lambda_x(\mathrm{Fin})$ if X is first countable. Hence $\Lambda_x(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I}) \subseteq \mathrm{L}_x$. We refer the reader to [26] for characterizations of $\mathcal{I}$-cluster points and [2] for their relation with $\mathcal{I}$-limit points. Lastly, we recall that the sequence x is said to be $\mathcal{I}$-*convergent* to $\ell \in X$, shortened as $x \to_{\mathcal{I}} \ell$, if

$$\{n \in \mathbf{N} : x_n \notin U\} \in \mathcal{I}$$

for each neighborhood U of ℓ. Assuming that X is first countable, it follows by [26, Corollary 3.2] that, if $x \to_{\mathcal{I}} \ell$ then $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = \{\ell\}$, provided that $\mathcal{I}$ is a P-ideal. In addition, if X is also compact, then $x \to_{\mathcal{I}} \ell$ is in fact equivalent to $\Gamma_x(\mathcal{I}) = \{\ell\}$, even if $\mathcal{I}$ is not a P-ideal, cf. [26, Corollary 3.4].

Let Σ be the sets of strictly increasing functions on $\mathbf{N}$, that is,

$$\Sigma := \{\sigma \in \mathbf{N}^{\mathbf{N}} : \forall n \in \mathbf{N}, \sigma(n) < \sigma(n+1)\};$$

also, let Π be the sets of permutations of $\mathbf{N}$, that is,

$$\Pi := \{\pi \in \mathbf{N}^{\mathbf{N}} : \pi \text{ is a bijection}\}.$$

Note that both Σ and Π are G_δ-subsets of the Polish space $\mathbf{N}^{\mathbf{N}}$, hence they are Polish spaces as well by Alexandrov's theorem; in particular, they are not meager in themselves, cf. [35, Chapter 2]. Given a sequence x and $\sigma \in \Sigma$, we denote by $\sigma(x)$ the subsequence $(x_{\sigma(n)})$. Similarly, given $\pi \in \Pi$, we write $\pi(x)$ for the rearranged sequence $(x_{\pi(n)})$. We identify each subsequence of (x_{k_n}) of x with the function $\sigma \in \Sigma$ defined by $\sigma(n) = k_n$ for all $n \in \mathbf{N}$ and, similarly, each rearranged sequence $(x_{\pi(n)})$ with the permutation $\pi \in \Pi$, cf. [1,3,29].

We will show that if $\mathcal{I}$ is a meager ideal and x is a sequence with values in a separable metric space then the set of subsequences (and permutations) of x which preserve the set of $\mathcal{I}$-cluster points of x is not meager if and only if every ordinary limit point of x is also an $\mathcal{I}$-cluster point of x (Theorem 2.2). A similar result holds for $\mathcal{I}$-limit points, provided that $\mathcal{I}$ is an analytic P-ideal (Theorem 2.9). Putting all together, this strenghtens all the results contained in [25] and answers an open question therein. As a byproduct, we obtain a characterization of meager ideals (Proposition 3.1). Lastly, the analogue statements fails for all maximal ideals (Example 2.6).

2. Main Results

2.1. $\mathcal{I}$-Cluster Points.

It has been shown in [25] that, from a topological viewpoint, almost all subsequences of x preserve the set of $\mathcal{I}$-cluster points, provided that $\mathcal{I}$ is "well separated" from its dual filter $\mathcal{I}^\star := \{A \subseteq \mathbf{N} : A^c \in \mathcal{I}\}$; that is,

$$\Sigma_x(\mathcal{I}) := \left\{\sigma \in \Sigma : \Gamma_{\sigma(x)}(\mathcal{I}) = \Gamma_x(\mathcal{I})\right\}$$

is comeager, cf. also [27] for the case $\mathcal{I} = \mathcal{Z}$ and [22] for a measure theoretic analogue. We will extend this result to all meager ideals. In addition, we will see that the same holds also for

$$\Pi_x(\mathcal{I}) := \left\{\pi \in \Pi : \Gamma_{\pi(x)}(\mathcal{I}) = \Gamma_x(\mathcal{I})\right\}.$$

Here, given $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{P}(\mathbf{N})$, we say that $\mathcal{A}$ is separated from $\mathcal{C}$ by $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$. In particular, an ideal $\mathcal{I}$ is F_σ-separated from its dual filter $\mathcal{I}^\star$ if there exists an F_σ-set $\mathcal{B} \subseteq \mathcal{P}(\mathbf{N})$ such that $\mathcal{I} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{I}^\star = \emptyset$ (with the language of [9,20], the filter $\mathcal{I}^\star$ has rank ≤ 1).

Theorem 2.1. *[25, Theorem 2.1] Let x be a sequence in a first countable space X such that all closed sets are separable and let $\mathcal{I}$ be an ideal which is F_σ-separated from its dual filter $\mathcal{I}^\star$. Then $\Sigma_x(\mathcal{I})$ is not meager if and only if $\Gamma_x(\mathcal{I}) = \mathrm{L}_x$. Moreover, in this case, it is comeager.*

As it has been shown in [34, Corollary 1.5], the family of ideals $\mathcal{I}$ which are F_σ-separated from $\mathcal{I}^\star$ includes all $F_{\sigma\delta}$-ideals. In addition, a Borel ideal is F_σ-separated from its dual filter if and only if it does not contain an isomorphic copy of $\mathrm{Fin} \times \mathrm{Fin}$ (which can be represented as an ideal on $\mathbf{N}$ as

$$\{A \subseteq \mathbf{N} : \forall^\infty n \in \mathbf{N}, \{a \in A : \nu_2(a) = n\} \in \mathrm{Fin}\}$$

where $\nu_2(n)$ stands for the 2-adic valuation of n), see [21, Theorem 4]. In particular, $\mathrm{Fin} \times \mathrm{Fin}$ is a $F_{\sigma\delta\sigma}$-ideal which is not F_σ-separated from its dual filter. For related results on F_σ-separation, see [11, Proposition 3.6] and [38].

We show that the analogue of Theorem 2.1 holds for all meager ideals. Hence, this includes new cases as, for instance, $\mathcal{I} = \mathrm{Fin} \times \mathrm{Fin}$.

It is worth noting that every meager ideal $\mathcal{I}$ is F_σ-separated from the Fréchet filter $\mathrm{Fin}^\star$ (see Proposition 3.1 below), hence $\mathcal{I}$ is F_σ-separated from $\mathcal{I}^\star$. This implies that our result is a proper generalization of Theorem 2.1:

Theorem 2.2. *Let x be a sequence in a first countable space X such that all closed sets are separable and let $\mathcal{I}$ be a meager ideal. Then the following are equivalent:*

(c1) $\Sigma_x(\mathcal{I})$ *is comeager in* Σ;
(c2) $\Sigma_x(\mathcal{I})$ *is not meager in* Σ;
(c3) $\Pi_x(\mathcal{I})$ *is comeager in* Π;
(c4) $\Pi_x(\mathcal{I})$ *is not meager in* Π;
(c5) $\Gamma_x(\mathcal{I}) = \mathrm{L}_x$.

It is worth to spend some comments on the class $\mathscr{C}$ of first countable spaces such that all closed sets are separable. It is clear that separable metric spaces belong to X. However, $\mathscr{C}$ contains also nonmetrizable spaces:

Example 2.3. It has been shown by Ostaszewski in [31] that there exists a topological space X which is hereditarily separable (i.e., all subsets of X are separable), first countable, countably compact, and not compact, cf. also [17]. In particular, $X \in \mathscr{C}$. However, X is not second countable (indeed, in the opposite, the notions of compactness and countably compactness would coincide). Considering that all separable metric spaces are second countable, it follows that X is not metrizable.

Moreover, there exists a separable first countable space outside $\mathscr{C}$:

Example 2.4. Let X be the Sorgenfrey plane, that is, the product of two copies of the real line $\mathbf{R}$ endowed with the lower limit topology. It is known that X is first countable and separable. Moreover, the anti-diagonal $\{(x, -x) : x \in \mathbf{R}\}$ is a closed uncountable discrete subspace of X, hence not separable, cf. [36, pp. 103–105]. Therefore $X \notin \mathscr{C}$. A similar example is the Moore plane with the tangent disk topology, cf. [36, p. 176].

We can also show that there exists a first countable space outside $\mathscr{C}$ which satisfies the statement of Theorem 2.2:

Example 2.5. Let X be an uncountable set, endowed the discrete topology. Then X is nonseparable first countable space, so that $X \notin \mathscr{C}$. Thanks to Remark 3.4 below, Theorem 2.2 holds if the separability of all closed sets is replaced by the condition that L_x is countable *for each* sequence x taking values in X. Indeed, the latter is verified because $\mathrm{L}_x \subseteq \{x_n : n \in \mathbf{N}\}$.

We leave as open question whether there exists a topological space X (necessarily outside $\mathscr{C}$) and a meager ideal $\mathcal{I}$ for which Theorem 2.2 fails. It is well possible that our main result extends beyond $\mathscr{C}$, as it happened very recently with related results, see e.g. the improvement of [23, Theorem 4.2] in [18, Lemma 2.2].

Lastly, one may ask whether Theorem 2.2 holds for all ideals. We show in the following example that the answer is negative:

Example 2.6. Let $\mathcal{I}$ be a maximal ideal. Hence there exists a unique $A \in \{2\mathbf{N}+1, 2\mathbf{N}+2\}$ such that $A \in \mathcal{I}$. Set $X = \mathbf{R}$. Let x be the characteristic function of A, i.e., $x_n = 1$ if $n \in A$ and $x_n = 0$ otherwise. Then $x \to_{\mathcal{I}} 0$. In particular, $\Gamma_x(\mathcal{I}) = \{0\}$. Note that a subsequence $\sigma(x)$ is $\mathcal{I}$-convergent to 0 if and only if $\Gamma_{\sigma(x)} = \{0\}$. Then

$$\Sigma_x(\mathcal{I}) = \{\sigma \in \Sigma : \sigma^{-1}[A] \in \mathcal{I}\}.$$

Considering that $\sigma^{-1}[A] \cup \sigma^{-1}[A-1]$ is cofinite, we have either $\sigma^{-1}[A] \in \mathcal{I}$ or $\sigma^{-1}[A-1] \in \mathcal{I}$. Let $T : \Sigma \to \Sigma$ be the embedding defined by $\sigma \mapsto \sigma + 1$, so that Σ is homeomorphic to the open set $T[\Sigma] = \{\sigma \in \Sigma : \sigma(1) \geq 2\}$. Notice that

$$\begin{aligned} T[\Sigma_x(\mathcal{I})] &= \{T(\sigma) : \sigma \in \Sigma_x(\mathcal{I})\} = \{\sigma + 1 \in \Sigma : \sigma^{-1}[A] \in \mathcal{I}\} \\ &= \{\sigma \in \Sigma : \sigma^{-1}[A-1] \in \mathcal{I}\} \cap T[\Sigma], \end{aligned}$$

which implies that the open set $T[\Sigma]$ is contained in $\Sigma_x(\mathcal{I}) \cup T[\Sigma_x(\mathcal{I})]$. Therefore both $\Sigma_x(\mathcal{I})$ and $T[\Sigma_x(\mathcal{I})]$ are not meager.

A similar example can be found for $\Pi_x(\mathcal{I})$, replacing the embedding T with the homeomorphism $H : \Pi \to \Pi$ defined by $H(\pi)(2n) = 2n-1$ and $H(\pi)(2n-1) = 2n$ for all $n \in \mathbf{N}$.

As noted by the referee, the equivalences (c1) $\iff$ (c2) and (c3) $\iff$ (c4), and their analogues in the next results, can be viewed a consequence a general topological 0-1 law stating that every tail set with the property of Baire is either meager or comeager, see [32, Theorem 21.4]. However, it seems rather difficult to show that the tail sets $\Sigma_x(\mathcal{I})$ and $\Pi_x(\mathcal{I})$ have the property of Baire, provided that $\mathcal{I}$ is meager (note that this is surely false if $\mathcal{I}$ is a maximal ideal, as it follows by Example 2.6).

As an application of our results, if x is $\mathcal{I}$-convergent to ℓ, then the set of subsequences [resp., rearrangements] of x which are $\mathcal{I}$-convergent to ℓ is not meager if and only if x is convergent (in the classical sense) to ℓ. This is somehow related to [1, Theorem 2.1] and [3, Theorem 1.1]; cf. also [28, Theorem 3] for a measure theoretical non-analogue.

Corollary 2.7. *Let x be a sequence in a first countable compact space X. Let $\mathcal{I}$ be a meager ideal and assume that x is $\mathcal{I}$-convergent to $\ell \in X$. Then the following are equivalent:*

(i1) $\{\sigma \in \Sigma : \sigma(x) \to_{\mathcal{I}} \ell\}$ *is comeager in* Σ;

(I2) $\{\sigma \in \Sigma : \sigma(x) \to_{\mathcal{I}} \ell\}$ *is not meager in* Π;
(I3) $\{\pi \in \Pi : \pi(x) \to_{\mathcal{I}} \ell\}$ *is comeager in* Σ;
(I4) $\{\pi \in \Pi : \pi(x) \to_{\mathcal{I}} \ell\}$ *is not meager in* Π;
(I5) $\lim_n x_n = \ell$.

The proofs of Theorem 2.2 and Corollary 2.7 follow in Sect. 3.

2.2. $\mathcal{I}$-Limit Points

Given a sequence x and an ideal $\mathcal{I}$, define

$$\tilde{\Sigma}_x(\mathcal{I}) := \{\sigma \in \Sigma : \Lambda_{\sigma(x)}(\mathcal{I}) = \Lambda_x(\mathcal{I})\}$$

and its analogue for permutations

$$\tilde{\Pi}_x(\mathcal{I}) := \{\pi \in \Pi : \Lambda_{\pi(x)}(\mathcal{I}) = \Lambda_x(\mathcal{I})\}.$$

It has been shown in [25] that, in the case of $\mathcal{I}$-limit points, the counterpart of Theorem 2.1 holds for generalized density ideals. Here, an ideal $\mathcal{I}$ is said to be a *generalized density ideal* if there exists a sequence (μ_n) of submeasures with finite and pairwise disjoint supports such that $\mathcal{I} = \{A \subseteq \mathbf{N} : \lim_n \mu_n(A) = 0\}$. More precisely:

Theorem 2.8. *[25, Theorem 2.3] Let x be a sequence in a first countable space X such that all closed sets are separable and let $\mathcal{I}$ be generalized density ideal. Then $\tilde{\Sigma}_x(\mathcal{I})$ is not meager if and only if $\Lambda_x(\mathcal{I}) = \mathrm{L}_x$. Moreover, in this case, it is comeager.*

See [24] for a measure theoretic analogue. It has been left as open question to check, in particular, whether the same statement holds for analytic P-ideals. We show that the answer is affirmative.

Note that this is strict generalization, as every generalized density ideal is an analytic P-ideal and there exists an analytic P-ideal which is not a generalized density ideal, see e.g. [5]. In addition, the same result holds for permutations.

Theorem 2.9. *Let x be a sequence in a first countable space X such that all closed sets are separable and let $\mathcal{I}$ be an analytic P-ideal. Then the following are equivalent:*

(L1) $\tilde{\Sigma}_x(\mathcal{I})$ *is comeager in* Σ;
(L2) $\tilde{\Sigma}_x(\mathcal{I})$ *is not meager in* Σ;
(L3) $\tilde{\Pi}_x(\mathcal{I})$ *is comeager in* Π;
(L4) $\tilde{\Pi}_x(\mathcal{I})$ *is not meager in* Π;
(L5) $\Gamma_x(\mathcal{I}) = \mathrm{L}_x$.

Note that the same Example 2.6 shows that the analogue of Theorem 2.9 fails for all maximal ideals. The proof of Theorem 2.9 follows in Sect. 4.

We leave as an open question to check whether Theorem 2.9 may be extended to all meager ideals.

3. Proofs for $\mathcal{I}$-Cluster Points

We start with a characterization of meager ideals (to the best of our knowledge, condition (M3) is novel). Here, a set $\mathcal{A} \subseteq \mathcal{P}(\mathbf{N})$ is called *hereditary* if it is closed under subsets.

Proposition 3.1. *Let $\mathcal{I}$ be an ideal on $\mathbf{N}$. Then the following are equivalent:*

(M1) *$\mathcal{I}$ is meager;*

(M2) *There exists a strictly increasing sequence (ι_n) of positive integers such that $A \notin \mathcal{I}$ whenever $\mathbf{N} \cap [\iota_n, \iota_{n+1}) \subseteq A$ for infinitely many $n \in \mathbf{N}$;*

(M3) *$\mathcal{I}$ is F_σ-separated from the Fréchet filter* $\mathrm{Fin}^\star$.

Proof. (M1) $\iff$ (M2) See [37, Theorem 2.1]; cf. also [4, Theorem 4.1.2].

(M2) $\implies$ (M3) Define $I_n := \mathbf{N} \cap [\iota_n, \iota_{n+1})$ for all $n \in \mathbf{N}$. Then $\mathcal{I} \subseteq F$, where $F := \bigcup_k F_k$ and

$$\forall k \in \mathbf{N}, \quad F_k := \bigcap_{n \geq k} \{A \subseteq \mathbf{N} : I_n \not\subseteq A\}. \tag{1}$$

Note that each F_k is closed and it does not contain any cofinite set. Therefore $\mathcal{I}$ is separated from $\mathrm{Fin}^\star$ by the F_σ-set F.

(M3) $\implies$ (M1) Suppose that there exists a sequence (F_k) of closed sets in $\{0,1\}^{\mathbf{N}}$ such that $\mathcal{I} \subseteq F := \bigcup_k F_k$ and $F \cap \mathrm{Fin}^\star = \emptyset$. Then each F_k has empty interior (otherwise it would contain a cofinite set). We conclude that $\mathcal{I}$ is contained in a countable union of nowhere dense sets. □

It is clear that condition (M3) is weaker than the extendability of $\mathcal{I}$ to a F_σ-ideal. For characterizations and related results of the latter property, see e.g. [16, Theorem 4.4] and [12, Theorem 3.3].

Lemma 3.2. *Let $\mathcal{I}$ be a meager ideal. Then*

$$\{\sigma \in \Sigma : \sigma^{-1}[A] \notin \mathcal{I}\} \quad \textit{and} \quad \{\pi \in \Pi : \pi^{-1}[A] \notin \mathcal{I}\}$$

are comeager for each infinite set $A \subseteq \mathbf{N}$.

Proof. Fix an infinite set $A \subseteq \mathbf{N}$. As in the proof of Proposition 3.1, we can define intervals $I_n := \mathbf{N} \cap [\iota_n, \iota_{n+1})$ for all $n \in \mathbf{N}$ such that a set $S \subseteq \mathbf{N}$ does not belong to $\mathcal{I}$ whenever S contains infinitely many intervals I_ns. At this point, for each $n, k \in \mathbf{N}$, define the sets

$$X_k := \{\sigma \in \Sigma : I_k \subseteq \sigma^{-1}[A]\} \quad \text{and} \quad Y_n := \textstyle\bigcup_{k \geq n} X_k.$$

Note that each X_k is open and that each Y_n is open and dense. Therefore the set $\bigcap_n Y_n$, which can be rewritten as $\{\sigma \in \Sigma : \sigma^{-1}[A] \notin \mathcal{I}\}$, is comeager. The proof that $\{\pi \in \Pi : \pi^{-1}[A] \notin \mathcal{I}\}$ is comeager is analogous. □

Lemma 3.3. *Let x be a sequence in a first countable space X and let $\mathcal{I}$ be a meager ideal. Then*

$$S(\ell) := \left\{\sigma \in \Sigma : \ell \in \Gamma_{\sigma(x)}(\mathcal{I})\right\} \quad \textit{and} \quad P(\ell) := \left\{\pi \in \Pi : \ell \in \Gamma_{\pi(x)}(\mathcal{I})\right\}$$

are comeager for each $\ell \in \mathrm{L}_x$.

Proof. Assume that $\mathrm{L}_x \neq \emptyset$, otherwise there is nothing to prove. Fix $\ell \in \mathrm{L}_x$ and let (U_m) be a decreasing local base at ℓ. For each $m \in \mathbf{N}$, define the infinite set $A_m := \{n \in \mathbf{N} : x_n \in U_m\}$. Thanks to Lemma 3.2, the set $B_m := \{\sigma \in \Sigma : \sigma^{-1}[A_m] \notin \mathcal{I}\}$ is comeager. Since $S(\ell)$ can be rewritten as $\bigcap_m B_m$, it follows that $S(\ell)$ is comeager. The proof that $P(\ell)$ is comeager is analogous. □

We are finally ready to prove Theorem 2.2.

Proof of Theorem 2.2. (c1) $\Longrightarrow$ (c2) It is obvious.

(c2) $\Longrightarrow$ (c5) Suppose that there exists $\ell \in \mathrm{L}_x \setminus \Gamma_x(\mathcal{I})$. Then $\Sigma_x(\mathcal{I})$ is contained in $\Sigma \setminus S(\ell)$, which is meager by Lemma 3.3.

(c5) $\Longrightarrow$ (c1) Suppose that $\mathrm{L}_x \neq \emptyset$, otherwise the claim is trivial. Let $\mathscr{L}$ be a countable dense subset of L_x, so that $\mathscr{L} \subseteq \Gamma_{\sigma(x)}(\mathcal{I})$ for each $\sigma \in S := \bigcap_{\ell \in \mathscr{L}} S(\ell)$, which is comeager by Lemma 3.3. Fix $\sigma \in S$. On the one hand, $\Gamma_{\sigma(x)}(\mathcal{I}) \subseteq \mathrm{L}_{\sigma(x)} \subseteq \mathrm{L}_x$. On the other hand, since $\Gamma_{\sigma(x)}(\mathcal{I})$ is closed by [26, Lemma 3.1(iv)], we get $\mathrm{L}_x \subseteq \Gamma_{\sigma(x)}(\mathcal{I})$. Therefore $S \subseteq \Sigma_x(\mathcal{I})$.

The implications (c3) $\Longrightarrow$ (c4) $\Longrightarrow$ (c5) $\Longrightarrow$ (c3) are analogous. □

Remark 3.4. As it is evident from the proof above, the hypothesis that "closed sets of X are separable" can be removed if, in addition, L_x is countable.

Lemma 3.5. *Let $\mathcal{I}$ be an ideal and x be a sequence in a first countable compact space. Then $x \to_{\mathcal{I}} \ell$ if and only if $\Gamma_x(\mathcal{I}) = \{\ell\}$.*

Proof. It follows by [26, Corollary 3.4]. □

Proof of Corollary 2.7. (i1) $\Longrightarrow$ (i2) It is obvious.

(i2) $\Longrightarrow$ (i5) By Lemma 3.5, the hypothesis can be rewritten as $\Gamma_x(\mathcal{I}) = \{\ell\}$. Hence, condition (i2) is equivalent to the nonmeagerness of $\{\sigma \in \Sigma : \Gamma_{\sigma(x)} = \Gamma_x(\mathcal{I}) = \{\ell\}\}$. The claim follows by Theorem 2.2 and Remark 3.4.

(i5) $\Longrightarrow$ (i1) If $x \to \ell$ then $\sigma(x) \to_{\mathcal{I}} \ell$ for all $\sigma \in \Sigma$.

The implications (i3) $\Longrightarrow$ (i4) $\Longrightarrow$ (i5) $\Longrightarrow$ (i3) are analogous. □

4. Proofs for $\mathcal{I}$-Limit Points

A lower semicontinuous submeasure (in short, lscsm) is a monotone subadditive function $\varphi : \mathcal{P}(\mathbf{N}) \to [0, \infty]$ such that $\varphi(\emptyset) = 0$, $\varphi(F) < \infty$ for all $F \in \mathrm{Fin}$, and $\varphi(A) = \lim_n \varphi(A \cap [1, n])$ for all $A \subseteq \mathbf{N}$. By a classical result of Solecki, an ideal $\mathcal{I}$ is an analytic P-ideal if and only if there exists a lscsm φ such that

$$\mathcal{I} = \mathrm{Exh}(\varphi) := \{A \subseteq \mathbf{N} : \|A\|_\varphi = 0\} \quad \text{and} \quad 0 < \|\mathbf{N}\|_\varphi \leq \varphi(\mathbf{N}) < \infty, \qquad (2)$$

where $\|A\|_\varphi := \lim_n \varphi(A \setminus [1, n])$ for all $A \subseteq \mathbf{N}$, see [33, Theorem 3.1]. Note that $\|\cdot\|_\varphi$ is a submeasure which is invariant modulo finite sets. Moreover, replacing φ with $\varphi/\|\mathbf{N}\|_\varphi$ in (2), we can assume without loss of generality that $\|\mathbf{N}\|_\varphi = 1$.

Given a sequence x in a first countable topological space X and an analytic P-ideal $\mathcal{I} = \mathrm{Exh}(\varphi)$, we define the function

$$\mathfrak{u} : \Sigma \times X \to \mathbf{R} : (\sigma, \ell) \mapsto \lim_{k\to\infty} \|\{n \in \mathbf{N} : x_{\sigma(n)} \in U_k\}\|_{\varphi}, \tag{3}$$

where (U_k) is a decreasing local base of neighborhoods at $\ell \in X$. Clearly, the limit in (3) exists and it is independent of the choice of (U_k).

Lemma 4.1. *Let x be a sequence in a first countable space X and let $\mathcal{I} = \mathrm{Exh}(\varphi)$ be an analytic P-ideal. Then, the section $\mathfrak{u}(\sigma, \cdot)$ is upper semicontinuous for each $\sigma \in \Sigma$. In particular, the set*

$$\Lambda_{\sigma(x)}(\mathcal{I}, q) := \{\ell \in X : \mathfrak{u}(\sigma, \ell) \geq q\}$$

is closed for all $q > 0$.

Proof. See [2, Lemma 2.1]. □

Lemma 4.2. *With the same hypotheses of Lemma 4.1, the set*

$$V(\ell, q) := \{\sigma \in \Sigma : \mathfrak{u}(\sigma, \ell) > q\}$$

is either comeager or empty for each $\ell \in X$ and $q \in (0, 1)$.

Proof. Suppose that $V(\ell, q) \neq \emptyset$, so that $\ell \in \mathrm{L}_x$, and note that

$$\begin{aligned}\Sigma \setminus V(\ell, q) &= \bigcup_{k\geq 1} \{\sigma \in \Sigma : \|\{n \in \mathbf{N} : x_{\sigma(n)} \in U_k\}\|_{\varphi} \leq q\} \\ &= \bigcup_{k\geq 1} \{\sigma \in \Sigma : \limsup_{t\to\infty} \varphi(\{n \geq t : x_{\sigma(n)} \in U_k\}) \leq q\} \\ &= \bigcup_{k\geq 1} \bigcup_{s\geq 1} \bigcap_{t\geq s} \{\sigma \in \Sigma : \varphi(\{n \geq t : x_{\sigma(n)} \in U_k\}) \leq q\}.\end{aligned}$$

Then, it is sufficient to show that

$$W_{k,s} := \bigcap_{t\geq s} \{\sigma \in \Sigma : \varphi(\{n \geq t : x_{\sigma(n)} \in U_k\}) \leq q\}$$

is nowhere dense for all $k, s \in \mathbf{N}$.

To this aim, for every nonempty open set $Z \subseteq \Sigma$, we need to prove that there exists a nonempty open subset $S \subseteq Z$ such that $S \cap W_{k,s} = \emptyset$. Fix a nonempty open set $Z \subseteq \Sigma$ and $\sigma_0 \in Z$ so that there exists $n_0 \in \mathbf{N}$ for which

$$Z' := \{\sigma \in \Sigma : \sigma \upharpoonright \{1, \ldots, n_0\} = \sigma_0 \upharpoonright \{1, \ldots, n_0\}\} \subseteq Z.$$

Since $\ell \in \mathrm{L}_x$, there exists $\sigma_1 \in Z'$ such that $\lim_n x_{\sigma_1(n)} = \ell$. Therefore

$$\begin{aligned}\varphi(\{n \geq n_1 : x_{\sigma_1(n)} \in U_k\}) &\geq \|\{n \geq n_1 : x_{\sigma_1(n)} \in U_k\}\|_{\varphi} \\ &= \|\{n \in \mathbf{N} : x_{\sigma_1(n)} \in U_k\}\|_{\varphi} = \mathfrak{u}(\sigma_1, \ell) = 1,\end{aligned}$$

where $n_1 := \max\{n_0 + 1, s\}$. At this point, since φ is a lscsm, it follows that there exists an integer $n_2 > n_1$ such that $\varphi(\{n \in \mathbf{N} \cap [n_1, n_2] : x_{\sigma_1(n)} \in$

$U_k\}) > q$. Therefore $S := \{\sigma \in Z' : \sigma \restriction \{n_1, \ldots, n_2\} = \sigma_1 \restriction \{n_1, \ldots, n_2\}\}$ is a nonempty open set contained in Z and disjoint from $W_{k,s}$. Indeed

$$\forall \sigma \in S, \quad \varphi(\{n \geq s : x_{\sigma(n)} \in U_k\}) \geq \varphi(\{n \in \mathbf{N} \cap [n_1, n_2] : x_{\sigma(n)} \in U_k\}) > q$$

by the monotonicity of φ. □

Lemma 4.3. *With the same hypotheses of Lemma 4.1, we have*

$$\forall \ell \in X, \quad \{\sigma \in \Sigma : \ell \in \Lambda_{\sigma(x)}(\mathcal{I})\} = \bigcup\nolimits_{q>0} V(\ell, q).$$

In addition, $\tilde{S}(\ell, q) := \{\sigma \in \Sigma : \ell \in \Lambda_{\sigma(x)}(\mathcal{I}, q)\}$ *contains* $V(\ell, q)$.

Proof. Fix $\ell \in X$ and $\sigma \in \tilde{S}(\ell)$, where

$$\tilde{S}(\ell) := \{\sigma \in \Sigma : \ell \in \Lambda_{\sigma(x)}(\mathcal{I})\}.$$

Then there exist $\tau \in \Sigma$ and $q > 0$ such that $\lim_n x_{\tau(\sigma(n))} = \ell$ and $\|\tau(\mathbf{N})\|_\varphi \geq 2q$. In particular, for each $k \in \mathbf{N}$ we have $x_{\tau(\sigma(n))} \in U_k$ for all large $n \in \mathbf{N}$. Hence

$$\|\{n \in \mathbf{N} : x_{\sigma(n)} \in U_k\}\|_\varphi \geq \|\{n \in \mathbf{N} : x_{\sigma(n)} \in U_k\} \cap \tau(\mathbf{N})\|_\varphi = \|\tau(\mathbf{N})\|_\varphi \geq 2q.$$

By the arbitrariness of k, it follows that $\mathfrak{u}(\sigma, \ell) \geq 2q > q$, that is, $\sigma \in V(\ell, q)$.

Conversely, fix $\ell \in X$, $\sigma \in \Sigma$, and $q > 0$ such that $\sigma \in V(\ell, q)$, hence $\|A_k\|_\varphi > q$ for all $k \in \mathbf{N}$, where $A_k := \{n \in \mathbf{N} : x_{\sigma(n)} \in U_k\}$. Let us define recursively a sequence (F_k) of finite subsets of $\mathbf{N}$ as it follows. Pick $F_1 \subseteq A_1$ such that $\varphi(F_1) \geq q$ (which is possibile since φ is a lscsm); then, for each integer $k \geq 2$, let F_k be a finite subset of A_k such that $\min F_k > \max F_{k-1}$ and $\varphi(F_k) \geq q$ (which is possible since $\|A_k \setminus [1, \max F_{k-1}]\|_\varphi = \|A_k\|_\varphi > q$). Let (y_n) be the increasing enumeration of the set $\bigcup_k F_k$, and define $\tau \in \Sigma$ such that $\tau(n) = y_n$ for all n. It follows by construction that

$$\lim_{n \to \infty} x_{\tau(\sigma(n))} = \ell \quad \text{and} \quad \|\tau(\mathbf{N})\|_\varphi \geq \liminf_{k \to \infty} \varphi(F_k) \geq q > 0.$$

Therefore $\ell \in \Lambda_{\sigma(x)}(\mathcal{I}, q) \subseteq \Lambda_{\sigma(x)}(\mathcal{I})$, which concludes the proof. □

Corollary 4.4. *With the same hypotheses of Lemma 4.1,* $\tilde{S}(\ell, q)$ *is comeager for each* $\ell \in \mathrm{L}_x$ *and* $q \in (0, 1)$.

Proof. Fix $\ell \in \mathrm{L}_x$ and $q \in (0, 1)$. Then $\tilde{S}(\ell, q)$ contains $V(\ell, q)$ by Lemma 4.3, which is comeager by Lemma 4.2. □

Corollary 4.5. *With the same hypotheses of Lemma 4.1,* $\tilde{S}(\ell)$ *is comeager for each* $\ell \in \mathrm{L}_x$.

Proof. Thanks to [2, Theorem 2.2], we have

$$\Lambda_{\sigma(x)}(\mathcal{I}) = \bigcup\nolimits_{q>0} \Lambda_{\sigma(x)}(\mathcal{I}, q). \tag{4}$$

Therefore $\tilde{S}(\ell)$ contains $\tilde{S}(\ell, \frac{1}{2})$, which is comeager by Corollary 4.4. □

Remark 4.6. All the analogues from Lemma 4.1 up to Corollary 4.5 hold for permutations, the only difference being in the last part of the proof of Lemma 4.2: let us show that

$$\widehat{W}_{k,s} := \bigcup_{t \geq s} \{\pi \in \Pi : \varphi(\{n \geq t : x_{\pi(n)} \in U_k\}) \leq q\}$$

is nowhere dense for all $k, s \in \mathbf{N}$. To this aim, fix $\pi_0 \in \Pi$ and $n_0 \in \mathbf{N}$ which defines the nonempty open set $G := \{\pi \in \Pi : \pi \upharpoonright \{1, \ldots, n_0\} = \pi_0 \upharpoonright \{1, \ldots, n_0\}\}$. Set $n_1 := \max\{n_0 + 1, s\}$ and let (y_n) be the increasing enumeration of the infinite set $\{n \in \mathbf{N} : x_n \in U_k\} \setminus \{\pi_0(1), \ldots, \pi_0(n_0)\}$. Since φ is a lscsm, there exists $n_2 \in \mathbf{N}$ such that $\varphi(\{s, s+1, \ldots, n_2\}) > q$. Lastly, let G' be the set of all $\pi \in G$ such that $\pi(n) = y_n$ for all $n \in \{s, s+1, \ldots, n_2\}$. We conclude that

$$\forall \pi \in G', \quad \varphi(\{n \geq s : x_{\pi(n)} \in U_k\}) \geq \varphi(\{s, s+1, \ldots, n_2\}) > q.$$

Therefore G' is a nonempty open subset of G which is disjoint from $\widehat{W}_{k,s}$.

Lastly, we prove Theorem 2.9.

Proof of Theorem 2.9. The implications (L1) $\Longrightarrow$ (L2) $\Longrightarrow$ (L5) are analogous to the ones in Theorem 2.2, replacing Lemma 3.3 with Corollary 4.5.

(L5) $\Longrightarrow$ (L1) Suppose that $\mathrm{L}_x \neq \emptyset$, otherwise the claim is trivial. Let $\mathscr{L}$ be a countable dense subset of L_x, so that $\mathscr{L} \subseteq \Lambda_{\sigma(x)}(\mathcal{I}, \frac{1}{2})$ for each $\sigma \in \tilde{S} := \bigcap_{\ell \in \mathscr{L}} \tilde{S}(\ell, \frac{1}{2})$, which is comeager by Corollary 4.4. Fix $\sigma \in \tilde{S}$. On the one hand, taking into account (4), we get $\Lambda_{\sigma(x)}(\mathcal{I}, \frac{1}{2}) \subseteq \Lambda_{\sigma(x)}(\mathcal{I}) \subseteq \mathrm{L}_{\sigma(x)} \subseteq \mathrm{L}_x$. On the other hand, since $\Lambda_{\sigma(x)}(\mathcal{I}, \frac{1}{2})$ is closed by Lemma 4.1, we obtain $\mathrm{L}_x \subseteq \Lambda_{\sigma(x)}(\mathcal{I})$. Therefore $\tilde{\Sigma}_x(\mathcal{I})$ contains the comeager set $\tilde{S}$.

The implications (L3) $\Longrightarrow$ (L4) $\Longrightarrow$ (L5) $\Longrightarrow$ (L3) are analogous, taking into account Remark 4.6. □

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Note added in proof

We realized that our main results (i.e., Theorem 2.2 and Theorem 2.9) have been also recently proved also in [30] for the case of subsequences of real sequences, also answering (in this case) the open question at the end of Section 2.

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Marek Balcerzak
Institute of Mathematics
Lodz University of Technology
ul. Wólczańska 215
93-005 Lodz
Poland
e-mail: marek.balcerzak@p.lodz.pl

Paolo Leonetti
Department of Decision Sciences
Università "Luigi Bocconi"
via Roentgen 1
20136 Milan
Italy
e-mail: leonetti.paolo@gmail.com
URL: https://sites.google.com/site/leonettipaolo/

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