Compressions of Multiplication Operators and Their Characterizations

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Abstract. Dual truncated Toeplitz operators and other restrictions of the multiplication by the independent variable M_z on the classical L^2 space on the unit circle are investigated. Commutators are calculated and commutativity is characterized. A necessary and sufficient condition for any operator to be a dual truncated Toeplitz operator is established. A formula for recovering its symbol is stated.

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1. Introduction

Let $L^2 := L^2(\partial\mathbb{D})$ be the space of all measurable and square integrable functions on the unit circle $\mathbb{T}$ with respect to the normalized Lebesgue's measure. Recall that for $\varphi \in L^\infty$ the multiplication operator M_φ is defined by $M_\varphi f = \varphi f$, $f \in L^2$. Let H^2 denote the classical Hardy space and $H_-^2 = L^2 \ominus H^2$, let P^+ stand for the orthogonal projection from L^2 onto H^2 and let $P^- = I_{L^2} - P^+$ be the orthogonal projection from L^2 onto H_-^2 . For $\varphi \in L^\infty$ recall the standard definitions

$$T_\varphi = P^+ M_\varphi|_{H^2}, \quad H_\varphi = P^- M_\varphi|_{H^2}.$$

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The operators: $T_\varphi \in \mathcal{B}(H^2)$ and $H_\varphi \in \mathcal{B}(H^2, H_-^2)$ are called *Toeplitz operator* and *Hankel operator* (with symbol φ), respectively. We will use the notation $\mathcal{B}(\mathcal{H})$ or $\mathcal{B}(\mathcal{H}, \mathcal{H})$ for the set of all bounded operators on the Hilbert space $\mathcal{H}$ or from $\mathcal{H}$ to $\mathcal{H}$.

In what follows let θ denote a nonconstant inner function. Recall that the *model space* K_θ is defined as the orthogonal complement of θH^2 in H^2 . One can therefore consider the decompositions

$$H^2 = K_\theta \oplus \theta H^2 \quad \text{and} \quad L^2 = K_\theta \oplus (K_\theta)^\perp = K_\theta \oplus \theta H^2 \oplus H_-^2.$$

According to the decomposition $L^2 = K_\theta \oplus (K_\theta)^\perp$ the operator M_φ , $\varphi \in L^\infty$, can be written as

$$M_\varphi = \begin{bmatrix} A_\varphi^\theta & (B_\varphi^\theta)^* \\ B_\varphi^\theta & D_\varphi^\theta \end{bmatrix}. \quad (1.1)$$

Although M_φ is bounded if and only if $\varphi \in L^\infty$, some restrictions of M_φ may still be bounded even if they do not have a bounded symbol (for example, for A_φ^θ see [1] and for B_φ^θ see Remark 4). Hence, now let $\varphi \in L^2$ and set the densely defined multiplication operator $M_\varphi : D(M_\varphi) \rightarrow L^2$ as $M_\varphi f = \varphi f$, where $D(M_\varphi) = \{f \in L^2 : \varphi f \in L^2\}$. Note that $L^\infty \subset D(M_\varphi)$ for all $\varphi \in L^2$. Let P_θ denote the orthogonal projection from L^2 onto K_θ and let $P_\theta^\perp = I_{L^2} - P_\theta$ be the orthogonal projection from L^2 onto $(K_\theta)^\perp$. Recall after [11] that $K_\theta^\infty := K_\theta \cap L^\infty$ is a dense subset of K_θ . Since $\bar{z}H^\infty$ is a dense subset of H_-^2 and θH^∞ is a dense subset of θH^2 , it follows that $K_\theta^\perp \cap L^\infty$ is a dense subset of $K_\theta^\perp$. We define

$$A_\varphi^\theta = P_\theta M_\varphi|_{K_\theta \cap L^\infty}, \quad B_\varphi^\theta = P_\theta^\perp M_\varphi|_{K_\theta \cap L^\infty} \quad \text{and} \quad D_\varphi^\theta = P_\theta^\perp M_\varphi|_{K_\theta^\perp \cap L^\infty}.$$

If A_φ^θ extends to the whole K_θ as a bounded operator, it is called a *truncated Toeplitz operator* (TTO). Similarly, if B_φ^θ extends to a bounded operator from K_θ to $K_\theta^\perp$, it is called a *big truncated Hankel operator* (THO) (see [17]), and if D_φ^θ extends to the whole $K_\theta^\perp$ as a bounded operator, it is called a *dual truncated Toeplitz operator* (DTTO). Let us fix the notation

$$\begin{aligned} \mathcal{T}(K_\theta) &= \{A_\varphi^\theta : \varphi \in L^2 \text{ and } A_\varphi^\theta \text{ is bounded}\}, \\ \mathcal{T}(K_\theta, K_\theta^\perp) &= \{B_\varphi^\theta : \varphi \in L^2 \text{ and } B_\varphi^\theta \text{ is bounded}\}, \\ \mathcal{T}(K_\theta^\perp) &= \{D_\varphi^\theta : \varphi \in L^2 \text{ and } D_\varphi^\theta \text{ is bounded}\}. \end{aligned}$$

Model spaces, which provide the natural setting for truncated Toeplitz operators, have generated enormous interest and they are relevant in connection with a variety of topics such as the Schrödinger operator, classical extremal problems in control theory, Hankel operators and Toeplitz matrices (see for instance [13] and [11]). Natural conjugations, which model spaces and the whole L^2 possess (see [5]), make model spaces even more natural in the context of physics [12]. Their orthogonal complements in L^2 also appear in numerous applications. In the equivalent setting of the real line [9, 16], using time and

frequency as the natural variables, and taking the inner function $\theta = \theta_\lambda$ with $\theta_\lambda(\xi) = \exp(i\lambda\xi)$ for $\xi \in \mathbb{R}$, they appear via the Fourier transform, for instance, as high frequency signals, which are of decisive importance in electronics, or as outputs of high-pass filters. Dual truncated Toeplitz operators, acting on these spaces have realizations, for example, in long distance communication links with several regenerators along the path that cancel low-frequency noise using high-pass filters, or in the description of wave propagation in the presence of finite-length obstacles.

Systematic study of truncated Toeplitz operators A_φ^θ (for general $\varphi \in L^2$) was started in [22] while the properties of dual truncated Toeplitz operators D_φ^θ were investigated in [8, 15, 20] and more recently in [2, 6, 7, 21]. Truncated Hankel operators were studied in [17] and also in [14], but there is a different definition of B_φ^θ (see also [3]).

The main purpose of this paper is to advance the study of dual truncated Toeplitz operators $\mathcal{T}(K_\theta^\perp)$ and big truncated Hankel operators $\mathcal{H}(K_\theta, K_\theta^\perp)$. The first result worth mentioning is a full characterization of rank-one elements in $\mathcal{T}(K_\theta, K_\theta^\perp)$ with L^∞ symbols—Sect. 3. Note that it is clear from [8, Property 2.2] that there are no rank-one operators in $\mathcal{T}(K_\theta^\perp)$.

The property of commutativity with the operator of multiplication by the independent variable is very important. In the classical situation, in the space L^2 , only multiplication operators M_φ with $\varphi \in L^\infty$ commute with M_z . In the case of the Hardy space H^2 , the operators which commute with the Toeplitz operator T_z (called *the unilateral shift*) are Toeplitz operators T_φ with φ being a bounded analytic symbol ($\varphi \in H^\infty$). In the case of a model space K_θ , where θ is a nonconstant inner function, it is shown in [23] that an operator commuting with the compression of the unilateral shift to the model space $P_\theta T_z|_{K_\theta} = A_z^\theta$ has to be a truncated Toeplitz operator with an analytic symbol. In Sect. 4 we discuss similar questions for dual truncated Toeplitz operators.

On the other hand, it is known that a bounded linear operator $T \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T = (T_z)^* T T_z$. Similar characterizations (in terms of compressions of M_z) are known for Hankel operators and dual Toeplitz operators. In [22] D. Sarason characterized bounded truncated Toeplitz operators in terms of the compressions of M_z to K_θ . For example, he characterized truncated Toeplitz operators expressing the difference $A - A_z^\theta A A_z^\theta$ as a specific rank-two operator. In Sect. 5 a relation similar to the one above is given for dual truncated Toeplitz operators. For any $D \in \mathcal{T}(K_\theta^\perp)$ we also express $D - D_z^\theta D (D_z^\theta)^*$ by rank-two operators. However, it does not lead to a characterization of DTTO. To obtain such a characterization we use specific restrictions of dual truncated Toeplitz operators in Sect. 6. Finally, in Sect. 7, Theorem 27 gives a necessary and sufficient condition for any operator $D \in \mathcal{B}(K_\theta^\perp)$ to be a DTTO. Moreover, using the formula given there we can easily recover its unique symbol.

2. Basic Properties

For any Hilbert space $\mathcal{H}$ and $h, g \in \mathcal{H}$ we define the rank-one operator $h \otimes g$ as $h \otimes g(f) = \langle f, g \rangle h$. Some rank-one operators are the commutators in $\mathcal{B}(L^2)$.

Lemma 1. *Let M_z be the operator of multiplication by the independent variable in L^2 . Then*

- (1) $P^+ M_z - M_{\bar{z}} P^+ = 1 \otimes \bar{z}$,
- (2) $P^- M_{\bar{z}} - M_z P^- = \bar{z} \otimes 1$.

2.1. Conjugation

There is a natural conjugation (an antiunitary involution) connected with a model space (see for instance [5, 12]). For an inner function θ define $C_\theta : L^2 \rightarrow L^2$ by

$$C_\theta f(z) = \theta(z) \overline{zf(z)}, \quad |z| = 1. \quad (2.1)$$

Then C_θ is an antilinear isometric involution on L^2 , which implies that $\langle C_\theta f, C_\theta g \rangle = \langle g, f \rangle$ for $f, g \in L^2$. One can easily verify that

$$C_\theta M_\varphi C_\theta = M_{\bar{\varphi}}. \quad (2.2)$$

It is well known ([12]) that C_θ preserves K_θ . Moreover, $C_\theta(\theta H^2) = H_-^2$ and $C_\theta(H_-^2) = \theta H^2$, so C_θ also preserves $(K_\theta)^\perp$. Hence,

$$C_\theta = \begin{bmatrix} C_{\theta|K_\theta} & 0 \\ 0 & C_{\theta|(K_\theta)^\perp} \end{bmatrix}.$$

Corollary 2. *Let $\varphi \in L^2$.*

- (1) *If $A_\varphi^\theta \in \mathcal{B}(K_\theta)$, then $C_\theta A_\varphi^\theta C_\theta = A_{\bar{\varphi}}^\theta$.*
- (2) *If $B_\varphi^\theta \in \mathcal{B}(K_\theta, K_\theta^\perp)$, then $B_{\bar{\varphi}}^\theta \in \mathcal{B}(K_\theta, K_\theta^\perp)$ and $C_\theta B_\varphi^\theta C_\theta = B_{\bar{\varphi}}^\theta$.*
- (3) *If $D_\varphi^\theta \in \mathcal{B}(K_\theta^\perp)$, then $C_\theta D_\varphi^\theta C_\theta = D_{\bar{\varphi}}^\theta$.*

Proof. Using the representation (1.1) and (2.2) on $(K_\theta \cap L^\infty) \oplus (K_\theta^\perp \cap L^\infty)$ the equality below holds:

$$\begin{aligned} \begin{bmatrix} A_{\bar{\varphi}}^\theta & (B_{\bar{\varphi}}^\theta)^* \\ B_{\bar{\varphi}}^\theta & D_{\bar{\varphi}}^\theta \end{bmatrix} &= \begin{bmatrix} C_{\theta|K_\theta} & 0 \\ 0 & C_{\theta|(K_\theta)^\perp} \end{bmatrix} \cdot \begin{bmatrix} A_\varphi^\theta & (B_\varphi^\theta)^* \\ B_\varphi^\theta & D_\varphi^\theta \end{bmatrix} \cdot \begin{bmatrix} C_{\theta|K_\theta} & 0 \\ 0 & C_{\theta|(K_\theta)^\perp} \end{bmatrix} \\ &= \begin{bmatrix} C_\theta A_\varphi^\theta C_\theta & C_\theta (B_\varphi^\theta)^* C_\theta \\ C_\theta B_\varphi^\theta C_\theta & C_\theta D_\varphi^\theta C_\theta \end{bmatrix}. \end{aligned}$$

In particular $C_\theta B_\varphi^\theta C_\theta = B_{\bar{\varphi}}^\theta$ on $K_\theta \cap L^\infty$. Since B_φ^θ is bounded, $B_{\bar{\varphi}}^\theta$ is bounded, too, and the equality holds on K_θ . The other equalities can be shown similarly. $\square$

Part (1) of Corollary 2 was proved by D. Sarason [22] while part (3) was shown in [15].

2.2. Zero Operators

Recall that $k_w = \frac{1}{1-\bar{w}z}$ is a reproducing kernel for all functions $f \in H^2$, i.e., $f(w) = \langle f, k_w \rangle$ for $w \in \mathbb{D}$. Let θ be a nonconstant inner function. Then $k_w^\theta = P_\theta k_w = (1 - \overline{\theta(w)}\theta)k_w$ is a reproducing kernel for all functions $f \in K_\theta$, i.e., $f(w) = \langle f, k_w^\theta \rangle$ for $w \in \mathbb{D}$. Denote $\tilde{k}_w^\theta = C_\theta k_w^\theta$, $\tilde{k}_w^\theta(z) = \frac{\theta(z) - \overline{\theta(w)}}{z - w}$. Note that $C_\theta f(w) = \langle f, \tilde{k}_w^\theta \rangle$ for $w \in \mathbb{D}$.

It is well known that a Toeplitz operator is uniquely determined by its symbol, that is, $T_\varphi = 0$ if and only if $\varphi = 0$. The same is true for dual Toeplitz operators, but is not for Hankel operators.

As for truncated Toeplitz operators, it was proved by D. Sarason in [22] that the symbol of a truncated Toeplitz operator is not uniquely determined. A similar result for truncated Hankel operators was obtained by P. Ma, F. Yan and Zhang in D. [17]. On the other hand, X. Ding and Y. Sang showed in [8] that the symbol of a dual truncated Toeplitz operator is unique. Summing up, we have the following.

Proposition 3. *Let θ be a nonconstant inner function and let $\varphi \in L^2$. Then*

- (1) $A_\varphi^\theta = 0$ if and only if $\varphi \in \theta H^2 + \overline{\theta H^2}$;
- (2) $B_\varphi^\theta = 0$ if and only if φ is a constant function;
- (3) $D_\varphi^\theta = 0$ if and only if $\varphi = 0$.

The proof of part (2) given in [17] was based on the fact that B_φ^θ can be expressed as a block operator built from certain products of classical Hankel operators, and only for $\varphi \in L^\infty$. But (2) can be proved in an alternative, simpler way. Namely, note that $B_\varphi^\theta = 0$ means that $P_\theta^\perp(\varphi f) = 0$ for all $f \in K_\theta$. The latter happens if and only if $\varphi f \in K_\theta$ for all $f \in K_\theta$, which means that the multiplication operator M_φ preserves K_θ . By [10, Proposition 2.2], this is equivalent to φ being a constant function.

2.3. Boundedness

It is a classical result that M_φ is bounded on L^2 if and only if $\varphi \in L^\infty$. It was proved in [8] that a dual truncated Toeplitz operator is bounded if and only if its symbol is bounded. Hence

$$\mathcal{T}(K_\theta^\perp) = \{D_\varphi^\theta : \varphi \in L^\infty\}.$$

On the other hand, it is known that there exist bounded truncated Toeplitz operators without bounded symbols [1].

Remark 4. The latter is true also for truncated Hankel operators. To see this it is enough to consider a finite dimensional model space K_θ (θ is a finite Blaschke product). Then B_φ^θ is clearly bounded for every $\varphi \in L^2$. But if $B_\varphi^\theta = B_\chi^\theta$ for some $\chi \in L^\infty$, then by Proposition 3 there exists $c \in \mathbb{C}$ such that $\varphi - \chi = c$ and so $\varphi \in L^\infty$.

Let $\varphi \in L^2$ and consider the matrix

$$M_\varphi = \begin{bmatrix} A_\varphi^\theta & (B_{\bar{\varphi}}^\theta)^* \\ B_\varphi^\theta & D_\varphi^\theta \end{bmatrix}. \quad (2.3)$$

Corollary 2 shows that B_φ^θ is bounded if and only if $(B_{\bar{\varphi}}^\theta)^*$ is bounded. Note also that if $\varphi \notin L^\infty$ even if $A_\varphi^\theta, B_\varphi^\theta$ are bounded M_φ is not bounded since D_φ^θ is not bounded.

3. Rank-One and Defect Operators

In [22] D. Sarason described all rank-one operators in $\mathcal{T}(K_\theta)$. Here we completely characterize rank-one operators in $\mathcal{T}(K_\theta, K_\theta^\perp)$.

Let f be analytic on some open set G . Denote the n -th Taylor polynomial of f at $a \in G$ as

$$\mathbf{P}_n(f, a)(z) = f(a) + \frac{1}{1!}f'(a)(z-a) + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(a)(z-a)^{n-1}.$$

Let now $f \in H_-^2$. Then $f_+ = \bar{z}\bar{f} \in H^2$ and it is a standard procedure to define the analytic extension of f to $\mathbb{D}^e = \mathbb{C} \setminus \bar{\mathbb{D}}$ as $f(z) = \frac{1}{z} \overline{f_+(\bar{z}^{-1})}$ for $z \in \mathbb{D}^e$. If $f \in H_-^2$ and $b \in \mathbb{D}^e$, then it is not hard to see that $\frac{f(\cdot) - f(b)}{z-b} \in H_-^2$. Moreover, using the formula $\lim_{z \rightarrow b} \left(f(z) - \mathbf{P}_n(f, b)(z) \right) (z-b)^{-n} = \frac{1}{n!} f^{(n)}(b)$, we have $\frac{f(\cdot) - \mathbf{P}_n(f, b)}{(z-b)^n} \in H_-^2$. In particular, we apply this below to $\bar{\theta}f \in H_-^2$ for $f \in K_\theta$.

Lemma 5. *Let θ be a nonconstant inner function and $n \in \mathbb{N}$. If $f \in K_\theta$, then*

- (1) $P_\theta^\perp(z^n f) = \theta z^{n-1} \overline{\mathbf{P}_n(C_\theta f, 0)}$;
- (2) if $a \in \mathbb{D}$ then $P_\theta^\perp((z-a)^{-n} f) = (z-a)^{-n} \mathbf{P}_n(f, a)$;
- (3) if $b \in \mathbb{D}^e$ then $P_\theta^\perp((z-b)^{-n} f) = \theta(z-b)^{-n} \mathbf{P}_n(\bar{\theta}f, b)$.

Proof. To see (1) note that $z^n f \in H^2$ and thus

$$P_\theta^\perp(z^n f) = \theta P^+ \bar{\theta} z^n f = \theta P^+ z^{n-1} \overline{C_\theta f} = \theta z^{n-1} \overline{\mathbf{P}_n(C_\theta f, 0)}.$$

For (2) note that $\bar{\theta}(z-a)^{-n} f \in H_-^2$. Hence

$$\begin{aligned} P_\theta^\perp((z-a)^{-n} f) &= (P^- + \theta P^+ \bar{\theta})((z-a)^{-n} f) \\ &= P^-((z-a)^{-n} f) = (z-a)^{-n} \mathbf{P}_n(f, a). \end{aligned}$$

To observe (3) note that $(z-b)^{-n} f \in H^2$, so

$$\begin{aligned} P_\theta^\perp((z-b)^{-n} f) &= \theta P^+((z-b)^{-n} \bar{\theta}f) \\ &= \theta P^+((z-b)^{-n} (\bar{\theta}f - \mathbf{P}_n(\bar{\theta}f, b))) \\ &\quad + \theta P^+((z-b)^{-n} \mathbf{P}_n(\bar{\theta}f, b)) \\ &= \theta(z-b)^{-n} \mathbf{P}_n(\bar{\theta}f, b). \end{aligned}$$

□

Before we state the result recall that $\dim K_\theta = 1$ if and only if $\theta(z) = \lambda \frac{a-z}{1-\bar{a}z}$ with $\lambda \in \mathbb{T}$, $a \in \mathbb{D}$, in particular for $\theta(z) = z$.

Theorem 6. *Let θ be a nonconstant inner function.*

- (1) *If $\dim K_\theta = 1$, then all nonzero operators from $\mathcal{T}(K_\theta, K_\theta^\perp)$ are of rank one.*
- (2) *Assume that $\dim K_\theta > 1$. If $B_\varphi^\theta \in \mathcal{T}(K_\theta, K_\theta^\perp)$ with $\varphi \in L^\infty$ is of rank one, then it is a constant multiple of one of the operators below*

$$B_{zk_w}^\theta = \theta k_w \otimes \tilde{k}_w, \quad \text{or} \quad B_{\bar{z}k_w}^\theta = \bar{z} \tilde{k}_w \otimes k_w, \quad \text{where } w \in \mathbb{D}. \quad (3.1)$$

Proof. Let B_φ^θ be a rank-one operator and let $\psi \in K_\theta^\perp$ span the range of B_φ^θ . For θ such that $\dim K_\theta > 1$, there exists $f_\theta \in K_\theta$ with $f_\theta(0) = 0$, so $\bar{z}f_\theta \in K_\theta$. Then $P_\theta^\perp(\varphi f_\theta) = \lambda_1 \psi$ and $P_\theta^\perp(\varphi \bar{z}f_\theta) = \lambda_2 \psi$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Thus $\lambda_2 P_\theta^\perp(\varphi f_\theta) - \lambda_1 P_\theta^\perp(\varphi \bar{z}f_\theta) = 0$ and hence $(\lambda_2 - \lambda_1 \bar{z})\varphi f_\theta = g_\theta \in K_\theta$.

If θ is a finite Blaschke product, then f_θ and g_θ are rational, and it follows that φ is also rational, without poles on the unit circle $\mathbb{T}$, because $\varphi \in L^\infty$. If θ is not a finite Blaschke product, it follows from [17, Theorem 1.2] that φ is also a rational function without poles on $\mathbb{T}$. Therefore φ is a linear combination of functions of the form z^n , $(z-a)^{-n}$ with $a \in \mathbb{D}$, $(z-b)^{-n}$ with $b \in \mathbb{D}^e$, $n \in \mathbb{N}$.

In view of Lemma 5 we see that B_φ^θ can have rank one if and only if $n = 1$, i.e., $\varphi = z$, $\varphi = (z-a)^{-1}$, $a \in \mathbb{D}$ or $\varphi = (z-b)^{-1}$, $b \in \mathbb{D}^e$. We will use the Taylor expansion with $n = 1$. Let $f \in K_\theta$ and consider the case $\varphi = z$. Then

$$B_z^\theta f = B_{zk_0}^\theta f = \overline{\theta C_\theta f(0)} = \langle f, \tilde{k}_0^\theta \rangle \theta = (\theta \otimes \tilde{k}_0^\theta) f.$$

Set $a = w$, and consider $\varphi = (z-w)^{-1} = \bar{z}(1-w\bar{z})^{-1} = \bar{z}k_w$. Then

$$\begin{aligned} B_{\bar{z}k_w}^\theta f &= f(w)(z-w)^{-1} = \langle f, k_w^\theta \rangle (z-w)^{-1} = ((z-w)^{-1} \otimes k_w^\theta) f \\ &= (\bar{z} \tilde{k}_w \otimes k_w^\theta) f. \end{aligned}$$

Now the last case is $\varphi = (z-b)^{-1}$, $b \in \mathbb{D}^e$. Let us set $w = \bar{b}^{-1}$ ($w \neq 0$). Then using the formula for the analytic extension we have $(\bar{\theta}f)(b) = \bar{w} C_\theta f(w)$. Since zk_w and $-\bar{w}^{-2}(z - \frac{1}{\bar{w}})^{-1}$ differ only by a constant, thus, by Lemma 5 (3) we have

$$\begin{aligned} B_{zk_w}^\theta f &= -\bar{w}^{-2} B_{(z-\frac{1}{\bar{w}})^{-1}}^\theta f = -\bar{w}^{-2} \theta (z - \frac{1}{\bar{w}})^{-1} (\bar{\theta}f)(\frac{1}{\bar{w}}) \\ &= \theta (1 - \bar{w}z)^{-1} \overline{(C_\theta f)(w)} = (\theta k_w \otimes \tilde{k}_w^\theta) f. \end{aligned}$$

□

The following is a consequence of Theorem 6.

Corollary 7. (1) $B_z^\theta = \theta \otimes \tilde{k}_0^\theta$;

$$(2) (B_z^\theta)^* = \tilde{k}_0^\theta \otimes \theta;$$

$$(3) B_{\bar{z}}^\theta = \bar{z} \otimes k_0^\theta;$$

$$(4) (B_{\bar{z}}^\theta)^* = k_0^\theta \otimes \bar{z}.$$

Recall from [22] that for truncated Toeplitz operators we have the following formulas for the defect operators:

$$A_z^\theta A_{\bar{z}}^\theta = I_{K_\theta} - k_0^\theta \otimes k_0^\theta \quad \text{and} \quad A_{\bar{z}}^\theta A_z^\theta = I_{K_\theta} - \tilde{k}_0^\theta \otimes \tilde{k}_0^\theta. \quad (3.2)$$

Similarly, we have the following.

Proposition 8. *Let θ be a nonconstant inner function. Then*

- (1) $D_z^\theta D_{\bar{z}}^\theta = I_{K_\theta^\perp} - (1 - |\theta(0)|^2)\theta \otimes \theta$;
- (2) $D_{\bar{z}}^\theta D_z^\theta = I_{K_\theta^\perp} - (1 - |\theta(0)|^2)\bar{z} \otimes \bar{z}$;
- (3) $B_z^\theta (B_{\bar{z}}^\theta)^* = \overline{\theta'(0)} \theta \otimes \bar{z}$;
- (4) $(\bar{B}_{\bar{z}}^\theta)^* B_z^\theta = 0$.

Proof. Note that for $f, g \in K_\theta^\perp$ we have

$$\begin{aligned} \langle D_z^\theta D_{\bar{z}}^\theta f, g \rangle &= \langle z(\bar{z}f - P_\theta(\bar{z}f)), g \rangle \\ &= \langle f - zP_\theta(\bar{z}f), g \rangle = \langle f, g \rangle - \langle zP_\theta(\bar{z}f), g \rangle. \end{aligned}$$

Since

$$\begin{aligned} P_\theta(\bar{z}f) &= P^+ \theta P^- \bar{\theta} (\bar{z}P^- f + \bar{z}P_{\theta H^2} f) \\ &= P^+ \theta P^- \bar{\theta} \bar{z} P_{\theta H^2} f = \langle \bar{\theta} f, 1 \rangle P^+ \theta \bar{z} = \langle f, \theta \rangle \tilde{k}_0^\theta, \end{aligned}$$

we have

$$\langle D_z^\theta D_{\bar{z}}^\theta f, g \rangle = \langle f, g \rangle - \langle f, \theta \rangle \langle z\tilde{k}_0^\theta, g \rangle = \langle f, g \rangle - \langle f, \theta \rangle \langle P_\theta^\perp(z\tilde{k}_0^\theta), g \rangle.$$

Note that $P_\theta^\perp(z\tilde{k}_0^\theta) = (1 - |\theta(0)|^2)\theta$. Hence

$$\begin{aligned} \langle D_z^\theta D_{\bar{z}}^\theta f, g \rangle &= \langle f, g \rangle - \langle \langle f, \theta \rangle (1 - |\theta(0)|^2)\theta, g \rangle \\ &= \langle (I_{(K_\theta)^\perp} - (1 - |\theta(0)|^2)\theta \otimes \theta)f, g \rangle. \end{aligned}$$

To prove (2) note that, by Corollary 2,

$$\begin{aligned} D_{\bar{z}}^\theta D_z^\theta &= C_\theta D_z^\theta D_{\bar{z}}^\theta C_\theta = C_\theta (I_{(K_\theta)^\perp} - (1 - |\theta(0)|^2)\theta \otimes \theta) C_\theta \\ &= I_{(K_\theta)^\perp} - (1 - |\theta(0)|^2)C_\theta \theta \otimes C_\theta \theta = I_{K_\theta^\perp} - (1 - |\theta(0)|^2)\bar{z} \otimes \bar{z}. \end{aligned}$$

Calculating (3) we will use Corollary 7 and the formula for multiplication of rank-one operators (see [19])

$$B_z^\theta (B_{\bar{z}}^\theta)^* = (\theta \otimes \tilde{k}_0^\theta)(k_0^\theta \otimes \bar{z}) = \langle k_0^\theta, \tilde{k}_0^\theta \rangle \theta \otimes \bar{z} = \overline{\theta'(0)} \theta \otimes \bar{z}.$$

We get the last formula similarly. $\square$

4. Commutativity with Restrictions of M_z

Starting the study of commutativity with $D_z^\theta = P_\theta^\perp M_z|_{K_\theta^\perp}$ we first give an example of an operator which commutes with D_z^θ and is not a dual truncated Toeplitz operator.

Example 9. Let θ be an inner function such that $\theta(0) = 0$. For $\varphi \in H^\infty \setminus \{0\}$ define $D \in \mathcal{B}(\theta H^2 \oplus H_-^2)$ by $D = P_{\theta H^2} M_{\varphi|_{\theta H^2}} \oplus 0$. Then for any $f, g \in H^2$ we have

$$D_z^\theta D(\theta f + \bar{z}\bar{g}) = D_z^\theta(\varphi\theta f) = z\varphi\theta f.$$

On the other hand,

$$\begin{aligned} D_z^\theta(\theta f + \bar{z}\bar{g}) &= P_\theta^\perp(z\theta f + \bar{g}) = z\theta f + \overline{g - g(0)} + P_\theta^\perp(\overline{g(0)}) \\ &= z\theta f + \overline{g - g(0)} + \overline{g(0)\theta\theta(0)} = z\theta f + \overline{g - g(0)}, \end{aligned}$$

since $\theta(0) = 0$. It follows that

$$DD_z^\theta(\theta f + \bar{z}\bar{g}) = D(z\theta f + \overline{g - g(0)}) = z\varphi\theta f$$

and so D commutes with D_z^θ . To see that $D \notin \mathcal{S}(K_\theta^\perp)$ observe that D does not satisfy condition (c) from Corollary 2, that is, $C_\theta DC_\theta \neq D^*$. Indeed, here $D^* = P_{\theta H^2} M_{\bar{\varphi}|_{\theta H^2}} \oplus 0$, while

$$C_\theta DC_\theta(\theta f + \bar{z}\bar{g}) = C_\theta D(\bar{z}\bar{f} + \theta g) = C_\theta(\varphi\theta g) = \overline{\varphi z}\bar{g},$$

which means that $C_\theta DC_\theta = 0 \oplus P^- M_{\bar{\varphi}|_{H_-^2}}$.

Below we calculate the commutator of a given dual truncated Toeplitz operator with D_z^θ . It shows how far is this relation from commutativity. Let us start with the following lemma.

Lemma 10. Let $\varphi \in L^2$, $\varphi = \overline{z(\chi_1 + \theta\chi_2)} + \psi_1 + \theta\psi_2$ with $\chi_1, \psi_1 \in K_\theta$, $\chi_2, \psi_2 \in H^2$ according to the decomposition $L^2 = \overline{z(K_\theta \oplus \theta H^2)} \oplus K_\theta \oplus \theta H^2$. Then

- (1) $B_\varphi^\theta k_0^\theta = \varphi k_0^\theta - P_\theta \varphi + \overline{\theta(0)} C_\theta P_\theta(\bar{z}\bar{\varphi})$;
- (2) $B_\varphi^\theta \bar{k}_0^\theta = C_\theta B_\varphi^\theta k_0^\theta = C_\theta P_\theta^\perp(\varphi k_0^\theta)$;
- (3) $(B_\varphi^\theta)^* \theta = P_\theta(\theta\varphi) = C_\theta \chi_1$;
- (4) $(B_\varphi^\theta)^* \bar{z} = C_\theta (B_\varphi^\theta)^* \theta = \chi_1 = C_\theta P_\theta(\theta\varphi)$.

Proof. Note that

$$\begin{aligned} B_\varphi^\theta k_0^\theta &= P_\theta^\perp(\varphi k_0^\theta) = P_\theta^\perp \varphi(1 - \overline{\theta(0)}\theta) \\ &= \varphi - P_\theta \varphi - \overline{\theta(0)} P_\theta^\perp(\theta \bar{z}\bar{\chi}_1 + \bar{z}\bar{\chi}_2 + \theta\psi_1 + \theta^2\psi_2) \\ &= \varphi - \psi_1 - \overline{\theta(0)}(\theta\varphi - C_\theta \chi_1) = \varphi k_0^\theta - \psi_1 + \overline{\theta(0)} C_\theta \chi_1. \end{aligned} \quad (4.1)$$

The second condition follows from Corollary 2. Condition (3) is straightforward, whereas (4) follows from (3) and Corollary 2. $\square$

The next theorem gives the formulas for the commutators.

Theorem 11. Let $\varphi \in L^2$. Then

- (1) if $D_\varphi^\theta \in \mathcal{B}(K_\theta^\perp)$, then $D_\varphi^\theta D_z^\theta - D_z^\theta D_\varphi^\theta = \theta \otimes C_\theta(P_\theta^\perp(\varphi k_0^\theta)) - (P_\theta^\perp(\varphi k_0^\theta)) \otimes \bar{z}$;
- (2) if $A_\varphi^\theta \in \mathcal{B}(K_\theta)$, then $A_\varphi^\theta A_z^\theta - A_z^\theta A_\varphi^\theta = k_0^\theta \otimes (C_\theta P_\theta(\theta\varphi)) - (P_\theta(\theta\varphi)) \otimes C_\theta k_0^\theta$;

(3) if $B_\varphi^\theta \in \mathcal{B}(K_\theta, K_\theta^\perp)$, then $B_\varphi^\theta A_z^\theta - D_z^\theta B_\varphi^\theta = \theta \otimes (C_\theta P_\theta(\varphi k_0^\theta)) - (P_\theta^\perp(\varphi\theta)) \otimes C_\theta k_0^\theta$.

Proof. Note that if $f \in D(M_\varphi)$, then also $zf \in D(M_\varphi)$ and as a consequence we get commutativity $M_z M_\varphi = M_\varphi M_z$ on $D(M_\varphi)$. Observe that by Corollary 7, $B_z^\theta(K_\theta^\infty) \subset \theta H^\infty \oplus \bar{z} \overline{H^\infty}$ and $(B_z^\theta)^*(\theta H^\infty \oplus \bar{z} \overline{H^\infty}) \subset K_\theta^\infty$. Moreover, for $f, g \in H^\infty$ and $h \in K_\theta^\infty$ we have

$$D_z^\theta(\theta f + \bar{z}g) = z\theta f + \overline{g - g(0)} + \overline{g(0)}\theta\bar{\theta}(0) \in \theta H^\infty \oplus \bar{z} \overline{H^\infty}$$

(see Example 9) and

$$A_z^\theta h = P_\theta(zh) = zh - \theta P^+(\bar{\theta}zh) = zh - \theta \cdot \overline{(C_\theta h)(0)} \in K_\theta^\infty,$$

that is, $D_z^\theta(\theta H^\infty \oplus \bar{z} \overline{H^\infty}) \subset \theta H^\infty \oplus \bar{z} \overline{H^\infty}$ and $A_z^\theta(K_\theta^\infty) \subset K_\theta^\infty$. Therefore, the equality $M_\varphi M_z = M_z M_\varphi$ can be expressed on $K_\theta^\infty \oplus (\theta H^\infty \oplus \bar{z} \overline{H^\infty})$ as

$$\begin{bmatrix} A_\varphi^\theta & (B_\varphi^\theta)^* \\ B_\varphi^\theta & D_\varphi^\theta \end{bmatrix} \cdot \begin{bmatrix} A_z^\theta & (B_z^\theta)^* \\ B_z^\theta & D_z^\theta \end{bmatrix} = \begin{bmatrix} A_z^\theta & (B_z^\theta)^* \\ B_z^\theta & D_z^\theta \end{bmatrix} \cdot \begin{bmatrix} A_\varphi^\theta & (B_\varphi^\theta)^* \\ B_\varphi^\theta & D_\varphi^\theta \end{bmatrix}. \quad (4.2)$$

Hence

$$B_\varphi^\theta (B_z^\theta)^* + D_\varphi^\theta D_z^\theta = B_z^\theta (B_\varphi^\theta)^* + D_z^\theta D_\varphi^\theta \quad \text{on } \theta H^\infty \oplus \bar{z} \overline{H^\infty}, \quad (4.3)$$

$$A_\varphi^\theta A_z^\theta + (B_\varphi^\theta)^* B_z^\theta = A_z^\theta A_\varphi^\theta + (B_z^\theta)^* B_\varphi^\theta \quad \text{on } K_\theta^\infty, \quad (4.4)$$

$$B_\varphi^\theta A_z^\theta + D_\varphi^\theta B_z^\theta = B_z^\theta A_\varphi^\theta + D_z^\theta B_\varphi^\theta \quad \text{on } K_\theta^\infty. \quad (4.5)$$

By Corollary 7 and Lemma 10, it follows from (4.3) that on $\theta H^\infty \oplus \bar{z} \overline{H^\infty}$,

$$\begin{aligned} D_\varphi^\theta D_z^\theta - D_z^\theta D_\varphi^\theta &= B_z^\theta (B_\varphi^\theta)^* - B_\varphi^\theta (B_z^\theta)^* \\ &= (\theta \otimes \tilde{k}_0^\theta)(B_\varphi^\theta)^* - B_\varphi^\theta(k_0^\theta \otimes \bar{z}) = \theta \otimes (B_\varphi^\theta \tilde{k}_0^\theta) - (B_\varphi^\theta k_0^\theta) \otimes \bar{z} \\ &= \theta \otimes C_\theta(P_\theta^\perp(\varphi k_0^\theta)) - (P_\theta^\perp(\varphi k_0^\theta)) \otimes \bar{z}. \end{aligned}$$

Hence (1) holds.

Similarly, from (4.4) it follows that on K_θ^∞ ,

$$\begin{aligned} A_\varphi^\theta A_z^\theta - A_z^\theta A_\varphi^\theta &= (B_z^\theta)^* B_\varphi^\theta - (B_\varphi^\theta)^* B_z^\theta \\ &= (k_0^\theta \otimes \bar{z}) B_\varphi^\theta - (B_\varphi^\theta)^*(\theta \otimes \tilde{k}_0^\theta) = k_0^\theta \otimes ((B_\varphi^\theta)^* \bar{z}) \\ &\quad - ((B_\varphi^\theta)^* \theta) \otimes \tilde{k}_0^\theta \\ &= k_0^\theta \otimes (C_\theta P_\theta(\theta \varphi)) - (P_\theta(\theta \varphi)) \otimes C_\theta k_0^\theta \end{aligned}$$

and (2) is satisfied.

Finally, from (4.5) we conclude that on K_θ^∞ ,

$$\begin{aligned} B_\varphi^\theta A_z^\theta - D_z^\theta B_\varphi^\theta &= B_z^\theta A_\varphi^\theta - D_\varphi^\theta B_z^\theta = (\theta \otimes \tilde{k}_0^\theta) A_\varphi^\theta - D_\varphi^\theta(\theta \otimes \tilde{k}_0^\theta) \\ &= \theta \otimes (A_\varphi^\theta C_\theta k_0^\theta) - (D_\varphi^\theta \theta) \otimes \tilde{k}_0^\theta = \theta \otimes (C_\theta A_\varphi^\theta k_0^\theta) \\ &\quad - (P_\theta^\perp(\varphi\theta)) \otimes \tilde{k}_0^\theta \\ &= \theta \otimes (C_\theta P_\theta(\varphi k_0^\theta)) - (P_\theta^\perp(\varphi\theta)) \otimes C_\theta k_0^\theta \end{aligned}$$

and (3) follows. $\square$

As a corollary we get a necessary and sufficient condition for a dual truncated Toeplitz operator to commute with D_z^θ .

Corollary 12. *Let $D_\varphi^\theta \in \mathcal{T}(K_\theta^\perp)$. Then D_φ^θ commutes with D_z^θ , i.e., $D_\varphi^\theta D_z^\theta = D_z^\theta D_\varphi^\theta$ if and only if $\varphi k_0^\theta = c\theta + P_\theta \varphi$ for $c \in \mathbb{C}$. In particular, if $\theta(0) = 0$, then $D_\varphi^\theta D_z^\theta = D_z^\theta D_\varphi^\theta$ if and only if $\varphi \in K_\theta + \mathbb{C}\theta$.*

Proof. By Theorem 11,

$$D_\varphi^\theta D_z^\theta = D_z^\theta D_\varphi^\theta$$

if and only if

$$\theta \otimes C_\theta(P_\theta^\perp(\varphi k_0^\theta)) = (P_\theta^\perp(\varphi k_0^\theta)) \otimes \bar{z} = (P_\theta^\perp(\varphi k_0^\theta)) \otimes C_\theta \theta.$$

The above holds if and only if there is $c \in \mathbb{C}$ such that $P_\theta^\perp(\varphi k_0^\theta) = c\theta$, which is equivalent to $C_\theta(P_\theta^\perp(\varphi k_0^\theta)) = \bar{c}\bar{z}$. In other words, it holds if and only if $\varphi k_0^\theta = c\theta + g$ for some $g \in K_\theta$. It follows that $\varphi(1 - \overline{\theta(0)}\theta) = \psi \in H^2$. Since $(1 - \overline{\theta(0)}\theta)^{-1}$ is a bounded analytic function, we also have $\varphi = (1 - \overline{\theta(0)}\theta)^{-1}\psi \in H^2$. Hence

$$P_\theta(\varphi) = P_\theta(\varphi k_0^\theta + \overline{\theta(0)}\theta\varphi) = P_\theta(c\theta + g + \overline{\theta(0)}\theta\varphi) = g,$$

that is, $\varphi k_0^\theta = c\theta + P_\theta(\varphi)$. $\square$

Remark 13. The known fact that a truncated Toeplitz operator which commutes with A_z^θ has an analytic symbol (see [23]) can also be obtained from Theorem 11. Recall that $A_\varphi^\theta = 0$ if and only if $\varphi \in \theta H^2 + \overline{\theta} H^2$ ([22, Theorem 3.1]). Hence we can assume that $A_\varphi^\theta = A_{\bar{z}\bar{\chi}+\psi}^\theta$ with $\chi, \psi \in K_\theta$. From Theorem 11 it follows that $A_\varphi^\theta A_z^\theta = A_z^\theta A_\varphi^\theta$ if and only if for some $c \in \mathbb{C}$ we have

$$c\tilde{k}_0^\theta = cC_\theta k_0^\theta = (B_\varphi^\theta)^* \bar{z} = \chi,$$

by Lemma 10. Now, since $A_\theta^\theta = 0$, we have

$$A_\varphi^\theta = A_{\bar{z}\bar{\chi}+\psi}^\theta = A_{\bar{c}(\bar{\theta}-\overline{\theta(0)})+\psi}^\theta = A_{\psi-\bar{c}\bar{\theta}(0)}^\theta$$

and A_φ^θ has an analytic symbol $\psi - \bar{c}\bar{\theta}(0)$.

Next we get a necessary and sufficient condition for an operator in $\mathcal{T}(K_\theta, K_\theta^\perp)$ to intertwine D_z^θ and A_z^θ .

Corollary 14. *Let $B_\varphi^\theta \in \mathcal{B}(K_\theta, K_\theta^\perp)$. The operator B_φ^θ intertwines A_z^θ and D_z^θ , i.e., $B_\varphi^\theta A_z^\theta = D_z^\theta B_\varphi^\theta$ if and only if*

- (1) $\varphi = c$ for some $c \in \mathbb{C}$, if $\theta(0) \neq 0$;
- (2) $\varphi \in \bar{\theta}K_\theta + \mathbb{C}$, if $\theta(0) = 0$.

Note that in the first case $B_\varphi^\theta = 0$ and in the second $B_\varphi^\theta = B_{\varphi_1}^\theta$ with $\varphi_1 \in \bar{\theta}K_\theta$.

Proof. Theorem 11 (3) says that $B_\varphi^\theta A_z^\theta = D_z^\theta B_\varphi^\theta$ if and only if

$$\theta \otimes (C_\theta P_\theta(\varphi k_0^\theta)) = (P_\theta^\perp(\varphi\theta)) \otimes C_\theta k_0^\theta.$$

There are two possible cases

- (i) $P_\theta(\varphi k_0^\theta) = 0$ and $P_\theta^\perp(\varphi\theta) = 0$, or
- (ii) $C_\theta P_\theta(\varphi k_0^\theta) = \bar{c}C_\theta k_0^\theta$ and $P_\theta^\perp(\varphi\theta) = c\theta$ for some $c \in \mathbb{C}$.

Consider the first case. Then $\varphi k_0^\theta = \varphi - \overline{\theta(0)}\varphi\theta \perp K_\theta$ and $\varphi\theta \perp \theta H^2$. Note that by the latter, $\varphi \perp H^2 \supset K_\theta$. Thus, if $\theta(0) \neq 0$, then $\varphi\theta \perp K_\theta$ which, together with $P_\theta^\perp(\varphi\theta) = 0$, implies that $\varphi\theta = 0$, i.e., $\varphi = 0$. If $\theta(0) = 0$, then $P_\theta^\perp(\varphi\theta) = 0$ implies $\varphi\theta \in K_\theta$, i.e., $\varphi \in \bar{\theta}K_\theta$.

If (ii) is fulfilled, then

$$P_\theta((\varphi - c)k_0^\theta) = 0 \quad \text{and} \quad P_\theta^\perp((\varphi - c)\theta) = 0,$$

which, reasoning as previously, gives (1) or (2). $\square$

5. Other Relations with Compressions of M_z

If we consider the whole space L^2 and an operator $A \in \mathcal{B}(L^2)$, then A fulfils the relation $A = M_z^* A M_z$ if and only if A commutes with M_z . On the other hand, it is not the case when we consider the space H^2 and the unilateral shift T_z as a compression of M_z . The classical Brown–Halmos result shows that a bounded linear operator $T \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T = (T_z)^* T T_z$. Similar characterizations (in terms of compressions of M_z) are known for Hankel operators and dual Toeplitz operators. In [22] D. Sarason characterized bounded truncated Toeplitz operators in terms of the compressions of M_z to K_θ . In particular, he proved that a bounded operator $A \in \mathcal{B}(K_\theta)$ is a truncated Toeplitz operator if and only if

$$A - A_z^\theta A A_{\bar{z}}^\theta = \psi \otimes k_0^\theta + k_0^\theta \otimes \chi \quad (5.1)$$

for some $\psi, \chi \in K_\theta$. In other words, the left hand side of (5.1) can be expressed as an operator of rank at most two. In this section our aim is to give similar expressions for operators from $\mathcal{T}(K_\theta, K_\theta^\perp)$ and $\mathcal{T}(K_\theta^\perp)$ using operators of rank at most two.

Proposition 15. *Let $D_\varphi^\theta \in \mathcal{T}(K_\theta^\perp)$. Then*

$$D_\varphi^\theta - D_z^\theta D_\varphi^\theta D_{\bar{z}}^\theta = C_\theta P_\theta^\perp(\bar{\varphi} \bar{z} k_0^\theta) \otimes \theta + \theta \otimes C_\theta P_\theta^\perp(\bar{z} P_\theta^\perp(\varphi k_0^\theta)). \quad (5.2)$$

Remark 16. The characterization (5.1) proved in [22] for truncated Toeplitz operators immediately gives a symbol of the truncated Toeplitz operator $A = A_{\psi+\bar{\chi}}^\theta$. Moreover, the relation between ψ and χ is simple, see [22, Corollary after Theorem 3.1]. However, for any dual truncated Toeplitz operator, the functions $\mu = C_\theta P_\theta^\perp(\bar{\varphi} \bar{z} k_0^\theta), \nu = C_\theta P_\theta^\perp(\bar{z} P_\theta^\perp(\varphi k_0^\theta)) \in K_\theta^\perp$ in the formula (5.2), strongly and in a very complicated way depend on each other. Moreover, in case of dual truncated Toeplitz operators, having the rank-two operator on

the right hand side of (5.2), $\mu \otimes \theta + \theta \otimes \nu$ with $\mu, \nu \in K_\theta^\perp$ we are far from obtaining the symbol of D . For this reason, to answer a natural question when an operator $D \in \mathcal{B}(K_\theta^\perp)$ is a DTTO and to find its symbol, we will consider restrictions of D to some subspaces of $K_\theta^\perp$. This will be done in Theorem 27.

Proof of Proposition 15. By Theorem 11 (1) we have since

$$D_\varphi^\theta D_z^\theta D_{\bar{z}}^\theta - D_z^\theta D_\varphi^\theta D_{\bar{z}}^\theta = \theta \otimes D_z^\theta C_\theta P_\theta^\perp(\varphi k_0^\theta) - P_\theta^\perp(\varphi k_0^\theta) \otimes D_z^\theta \bar{z}.$$

$D_z^\theta \bar{z} = P_\theta^\perp 1 = \overline{\theta(0)}\theta$, using Proposition 8 (1) we get

$$\begin{aligned} D_\varphi^\theta - D_z^\theta D_\varphi^\theta D_{\bar{z}}^\theta \\ = ((1 - |\theta(0)|^2)D_\varphi^\theta \theta - \theta(0)P_\theta^\perp(\varphi k_0^\theta)) \otimes \theta + \theta \otimes D_z^\theta C_\theta P_\theta^\perp(\varphi k_0^\theta). \end{aligned} \quad (5.3)$$

Since $D_\varphi^\theta(\theta) = P_\theta^\perp(\theta\varphi)$, we obtain

$$\begin{aligned} (1 - |\theta(0)|^2)D_\varphi^\theta \theta - \theta(0)P_\theta^\perp(\varphi k_0^\theta) &= P_\theta^\perp(\theta\varphi - |\theta(0)|^2\theta\varphi - \theta(0)\varphi k_0^\theta) \\ &= P_\theta^\perp(\theta\varphi - |\theta(0)|^2\theta\varphi - \theta(0)\varphi + |\theta(0)|^2\theta\varphi) \\ &= P_\theta^\perp(\varphi(\theta - \theta(0))) = P_\theta^\perp C_\theta(\bar{\varphi}\bar{z}k_0^\theta) = C_\theta P_\theta^\perp(\bar{\varphi}\bar{z}k_0^\theta). \end{aligned}$$

The proof will be completed with

$$D_z^\theta C_\theta P_\theta^\perp(\varphi k_0^\theta) = P_\theta^\perp(zC_\theta P_\theta^\perp(\varphi k_0^\theta)) = C_\theta P_\theta^\perp(\bar{z}P_\theta^\perp(\varphi k_0^\theta)).$$

□

Proposition 17. Let $B_\varphi^\theta \in \mathcal{T}(K_\theta, K_\theta^\perp)$. Then

$$B_\varphi^\theta - D_z^\theta B_\varphi^\theta A_{\bar{z}}^\theta = \theta \otimes (P_\theta(zP_\theta(\bar{\varphi}\tilde{k}_0^\theta))) + (P_\theta^\perp\varphi) \otimes k_0^\theta. \quad (5.4)$$

Proof. By Theorem 11 (3) we have

$$B_\varphi^\theta A_z^\theta A_{\bar{z}}^\theta - D_z^\theta B_\varphi^\theta A_{\bar{z}}^\theta = \theta \otimes (A_z^\theta C_\theta P_\theta(\varphi k_0^\theta)) - (P_\theta^\perp(\varphi\theta)) \otimes A_z^\theta C_\theta k_0^\theta.$$

By (3.2) it follows that

$$B_\varphi^\theta(I_{K_\theta} - k_0^\theta \otimes k_0^\theta) - D_z^\theta B_\varphi^\theta A_{\bar{z}}^\theta = \theta \otimes (A_z^\theta P_\theta(\bar{\varphi}\tilde{k}_0^\theta)) - (P_\theta^\perp(\varphi\theta)) \otimes (A_z^\theta \tilde{k}_0^\theta).$$

Since $A_z^\theta \tilde{k}_0^\theta = -\theta(0)k_0^\theta$ (see [22, Lemma 2.2]) and $A_z^\theta P_\theta(\bar{\varphi}\tilde{k}_0^\theta) = P_\theta(zP_\theta(\bar{\varphi}\tilde{k}_0^\theta))$, we have finally,

$$B_\varphi^\theta - D_z^\theta B_\varphi^\theta A_{\bar{z}}^\theta = \theta \otimes (P_\theta(zP_\theta(\bar{\varphi}\tilde{k}_0^\theta))) + \overline{\theta(0)}(P_\theta^\perp(\varphi\theta)) \otimes k_0^\theta + (B_\varphi^\theta k_0^\theta) \otimes k_0^\theta.$$

$$B_\varphi^\theta k_0^\theta + \overline{\theta(0)}P_\theta^\perp(\varphi\theta) = P_\theta^\perp(\varphi(1 - \overline{\theta(0)}\theta)) + \overline{\theta(0)}P_\theta^\perp(\varphi\theta) = P_\theta^\perp\varphi,$$

which completes the proof. □

Corollary 18. For each $\varphi \in L^2$:

1. if $D_\varphi^\theta \in \mathcal{T}(K_\theta^\perp)$, then $D_\varphi^\theta - D_{\bar{z}}^\theta D_\varphi^\theta D_z^\theta = P_\theta^\perp(\varphi\bar{z}k_0^\theta) \otimes \bar{z} + \bar{z} \otimes P_\theta^\perp(\bar{z}P_\theta^\perp(\bar{\varphi}k_0^\theta))$;
2. if $B_\varphi^\theta \in \mathcal{T}(K_\theta, K_\theta^\perp)$, then $B_\varphi^\theta - D_{\bar{z}}^\theta B_\varphi^\theta A_z^\theta = C_\theta P_\theta^\perp\bar{\varphi} \otimes \tilde{k}_0^\theta + \bar{z} \otimes P_\theta(\bar{z}P_\theta(\bar{\varphi}k_0^\theta))$.

Proof. We apply the conjugation C_θ on (5.2) and (5.4) substituting φ with $\bar{\varphi}$. Then by Corollary 2, we have

$$D_\varphi^\theta - D_{\bar{z}}^\theta D_\varphi^\theta D_z^\theta = P_\theta^\perp (\varphi \bar{z} k_0^\theta) \otimes C_\theta \theta + C_\theta \theta \otimes P_\theta^\perp (\bar{z} P_\theta^\perp (\bar{\varphi} k_0^\theta)).$$

Now (1) follows, since $C_\theta \theta = \bar{z}$. Similarly,

$$B_\varphi^\theta - D_{\bar{z}}^\theta B_\varphi^\theta A_z^\theta = \bar{z} \otimes C_\theta (P_\theta (z P_\theta (\bar{\varphi} \tilde{k}_0^\theta))) + C_\theta P_\theta^\perp \bar{\varphi} \otimes \tilde{k}_0^\theta.$$

We obtain (2) showing that

$$\begin{aligned} C_\theta (P_\theta (z P_\theta (\varphi \tilde{k}_0^\theta))) &= P_\theta C_\theta (z P_\theta C_\theta (\bar{\varphi} k_0^\theta)) \\ &= P_\theta C_\theta (z C_\theta P_\theta (\bar{\varphi} k_0^\theta)) = P_\theta (\bar{z} P_\theta (\bar{\varphi} k_0^\theta)). \end{aligned}$$

□

6. Restrictions of D_φ^θ

Let θ be an inner function, and let $D \in \mathcal{B}(K_\theta^\perp)$. Using the decomposition $K_\theta^\perp = \theta H^2 \oplus H_-^2$ we can write D as a matrix

$$D = \begin{bmatrix} P_{\theta H^2} D|_{\theta H^2} & P_{\theta H^2} D|_{H_-^2} \\ P^- D|_{\theta H^2} & P^- D|_{H_-^2} \end{bmatrix}.$$

In particular, for $\varphi \in L^\infty$, we have

$$D_\varphi^\theta = \begin{bmatrix} \hat{T}_\varphi^\theta & \check{T}_\varphi^\theta \\ \hat{\Gamma}_\varphi^\theta & \check{\Gamma}_\varphi^\theta \end{bmatrix} = \begin{bmatrix} \hat{T}_\varphi^\theta & (\hat{\Gamma}_\varphi^\theta)^* \\ \hat{\Gamma}_\varphi^\theta & \check{\Gamma}_\varphi^\theta \end{bmatrix},$$

where

$$\begin{aligned} \hat{T}_\varphi^\theta &= P_{\theta H^2} M_{\varphi|_{\theta H^2}}, & \check{T}_\varphi^\theta &= P_{\theta H^2} M_{\varphi|_{H_-^2}}, \\ \hat{\Gamma}_\varphi^\theta &= P^- M_{\varphi|_{\theta H^2}}, & \check{\Gamma}_\varphi^\theta &= P^- M_{\varphi|_{H_-^2}}. \end{aligned}$$

This notation will be used in the following sections. We also set the notation

$$\begin{aligned} \mathcal{T}(\theta H^2) &= \{\hat{T} \in \mathcal{B}(\theta H^2) : \hat{T} = \hat{T}_\varphi^\theta \text{ for some } \varphi \in L^\infty\}, \\ \mathcal{T}(H_-^2) &= \{\check{T} \in \mathcal{B}(H_-^2) : \check{T} = \check{T}_\varphi^\theta \text{ for some } \varphi \in L^\infty\}, \\ \mathcal{T}(\theta H^2, H_-^2) &= \{\hat{\Gamma} \in \mathcal{B}(\theta H^2, H_-^2) : \hat{\Gamma} = \hat{\Gamma}_\varphi^\theta = P^- M_{\varphi|_{\theta H^2}} \text{ for } \varphi \in L^\infty\}, \\ \mathcal{T}(H_-^2, \theta H^2) &= \{\check{\Gamma} \in \mathcal{B}(H_-^2, \theta H^2) : \check{\Gamma} = \check{\Gamma}_\varphi^\theta = P_{\theta H^2} M_{\varphi|_{H_-^2}} \text{ for } \varphi \in L^\infty\}. \end{aligned}$$

Note that $\check{\Gamma}_\varphi^\theta = (\hat{\Gamma}_\varphi^\theta)^*$ and so $\check{\Gamma} \in \mathcal{T}(H_-^2, \theta H^2)$ if and only if $\check{\Gamma}^* \in \mathcal{T}(\theta H^2, H_-^2)$.

It is a part of common knowledge that the space of classical Toeplitz operators $\mathcal{T}(H^2)$ is isomorphic to the space $\mathcal{T}(\theta H^2)$ and to $\mathcal{T}(H_-^2)$, but we present some lemmas for completeness and to fix the notations of these spacial isomorphisms. Recall also the notation $\mathcal{H}(H^2, H_-^2)$ for the space of all Hankel operators.

Consider the operator J , where $J: L^2 \rightarrow L^2$, $Jf(z) = \overline{zf(z)}$, $z \in \mathbb{T}$. Note that J maps antilinearly H^2 onto H_-^2 and conversely bijectively. Moreover,

$J^{-1} = J = J^\sharp$ (by $\sharp$ we denote the antilinear adjoint). For more properties of antilinear operators authors send the reader to [19]. Consider also the operator M_θ and note that M_θ maps H^2 onto θH^2 bijectively. Moreover $M_\theta^{-1} = M_{\bar{\theta}}$.

The following properties can be easily verified.

Proposition 19. (1) $\langle f_1, f_2 \rangle = \langle \theta f_1, \theta f_2 \rangle = \langle Jf_2, Jf_1 \rangle$ for $f_1, f_2 \in L^2$;

(2) $P_{\theta H^2} = M_\theta P^+ M_{\bar{\theta}}$;

(3) $P^- = JP^+J$;

(4) $M_\theta(f_1 \otimes f_2)M_{\bar{\theta}} = \theta f_1 \otimes \theta f_2$ for $f_1, f_2 \in L^2$;

(5) $J(f_1 \otimes f_2)J = Jf_1 \otimes Jf_2$ for $f_1, f_2 \in L^2$;

(6) $M_\theta JM_\theta = J$;

(7) $JM_\varphi = M_{\bar{\varphi}}J$ for $\varphi \in L^\infty$.

In particular $J1 = \bar{z}$ and $J(1 \otimes 1)J = \bar{z} \otimes \bar{z}$. From these properties we get:

Proposition 20. Let $\varphi \in L^\infty$. Then

(1) $\hat{T}_\varphi^\theta = P_{\theta H^2} M_{\varphi|_{\theta H^2}} = M_\theta T_\varphi M_{\bar{\theta}}|_{\theta H^2}$;

(2) $\hat{T}_\varphi^\theta = M_{\varphi|_{\theta H^2}} = T_{\varphi|_{\theta H^2}}$ if $\varphi \in H^\infty$;

(3) $\hat{T}_\varphi = P^- M_{\varphi|_{H_-^2}} = JT_{\bar{\varphi}}J|_{H_-^2}$;

(4) $\hat{T}_\varphi = M_{\varphi|_{H_-^2}}$, if $\varphi \in \overline{H^\infty}$;

(5) $\hat{I}_\varphi^\theta = P^- M_{\varphi|_{\theta H^2}} = H_{\varphi\theta} M_{\bar{\theta}}|_{\theta H^2}$;

(6) $\check{I}_\varphi^\theta = P_{\theta H^2} M_{\varphi|_{H_-^2}} = M_\theta H_{\bar{\theta}\bar{\varphi}}^*$.

This proposition shows that the symbols of $\hat{T}_\varphi^\theta$ and $\check{T}_\varphi$ are unique, and that $\hat{I}_\varphi^\theta$ and $\check{I}_\varphi^\theta$ are uniquely determined by $P^-(\varphi\theta)$ and $P^-(\theta\bar{\varphi})$, respectively. Note that for $\varphi \in L^\infty$, $\bar{\theta}P^-(\theta\varphi)$ is the orthogonal projection of φ onto $\overline{\theta z H^2}$. Thus

$$\hat{I}_\varphi^\theta = \hat{I}_\psi \quad \text{if and only if} \quad (\varphi - \psi) \perp \overline{\theta z H^2}. \quad (6.1)$$

In particular, $\hat{I}_\varphi^\theta = 0$ if $\varphi \in \bar{\theta}H^\infty \supset H^\infty$. Similarly, $\check{I}_\varphi^\theta = 0$ if $\varphi \in \theta\overline{H^\infty} \supset \overline{H^\infty}$.

Proposition 20 implies the following.

Corollary 21. (1) $\hat{T}_z^\theta = M_{z|_{\theta H^2}}$, $\hat{T}_{\bar{z}}^\theta = M_{\bar{z}|_{\theta H^2}} - (\theta\bar{z} \otimes \theta)|_{\theta H^2}$;

(2) $\check{T}_z = M_{z|_{H_-^2}} - (1 \otimes \bar{z})|_{H_-^2}$, $\check{T}_{\bar{z}} = M_{\bar{z}|_{H_-^2}}$;

(3) $\hat{I}_z^\theta = 0$, $\hat{I}_{\bar{z}}^\theta = (\bar{z} \otimes 1)|_{\theta H^2}$, $\hat{I}_{\bar{z}}^\theta = 0$ if $\theta(0) = 0$;

(4) $\check{I}_z^\theta = (\overline{\theta(0)} \theta \otimes \bar{z})|_{H_-^2}$, $\check{I}_{\bar{z}}^\theta = 0$, $\check{I}_{\bar{z}}^\theta = 0$ if $\theta(0) = 0$.

Proof. For (1) take $f \in \theta H^2$. Then

$$\begin{aligned} \hat{T}_z^\theta f &= \theta P^+ \bar{\theta} z f = \theta P^+ \bar{z}(\bar{\theta} f) = \theta \bar{z}(\bar{\theta} f - (\bar{\theta} f)(0)) \\ &= \bar{z} f - \theta \bar{z} \langle \bar{\theta} f, 1 \rangle = \bar{z} f - \theta \bar{z} \langle f, \theta \rangle = \bar{z} f - (\theta \bar{z} \otimes \theta) f. \end{aligned}$$

To obtain (2) take $f \in H_-^2$,

$$\check{T}_z f = P^- z f = z f - P^+(z f) = z f - \langle f, \bar{z} \rangle = z f - (1 \otimes \bar{z}) f.$$

Observe that (3) follows, since for $f \in H^2$, we have

$$\hat{I}_{\bar{z}}^\theta(\theta f) = P^- \bar{z} \theta f = \theta(0) f(0) \bar{z} = \langle \theta f, 1 \rangle \bar{z} = (\bar{z} \otimes 1)(\theta f).$$

To note (4) take $f \in H_-^2$,

$$\begin{aligned} \check{I}_z^\theta f &= \theta P^+ \bar{\theta} z f = \theta P^+ z(\bar{\theta} f) = \theta \langle \bar{\theta} f, \bar{z} \rangle = \theta \langle f, \theta \bar{z} \rangle \\ &= \langle f, P^-(\theta \bar{z}) \rangle \theta = \langle f, \theta(0) \bar{z} \rangle \theta = \overline{\theta(0)} \langle f, \bar{z} \rangle \theta = \overline{\theta(0)} (\theta \otimes \bar{z}) f. \end{aligned}$$

□

The conjugation C_θ (see (2.1)) can be expressed as $C_\theta = M_\theta J = J M_{\bar{\theta}}$, hence

Proposition 22. For $\varphi \in L^\infty$,

- (1) $\hat{T}_\varphi^\theta = C_\theta \check{T}_{\bar{\varphi}} C_{\theta|_{\theta H^2}};$
- (2) $\check{I}_\varphi^\theta = C_\theta \hat{I}_{\bar{\varphi}}^\theta C_{\theta|_{\theta H^2}}.$

Proof. To prove (1), note that by Propositions 19 and 20 we have

$$\begin{aligned} \hat{T}_\varphi^\theta &= M_\theta T_\varphi M_{\bar{\theta}|_{\theta H^2}} = M_\theta (J P^- J) M_\varphi J (J M_{\bar{\theta}})|_{\theta H^2} \\ &= (M_\theta J) P^- M_{\bar{\varphi}} P^- J M_{\bar{\theta}|_{\theta H^2}} = C_\theta \check{T}_{\bar{\varphi}} C_{\theta|_{\theta H^2}}. \end{aligned}$$

The condition (2) can be also obtained from the propositions above, since

$$\begin{aligned} \hat{I}_\varphi^\theta &= P^- M_{\varphi|_{\theta H^2}} = (J P^+ J) M_\varphi M_\theta J (J M_{\bar{\theta}})|_{\theta H^2} = J P^+ M_{\bar{\theta}} M_{\bar{\varphi}} C_{\theta|_{\theta H^2}} \\ &= M_\theta J (M_\theta P^+ M_{\bar{\theta}}) M_{\bar{\varphi}} P^- C_{\theta|_{\theta H^2}} = C_\theta P_{\theta H^2} M_{\bar{\varphi}} P^- C_{\theta|_{\theta H^2}} = C_\theta \check{I}_{\bar{\varphi}}^\theta C_{\theta|_{\theta H^2}}. \end{aligned}$$

□

From Proposition 20 (1), (2) we obtain, in particular, that

Corollary 23. For $\varphi_1, \varphi_2 \in L^\infty$,

- (1) $\hat{T}_{\varphi_1}^\theta \hat{T}_{\varphi_2}^\theta = M_\theta T_{\varphi_1} T_{\varphi_2} M_{\bar{\theta}|_{\theta H^2}};$
- (2) $\check{T}_{\varphi_1} \check{T}_{\varphi_2} = J T_{\bar{\varphi}_1} T_{\bar{\varphi}_2} J;$
- (3) $\hat{T}_{\bar{z}}^\theta \hat{T}_z^\theta = I_{\theta H^2};$
- (4) $\hat{T}_z^\theta \hat{T}_{\bar{z}}^\theta = M_\theta (I - 1 \otimes 1) M_{\bar{\theta}|_{\theta H^2}} = I_{\theta H^2} - \theta \otimes \theta|_{\theta H^2};$
- (5) $\check{T}_{\bar{z}} \check{T}_z = J (I - 1 \otimes 1) J|_{H_-^2} = I_{H_-^2} - \bar{z} \otimes \bar{z}|_{H_-^2};$
- (6) $\check{T}_z \check{T}_{\bar{z}} = I_{H_-^2}.$

Proposition 24. Let θ be a nonconstant inner function. Then

- (1) $\hat{T} \in \mathcal{T}(\theta H^2)$ if and only if $M_{\bar{\theta}} \hat{T} M_{\theta|_{H^2}} \in \mathcal{T}(H^2);$
- (2) $\check{T} \in \mathcal{T}(H_-^2)$ if and only if $J \check{T} J|_{H^2} \in \mathcal{T}(H^2);$
- (3) $\hat{I} \in \mathcal{T}(\theta H^2, H_-^2)$ if and only if $\hat{I} M_{\theta|_{H^2}} \in \mathcal{H}(H^2, H_-^2);$
- (4) $\check{I} \in \mathcal{T}(H_-^2, \theta H^2)$ if and only if $(M_{\bar{\theta}} \check{I})^* \in \mathcal{H}(H^2, H_-^2).$

Recall that

- (A) if $T \in \mathcal{B}(H^2)$, then $T \in \mathcal{T}(H^2)$ if and only if $T = T_z^* T T_z$ and in that case $T = T_\varphi$ with $\varphi = T(1) + \overline{T^*(1)} - \langle T^*1, 1 \rangle$, (it is Brown–Halmos result, see [11, Theorem 4.16]);
- (B) if $H \in \mathcal{B}(H^2, H_-^2)$, then $H \in \mathcal{H}(H^2, H_-^2)$ if and only if $P^- z H = H T_z$ and in that case $P^- \varphi = H(1)$, see [18, Theorem 1.8, Chapter 1].

Using these results and Proposition 24 we get the following characterizations.

Theorem 25. *Let θ be a nonconstant inner function.*

- (a) Let $\hat{T} \in \mathcal{B}(\theta H^2)$. Then $\hat{T} \in \mathcal{T}(\theta H^2)$ if and only if $\hat{T} = \hat{T}_{\bar{z}}^\theta \hat{T} \hat{T}_z^\theta$ and in that case $\hat{T} = \hat{T}_\varphi^\theta$ with $\varphi = \bar{\theta} \hat{T}(\theta) + \theta \hat{T}^*(\theta) - \langle \hat{T}^* \theta, \theta \rangle$.
- (b) Let $\check{T} \in \mathcal{B}(H_-^2)$. Then $\check{T} \in \mathcal{T}(H_-^2)$ if and only if $\check{T} = \check{T}_{\bar{z}} \check{T} \check{T}_{\bar{z}}$ and, in that case, $\check{T} = \check{T}_\varphi$ with $\varphi = z \check{T} \bar{z} + \bar{z} \check{T}^* \bar{z} - \langle \check{T} \bar{z}, \bar{z} \rangle$.
- (c) Let $\hat{\Gamma} \in \mathcal{B}(\theta H^2, H_-^2)$. Then $\hat{\Gamma} \in \mathcal{T}(\theta H^2, H_-^2)$ if and only if $\hat{T}_z \hat{\Gamma} = \hat{\Gamma} \hat{T}_z^\theta$ and in that case $\hat{\Gamma} = \hat{\Gamma}_\varphi^\theta$ with $P^-(\theta \varphi) = \hat{\Gamma} \theta$.
- (d) Let $\check{\Gamma} \in \mathcal{B}(H_-^2, \theta H^2)$. Then $\check{\Gamma} \in \mathcal{T}(H_-^2, \theta H^2)$ if and only if $\check{\Gamma} \check{T}_{\bar{z}} = \hat{T}_{\bar{z}}^\theta \check{\Gamma}$ and in that case $\check{\Gamma} = \check{\Gamma}_\varphi^\theta$ with $P^-(\theta \bar{\varphi}) = \check{\Gamma}^* \theta$.

Proof. By Proposition 24 (1) and (A), if $\hat{T} \in \mathcal{T}(\theta H^2)$, then $\mathcal{T}(H^2) \ni M_{\bar{\theta}} \hat{T} M_{\theta|H^2} = T_z^* M_{\bar{\theta}} \hat{T} M_{\theta} T_z$. Equivalently $\hat{T} = (M_{\theta} T_z M_{\bar{\theta}}) \hat{T} (M_{\theta} T_z M_{\bar{\theta}})_{|\theta H^2} = \hat{T}_{\bar{z}}^\theta \hat{T} \hat{T}_z^\theta$.

The symbol of $\hat{T}$ is the same as the symbol of $T = M_{\bar{\theta}} \hat{T} M_{\theta}$, so

$$\begin{aligned} \varphi &= (M_{\bar{\theta}} \hat{T} M_{\theta})(1) + \overline{(M_{\bar{\theta}} \hat{T} M_{\theta})^*(1)} - \langle (M_{\bar{\theta}} \hat{T} M_{\theta})^* 1, 1 \rangle \\ &= \bar{\theta} \hat{T} \theta + \overline{\bar{\theta} \hat{T}^* \theta} - \langle \bar{\theta} \hat{T}^* \theta, 1 \rangle = \bar{\theta} \hat{T} \theta + \theta \hat{T}^* \theta - \langle \hat{T}^* \theta, \theta \rangle. \end{aligned}$$

To see (b) note that by Proposition 24 (2) and (A), if $\check{T} \in \mathcal{T}(H_-^2)$, then $\mathcal{T}(H^2) \ni J \check{T} J_{|H^2} = T_z^* J \check{T} J T_z$. Equivalently $\check{T} = (J T_z J) \check{T} J T_z J_{|H_-^2} = \check{T}_{\bar{z}} \check{T} \check{T}_{\bar{z}}$. In that case its symbol is the conjugate of the symbol of $T = J \check{T} J \in \mathcal{T}(H^2)$, hence

$$\bar{\varphi} = J \check{T} J(1) + \overline{J \check{T}^* J(1)} - \langle J \check{T}^* J 1, 1 \rangle = \bar{z} \check{T}(\bar{z}) + z \check{T}^*(\bar{z}) - \langle \check{T} \bar{z}, \bar{z} \rangle.$$

To prove (c) we apply Proposition 24 (3) and (B). We have that $\hat{\Gamma} \in \mathcal{T}(\theta H^2, H_-^2)$ if and only if $P^- z \hat{\Gamma} M_{\theta|H^2} = \hat{\Gamma} M_{\theta} T_z$. Equivalently,

$$\check{T}_z \hat{\Gamma} = P^- z P^- \hat{\Gamma} = \hat{\Gamma} M_{\theta} T_z M_{\bar{\theta}|H^2} = \hat{\Gamma} \hat{T}_z^\theta.$$

In that case $\hat{\Gamma} = \hat{\Gamma}_\varphi$ where $\theta \varphi$ is a symbol for the Hankel operator $\hat{\Gamma} M_{\theta|H^2}$ (by Proposition 20 (5)), thus $P^-(\theta \varphi) = \hat{\Gamma} M_{\theta}(1) = \hat{\Gamma} \theta$.

To obtain the last condition we apply Proposition 24 (4) and (B). Note that $\check{\Gamma} \in \mathcal{T}(H_-^2, \theta H^2)$ if and only if $\check{\Gamma}^* M_{\theta|H^2} \in \mathcal{H}(H^2, H_-^2)$. Equivalently, $P^- z \check{\Gamma}^* M_{\theta|H^2} = \check{\Gamma}^* M_{\theta} T_z$. Hence $\check{T}_z \check{\Gamma}^* = \check{\Gamma}^* M_{\theta} T_z M_{\bar{\theta}|H^2}$. Finally, $\check{T}_z \check{\Gamma}^* = \check{\Gamma}^* \hat{T}_z^\theta$, which is the same as $\check{\Gamma} \check{T}_{\bar{z}} = \hat{T}_{\bar{z}}^\theta \check{\Gamma}$. In that case $\check{\Gamma} = \check{\Gamma}_\varphi^\theta$ where $\theta \bar{\varphi}$ is a

symbol of the Hankel operator $(M_{\bar{\theta}}\check{I})^* = \check{I}^*M_{\theta}|_{H^2}$, so $P^-(\theta\bar{\varphi}) = \check{I}^*M_{\theta}1 = \check{I}^*\theta$. $\square$

At the end of this section we will give formulas using rank-two operators corresponding to relations in Lemma 1.

Proposition 26. *Let $\varphi \in L^\infty$.*

- (1) *If $\hat{T} = \hat{T}_\varphi^\theta \in \mathcal{T}(\theta H^2)$, then $\hat{T} - \hat{T}_z^\theta \hat{T} \hat{T}_z^\theta = (\theta \otimes h + g \otimes \theta)|_{\theta H^2}$ with $h = \theta \overline{P^-} \varphi \in \theta H^2$, $g = \theta P^+ \varphi \in \theta H^2$.*
- (2) *If $\hat{T} = \hat{T}_\varphi \in \mathcal{T}(H_-^2)$, then $\hat{T} - \hat{T}_z \hat{T} \hat{T}_z = (\bar{z} \otimes h + g \otimes \bar{z})|_{H_-^2}$ with $h = \bar{z} P^- \bar{\varphi} \in H_-^2$, $g = P^-(\bar{z} \varphi) \in H_-^2$.*
- (3) *If $\hat{I} = \hat{I}_\varphi^\theta \in \mathcal{T}(\theta H^2, H_-^2)$, then $\hat{T}_z \hat{I} - \hat{I} \hat{T}_z^\theta = (\bar{z} \otimes h + g \otimes \bar{z})|_{\theta H^2}$ with $h = -P_{\theta H^2} \bar{\varphi} \in \theta H^2$, $g = P^-(C_\theta \bar{\varphi}) \in H_-^2$.*
- (4) *If $\check{I} = \check{I}_\varphi^\theta \in \mathcal{T}(H_-^2, \theta H^2)$, then $\check{I} \hat{T}_z - \hat{T}_z^\theta \check{I} = (\theta \otimes h + g \otimes \bar{z})|_{H_-^2}$ with $h = P^-(C_\theta \varphi) \in H_-^2$, $g = -P_{\theta H^2} \varphi \in \theta H^2$.*

Proof. For the proof of (1) note that by Proposition 20, $\hat{T}_\varphi^\theta = M_\theta T_\varphi M_{\bar{\theta}}|_{\theta H^2}$, and so we have $\hat{T}_\varphi^\theta - \hat{T}_z^\theta \hat{T}_\varphi^\theta \hat{T}_z^\theta = M_\theta (T_\varphi - T_z T_\varphi T_z) M_{\bar{\theta}}$. Since

$$T_\varphi - T_z T_\varphi T_z = 1 \otimes \overline{P^-} \varphi + P^+ \varphi \otimes 1,$$

by Proposition 19 (4) we get

$$\hat{T}_\varphi^\theta - \hat{T}_z^\theta \hat{T}_\varphi^\theta \hat{T}_z^\theta = \theta \otimes \overline{\theta P^-} \varphi + \theta P^+ \varphi \otimes \theta.$$

To obtain (2) note that

$$\begin{aligned} \hat{T}_\varphi - \hat{T}_z \hat{T}_\varphi \hat{T}_z &= J(T_\varphi - T_z T_\varphi T_z)J = J(1 \otimes \overline{P^-} \varphi + P^+ \varphi \otimes 1)J \\ &= \bar{z} \otimes \overline{J P^-} \varphi + J P^+ \varphi \otimes \bar{z} = \bar{z} \otimes \bar{z} P^- \bar{\varphi} + P^- J \bar{\varphi} \otimes \bar{z} \\ &= \bar{z} \otimes \bar{z} P^- \bar{\varphi} + P^-(\bar{z} \varphi) \otimes \bar{z} \end{aligned}$$

by Propositions 19 and 20 (5). Now to show (3) take $f \in \theta H^2$. Then by Lemma 1 we have

$$\begin{aligned} (\hat{T}_z \hat{I}_\varphi^\theta - \hat{I}_\varphi^\theta \hat{T}_z^\theta) f &= \bar{z} P^- \varphi f - P^- \varphi (\bar{z} f - \theta \bar{z} (\bar{\theta} f)(0)) \\ &= \bar{z} P^- \varphi f - P^- \bar{z} \varphi f + (\bar{\theta} f)(0) P^- \theta \bar{z} \varphi = -\langle \varphi f, 1 \rangle \bar{z} + \langle \bar{\theta} f, 1 \rangle P^-(C_\theta \bar{\varphi}) \\ &= -\langle f, P_{\theta H^2} \bar{\varphi} \rangle \bar{z} + \langle f, \theta \rangle P^-(C_\theta \bar{\varphi}) = (-\bar{z} \otimes P_{\theta H^2} \bar{\varphi} + P^-(C_\theta \bar{\varphi}) \otimes \theta) f \end{aligned}$$

using Proposition 20 and Corollary 21 (1). To prove (4) take $f \in H_-^2$ and note that, by Lemma 1 (2) and Proposition 20, we have

$$\begin{aligned} (\check{I}_\varphi \hat{T}_z - \hat{T}_z^\theta \check{I}_\varphi) f &= \theta P^+ \bar{\theta} \varphi \check{I}_z f - z \theta P^+ \bar{\theta} \varphi f = \theta (P^+ \bar{\theta} \varphi (z f - \langle f, \bar{z} \rangle) - z P^+ \bar{\theta} \varphi f) \\ &= \theta ((P^+ z - z P^+) \bar{\theta} \varphi f - \langle f, \bar{z} \rangle P^+ \bar{\theta} \varphi) = \theta (\langle \bar{\theta} \varphi f, \bar{z} \rangle - \langle f, \bar{z} \rangle P^+ (\bar{\theta} \varphi)) \\ &= \langle f, C_\theta \varphi \rangle \theta - \langle f, \bar{z} \rangle \theta P^+ (\bar{\theta} \varphi) = (\theta \otimes P^-(C_\theta \varphi) - P_{\theta H^2}(\varphi) \otimes \bar{z}) f. \end{aligned}$$

$\square$

7. Characterization of Dual Truncated Toeplitz Operators

The restrictions of dual truncated Toeplitz operators considered in Sect. 6 will be used to give a necessary and sufficient condition for any operator $D \in \mathcal{B}(K_\theta^\perp)$ to be a DTTO. Moreover, we can easily recover its unique symbol.

Theorem 27. *Let θ be an inner function and let $D \in \mathcal{B}(K_\theta^\perp)$. Then D is a dual truncated Toeplitz operator, $D \in \mathcal{T}(K_\theta^\perp)$, if and only if the following conditions hold*

- (1) $P_{\theta H^2} D|_{\theta H^2} = \hat{T}_z^\theta P_{\theta H^2} D|_{\theta H^2} \hat{T}_z^\theta$;
- (2) $P^- D|_{H_-^2} = C_\theta P_{\theta H^2} D|_{\theta H^2}^* C_\theta$;
- (3) $P^- D|_{\theta H^2} \hat{T}_z^\theta = \hat{T}_z P^- D|_{\theta H^2}$ and $(P_{\theta H^2} D|_{H_-^2})^* \hat{T}_z^\theta = \hat{T}_z (P_{\theta H^2} D|_{H_-^2})^*$;
- (4) $P^-(D(\theta)) = P^-(\theta^2 \overline{D^*(\theta)})$ and $P^-(D^*(\theta)) = P^-(\theta^2 \overline{D(\theta)})$.

In that case, $D = D_\varphi^\theta$ with $\varphi \in L^\infty$ given by

$$\varphi = P^+(\bar{\theta} D(\theta)) + \overline{P^+(\bar{\theta} D^*(\theta))} - \langle D(\theta), \theta \rangle. \quad (7.1)$$

Remark 28. Equality (7.1) gives a formula for the symbol φ of a dual truncated Toeplitz operator D by using the value of D and its adjoint D^* on the function θ . The symbol $\varphi \in L^\infty$ can be equivalently calculated using different formulas. Let $\varphi = \varphi^- + \varphi^+$, $\varphi^- \in H_-^2$, $\varphi^+ \in H^2$ and let $\hat{\varphi}(0)$ denote the 0-th Fourier coefficient of φ . Then using Corollary 2 we have

- (1) $\hat{\varphi}(0) = \langle D\theta, \theta \rangle = \langle D\bar{z}, \bar{z} \rangle$,
- (2) $\varphi^+ = P^+ \bar{\theta} D\theta = \bar{\theta} P_{\theta H^2}(D\theta) = \overline{z P^-(D^* \bar{z})}$,
- (3) $\varphi^- = \overline{P^+(\bar{\theta} D^*(\theta))} - \hat{\varphi}(0) = \overline{z P^+(\bar{z} \bar{\theta} D^*(\theta))} = z P^-(D(\bar{z})) - \hat{\varphi}(0) = P^-(z D(\bar{z}))$.

Proof of Theorem 27. Assume firstly that $D = D_\varphi^\theta$ with $\varphi \in L^\infty$. Then

$$D_\varphi^\theta = \begin{bmatrix} \hat{T}_\varphi^\theta & \tilde{I}_\varphi^\theta \\ \hat{I}_\varphi^\theta & \tilde{T}_\varphi^\theta \end{bmatrix} = \begin{bmatrix} \hat{T}_\varphi^\theta & (\hat{I}_\varphi^\theta)^* \\ \hat{I}_\varphi^\theta & \tilde{T}_\varphi^\theta \end{bmatrix}.$$

Hence (1) is satisfied by Theorem 25 (a), (2) is satisfied by Proposition 22 and (3) is satisfied by Theorem 25 (c), (d). Moreover,

$$D(\theta) = P_\theta^\perp(\varphi\theta) = \theta P^+(\varphi) + P^-(\varphi\theta)$$

and

$$D^*(\theta) = P_\theta^\perp(\bar{\varphi}\theta) = \theta P^+(\bar{\varphi}) + P^-(\bar{\varphi}\theta).$$

It follows that $P^-(D(\theta)) = P^-(\varphi\theta)$ and

$$P^-(\theta^2 \overline{D^*(\theta)}) = P^-(\theta \overline{P^+(\bar{\varphi})}) = P^-(\varphi\theta) = P^-(D(\theta)).$$

Similarly, $P^-(D^*(\theta)) = P^-(\bar{\varphi}\theta)$ and

$$P^-(\theta^2 \overline{D(\theta)}) = P^-(\theta \overline{P^+(\varphi)}) = P^-(\bar{\varphi}\theta) = P^-(D^*(\theta)).$$

Thus (4) is also satisfied.

Assume now that $D = \begin{bmatrix} P_{\theta H^2} D_{|\theta H^2} & P_{\theta H^2} D_{|H_-^2} \\ P^- D_{|\theta H^2} & P^- D_{|H_-^2} \end{bmatrix} \in \mathcal{B}(K_\theta^\perp)$ satisfies (1)–(4). It then follows from (1) and Theorem 25 (a) that $P_{\theta H^2} D_{|\theta H^2} = \hat{T}_\varphi^\theta$ for $\varphi \in L^\infty$ given by

$$\varphi = \bar{\theta} P_{\theta H^2} D_{|\theta H^2}(\theta) + \overline{\theta(P_{\theta H^2} D_{|\theta H^2}^*(\theta))} - \langle P_{\theta H^2} D_{|\theta H^2}(\theta), \theta \rangle. \quad (7.2)$$

By (2) and Proposition 22,

$$P^- D_{|H_-^2} = C_\theta P_{\theta H^2} D_{|\theta H^2}^* C_\theta = C_\theta \hat{T}_\varphi^\theta C_\theta = \check{T}_\varphi.$$

By (3) and Theorem 25 (c), (d) there exist $\psi, \chi \in L^\infty$ such that $P^- D_{|\theta H^2} = \hat{\Gamma}_\psi^\theta$ with $P^-(\theta\psi) = P^- D_{|\theta H^2}(\theta)$ and similarly $(P_{\theta H^2} D_{|H_-^2})^* = \hat{\Gamma}_\chi^\theta$ with $P^-(\theta\chi) = (P_{\theta H^2} D_{|H_-^2})^*(\theta)$. We will now use (4) to show that

$$(\varphi - \psi) \perp \overline{\theta z H^2} \quad \text{and} \quad (\bar{\varphi} - \chi) \perp \overline{\theta z H^2}. \quad (7.3)$$

Since φ is given by (7.2), using the first equality in (4) we get

$$\begin{aligned} P^-(\theta\varphi) &= P^-(\theta^2 \overline{(P_{\theta H^2} D_{|\theta H^2}^*(\theta))}) = P^-(\theta^2 \overline{D^*(\theta)}) \\ &= P^-(D(\theta)) = P^- D_{|\theta H^2}(\theta) = P^-(\theta\psi), \end{aligned}$$

and so $(\theta\varphi - \theta\psi) \perp H_-^2$. Similarly, using the second equality in (4) we get

$$\begin{aligned} P^-(\theta\bar{\varphi}) &= P^-(\theta^2 \overline{(P_{\theta H^2} D_{|\theta H^2}(\theta))}) = P^-(\theta^2 \overline{D(\theta)}) \\ &= P^-(D^*(\theta)) = (P_{\theta H^2} D_{|H_-^2})^*(\theta) = P^-(\theta\chi), \end{aligned}$$

hence $(\theta\bar{\varphi} - \theta\chi) \perp H_-^2$. Thus, by (6.1), we proved that $P^- D_{|\theta H^2} = \hat{\Gamma}_\varphi^\theta$ and $(P_{\theta H^2} D_{|H_-^2})^* = \hat{\Gamma}_{\bar{\varphi}}^\theta$, that is, $P_{\theta H^2} D_{|H_-^2} = (\hat{\Gamma}_{\bar{\varphi}}^\theta)^* = \check{\Gamma}_\varphi^\theta$. Therefore,

$$D_\varphi^\theta = \begin{bmatrix} \hat{\Gamma}_\varphi^\theta & \check{\Gamma}_\varphi^\theta \\ \hat{\Gamma}_{\bar{\varphi}}^\theta & \check{\Gamma}_\varphi^\theta \end{bmatrix}.$$

Moreover,

$$\begin{aligned} \varphi &= P^+(\bar{\theta}(P_{\theta H^2} D_{|\theta H^2}(\theta))) + \overline{P^+(\bar{\theta} P_{\theta H^2} D_{|\theta H^2}^*(\theta))} - \langle P_{\theta H^2} D_{|\theta H^2}(\theta), \theta \rangle \\ &= P^+(\bar{\theta} D(\theta)) + \overline{P^+(\bar{\theta} D^*(\theta))} - \langle D(\theta), \theta \rangle. \end{aligned}$$

□

Example 29. Let $\theta(z) = z^n$, take $D \in \mathcal{B}(K_\theta^\perp)$ and denote by $[a_{i,j}]$ the matrix of D with respect to the monomial basis $\{z^k : k \geq n \vee k < 0\}$. Then the conditions in (4) can be written as

$$a_{-k,n} = a_{n,k+2n} \quad \text{and} \quad a_{n,-k} = a_{k+2n,n} \quad \text{for all } k > 0.$$

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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