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Results in Mathematics

On Ulam Stability of a Functional Equation

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Abstract. In this note, we study the Ulam stability of a functional equation both in Banach and m-Banach spaces. Particular cases of this equation are, among others, equations which characterize multi-additive and multi-Jensen functions. Moreover, it is satisfied by the so-called multi-linear mappings.

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1. Background and Motivation

Assume that X is a linear space over the field $\mathbb{F}$, and Y is a linear space over the field $\mathbb{K}$. Let us recall (see for instance [20]) that a mapping $f : X \to Y$ satisfies a *linear functional equation* provided

$$f(a_1x + a_2y) = A_1f(x) + A_2f(y), \qquad x, y \in X \tag{1}$$

for some $a_1, a_2 \in \mathbb{F}$ and $A_1, A_2 \in \mathbb{K}$.

It is obvious that the functional equation

$$f(x + y) = f(x) + f(y) \tag{2}$$

and, under the additional assumption that the characteristics of $\mathbb{F}$ and $\mathbb{K}$ are different from 2, the equation

$$f\Big(\frac{x + y}{2}\Big) = \frac{f(x) + f(y)}{2}, \tag{3}$$

(their solutions are said to be additive and Jensen mappings, respectively) are particular cases of (1). For more information about equations (2) and (3) and some applications of them ((3) is called the Jensen equation and it is connected with the notion of convexity) we refer the reader for example to [14,15].

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Given an $n \in \mathbb{N}$ (throughout this note $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) such that $n \geq 2$, we will say that a function $f : X^n \to Y$ is *n-linear* (roughly, *multi-linear*) if it satisfies the linear functional equation in each of its arguments, i.e.

$$\begin{aligned}
&f(x_1, \dots, x_{i-1}, a_{i1}x_{i1} + a_{i2}x_{i2}, x_{i+1}, \dots, x_n)\\
&= A_{i1}f(x_1, \dots, x_{i-1}, x_{i1}, x_{i+1}, \dots, x_n)\\
&\quad + A_{i2}f(x_1, \dots, x_{i-1}, x_{i2}, x_{i+1}, \dots, x_n),\\
&i \in \{1, \dots, n\},\ x_1, \dots, x_{i-1}, x_{i1}, x_{i2}, x_{i+1}, \dots, x_n \in X
\end{aligned}$$

with some $a_{i1}, a_{i2} \in \mathbb{F}$ and $A_{i1}, A_{i2} \in \mathbb{K}$.

It is clear that multi-additive functions, introduced by S. Mazur and W. Orlicz (see for example [15], where one can also find their application to the representation of polynomial mappings), and multi-Jensen functions, defined in 2005 by W. Prager and J. Schwaiger (see for instance [21]) with the connection with generalized polynomials, are multi-linear. Moreover, with $k \in \mathbb{N}$ such that $1 \leq k < n$, $a_{11} = a_{12} = \ldots = a_{k1} = a_{k2} = 1,\ a_{k+11} = a_{k+12} = \ldots = a_{n1} = a_{n2} = \frac{1}{2}$ and $A_{11} = A_{12} = \ldots = A_{k1} = A_{k2} = 1,\ A_{k+11} = A_{k+12} = \ldots = A_{n1} = A_{n2} = \frac{1}{2}$ in the above definition we obtain the notion of a k-Cauchy and $n-k$-Jensen (briefly, multi-Cauchy-Jensen) function (see [1,2]).

Let $a_{11}, a_{12}, \dots, a_{n1}, a_{n2} \in \mathbb{F}$ and $A_{i_1,\dots,i_n} \in \mathbb{K}$ for $i_1, \dots, i_n \in \{1, 2\}$ be given scalars. In this paper, we deal with the following quite general functional equation

$$\begin{aligned}
&f(a_{11}x_{11} + a_{12}x_{12}, \dots, a_{n1}x_{n1} + a_{n2}x_{n2}) =\\
&\textstyle\sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,\dots,i_n} f(x_{1i_1}, \dots, x_{ni_n}).
\end{aligned} \tag{4}$$

Now, we present three functional equations which are special cases of the considered equation.

Example 1. Equation (4) with $a_{11} = a_{12} = \ldots = a_{n1} = a_{n2} = 1$ and $A_{i_1,\dots,i_n} = 1$ for $i_1, \dots, i_n \in \{1, 2\}$ leads to the following functional equation

$$f(x_{11} + x_{12}, \dots, x_{n1} + x_{n2}) = \sum_{i_1,\dots,i_n \in \{1,2\}} f(x_{1i_1}, \dots, x_{ni_n}), \tag{5}$$

which (see [9]) characterizes multi-additive mappings.

Example 2. Another particular case of equation (4), i.e. the functional equation

$$f(\frac{1}{2}x_{11} + \frac{1}{2}x_{12}, \dots, \frac{1}{2}x_{n1} + \frac{1}{2}x_{n2}) = \sum_{i_1,\dots,i_n \in \{1,2\}} \frac{1}{2^n} f(x_{1i_1}, \dots, x_{ni_n}) \tag{6}$$

was introduced and investigated in [21].

Example 3. The functional equation

$$f(x_{11}+x_{12},\dots,x_{k1}+x_{k2},\tfrac{1}{2}x_{k+11}+\tfrac{1}{2}x_{k+12},\dots,\tfrac{1}{2}x_{n1}+\tfrac{1}{2}x_{n2}) = \sum_{i_1,\dots,i_n\in\{1,2\}}\frac{1}{2^{n-k}}f(x_{1i_1},\dots,x_{ni_n}), \tag{7}$$

which characterizes multi-Cauchy-Jensen mappings (see [1]), is also a special case of equation (4).

Next, note that we have the following.

Proposition 1. *If $f : X^n \to Y$ is a multi-linear mapping, then there exist $A_{i_1,\dots,i_n} \in \mathbb{K}$ for $i_1,\dots,i_n \in \{1,2\}$ such that equation (4) holds for any $x_{11},x_{12},\dots,x_{n1},x_{n2} \in X$.*

The question about an error we commit replacing an object fulfilling some properties only approximately by an actual object possessing these properties is natural and interesting in many scientific investigations. To deal with it one can use the notion of the Ulam stability.

In 1940, S.M. Ulam posed the problem of the stability of homomorphisms of metric groups (a year later, D.H. Hyers gave its solution in the case of Banach spaces). Another very important example is a question concerning the stability of isometries. This problem was investigated for instance in [3,11,13,19,23] (see also [18] for more information and references on this topic).

Let us recall that an equation is said to be *Ulam stable* in a class of functions provided each function from this class fulfilling our equation "approximately" is "near" to its actual solution.

In recent years, the stability of various objects has been studied by many researchers (for more information on the notion of the Ulam stability as well as its applications we refer the reader to [4,6,7,12,18]). Furthermore, some stability results on equations (5), (6) and (7) can be found, among others, in [1,2,9,21].

In this note, the Ulam stability of equation (4) is shown. Moreover, we apply our main results (Theorems 2 and 7) to get some stability outcomes on functional equations (5) and (7).

Let us finally mention that as the concepts of an approximate solution and the nearness of two mappings can be obviously understood in various ways, we deal with the stability of the considered functional equations not only in classical Banach spaces, but also in m-Banach spaces (i.e. spaces with non-standard measures of the distance, namely the ones given by m-norms).

2. Stability in Banach Spaces

In this section, we prove the Ulam stability of functional equations (4), (5) and (7) in Banach spaces.

2.1. Main Result

We start with equation (4).

Theorem 2. *Assume that Y is a Banach space, $\varepsilon > 0$ and*

$$|\sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n}| > 1. \tag{8}$$

If $f : X^n \to Y$ is a function satisfying

$$\begin{aligned}&\|f(a_{11}x_{11}+a_{12}x_{12},\ldots,a_{n1}x_{n1}+a_{n2}x_{n2})-\\&\textstyle\sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n} f(x_{1i_1},\ldots,x_{ni_n})\| \le \varepsilon\end{aligned} \tag{9}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$, then there is a unique mapping $F : X^n \to Y$ fulfilling equation (4) and

$$\|f(x_1,\ldots,x_n) - F(x_1,\ldots,x_n)\| \le \frac{\varepsilon}{|\sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n}| - 1} \tag{10}$$

for $x_1,\ldots,x_n \in X$.

Proof. Put

$$A := \sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n}, \qquad a_i := a_{i1}+a_{i2}, \quad i\in\{1,\ldots,n\}.$$

Let us first note that (9) with $x_{i2} = x_{i1}$ for $i \in \{1,\ldots,n\}$ gives

$$\|f(a_1x_{11},\ldots,a_nx_{n1}) - Af(x_{11},\ldots,x_{n1})\| \le \varepsilon, \qquad (x_{11},\ldots,x_{n1}) \in X^n,$$

and consequently

$$\begin{aligned}&\left\|\tfrac{f(a_1^{k+1}x_{11},\ldots,a_n^{k+1}x_{n1})}{A^{k+1}} - \tfrac{f(a_1^kx_{11},\ldots,a_n^kx_{n1})}{A^k}\right\| \le \tfrac{\varepsilon}{|A|^{k+1}},\\&(x_{11},\ldots,x_{n1}) \in X^n,\ k\in\mathbb{N}_0.\end{aligned} \tag{11}$$

Fix $l, p \in \mathbb{N}_0$ such that $l < p$. Then

$$\begin{aligned}&\left\|\tfrac{f(a_1^px_{11},\ldots,a_n^px_{n1})}{A^p} - \tfrac{f(a_1^lx_{11},\ldots,a_n^lx_{n1})}{A^l}\right\| \le \sum_{j=l}^{p-1}\tfrac{\varepsilon}{|A|^{j+1}},\\&(x_{11},\ldots,x_{n1}) \in X^n,\end{aligned} \tag{12}$$

and thus for each $(x_{11},\ldots,x_{n1}) \in X^n$, $\left(\frac{f(a_1^kx_{11},\ldots,a_n^kx_{n1})}{A^k}\right)_{k\in\mathbb{N}_0}$ is a Cauchy sequence. Using the fact that Y is a Banach space we conclude that this sequence is convergent, which allows us to define

$$F(x_{11},\ldots,x_{n1}) := \lim_{k\to\infty}\frac{f(a_1^kx_{11},\ldots,a_n^kx_{n1})}{A^k}, \qquad (x_{11},\ldots,x_{n1}) \in X^n. \tag{13}$$

Putting now $l = 0$ and letting $p \to \infty$ in (12) we see that

$$\|f(x_{11},\ldots,x_{n1}) - F(x_{11},\ldots,x_{n1})\| \le \frac{\varepsilon}{|A|-1}, \qquad (x_{11},\ldots,x_{n1}) \in X^n,$$

i.e. condition (10) is satisfied.

Let us next observe that from (9) we get

$$\left\| \tfrac{f(a_1^k(a_{11}x_{11}+a_{12}x_{12}),\ldots,a_n^k(a_{n1}x_{n1}+a_{n2}x_{n2}))}{A^k} - \right.$$

$$\left. \textstyle\sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n} \tfrac{f(a_1^k x_{1i_1},\ldots,a_n^k x_{ni_n})}{A^k} \right\| \le \tfrac{\varepsilon}{|A|^k}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $k \in \mathbb{N}_0$. Letting now $k \to \infty$ and applying definition (13) we deduce that

$$\|F(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) -$$

$$\textstyle\sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n} F(x_{1i_1}, \ldots, x_{ni_n})\| \le 0$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$, and thus we see that the mapping $F : X^n \to Y$ is a solution of functional equation (4).

Let us finally assume that $G : X^n \to Y$ is another mapping fulfilling equation (4) and inequality (10) for $x_1, \ldots, x_n \in X$. Then for any $l \in \mathbb{N}$ and $(x_{11}, \ldots, x_{n1}) \in X^n$ we have

$$\|F(x_{11}, \ldots, x_{n1}) - G(x_{11}, \ldots, x_{n1})\| =$$

$$\left\| \tfrac{F(a_1^l x_{11},\ldots,a_n^l x_{n1})}{A^l} - \tfrac{G(a_1^l x_{11},\ldots,a_n^l x_{n1})}{A^l} \right\| \le$$

$$\tfrac{1}{|A|^l} \big(\|F(a_1^l x_{11}, \ldots, a_n^l x_{n1}) - f(a_1^l x_{11}, \ldots, a_n^l x_{n1})\| +$$

$$\|f(a_1^l x_{11}, \ldots, a_n^l x_{n1}) - G(a_1^l x_{11}, \ldots, a_n^l x_{n1})\| \big) \le \tfrac{1}{|A|^l} \tfrac{2\varepsilon}{|A|-1}.$$

Letting now $l \to \infty$ and using (8) we conclude that $G = F$. □

2.2. Corollaries

Now, we present some consequences of the main result.

First, consider the case $a_{11} = a_{12} = \ldots = a_{n1} = a_{n2} = 1$ and $A_{i_1,\ldots,i_n} = 1$ for $i_1, \ldots, i_n \in \{1, 2\}$. Then from Theorem 2 we get the following outcome on the Ulam stability of functional equation (5).

Corollary 3. *Assume that Y is a Banach space and $\varepsilon > 0$. If $f : X^n \to Y$ is a function satisfying*

$$\|f(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) - \sum_{i_1,\ldots,i_n\in\{1,2\}} f(x_{1i_1}, \ldots, x_{ni_n})\| \le \varepsilon$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$, then there is a unique solution $F : X^n \to Y$ of equation (5) such that

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n)\| \le \frac{\varepsilon}{2^n - 1}, \qquad (x_1, \ldots, x_n) \in X^n.$$

Another consequence of Theorem 2 is a result on the stability of equation (7). Namely, we have the following.

Corollary 4. *Assume that Y is a Banach space and $\varepsilon > 0$. If $f : X^n \to Y$ is a function satisfying*

$$\Big\| f(x_{11} + x_{12}, \ldots, x_{k1} + x_{k2}, \tfrac{1}{2}x_{k+11} + \tfrac{1}{2}x_{k+12}, \ldots, \tfrac{1}{2}x_{n1} + \tfrac{1}{2}x_{n2}) -$$

$$\textstyle\sum_{i_1,\ldots,i_n\in\{1,2\}} \frac{1}{2^{n-k}} f(x_{1i_1}, \ldots, x_{ni_n}) \Big\| \le \varepsilon$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$, then there is a unique solution $F : X^n \to Y$ of equation (7) such that

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n)\| \le \frac{\varepsilon}{2^k - 1}, \qquad (x_1, \ldots, x_n) \in X^n.$$

Let us finally observe that a particular case of Theorem 2 is Theorem 2.1 in [10].

3. Stability in m-Banach Spaces

In this section, we deal with the Ulam stability of the considered equations in m-Banach spaces.

3.1. Preliminaries

In 1989, A. Misiak (see [17]) generalized the notion of 2-normed space introduced by S. Gähler a quarter of a century earlier and defined m-normed spaces. Now, we recall (see for instance [5,17,22]) some basic definitions and facts concerning such spaces.

Let $m \in \mathbb{N}$ be such that $m \ge 2$ and Y be an at least m-dimensional real linear space. If a mapping $\|\cdot, \ldots, \cdot\| : Y^m \to \mathbb{R}$ fulfils the following four conditions:

(i) $\|x_1, \ldots, x_m\| = 0$ if and only if $x_1, \ldots, x_m$ are linearly dependent,

(ii) $\|x_1, \ldots, x_m\|$ is invariant under permutation,

(iii) $\|\alpha x_1, \ldots, x_m\| = |\alpha| \|x_1, \ldots, x_m\|$,

(iv) $\|x + y, x_2, \ldots, x_m\| \le \|x, x_2, \ldots, x_m\| + \|y, x_2, \ldots, x_m\|$

for any $\alpha \in \mathbb{R}$ and $x, y, x_1, \ldots, x_m \in Y$, then it is said to be an *m-norm* on Y, whereas the pair $(Y, \|\cdot, \ldots, \cdot\|)$ is called an *m-normed space.*

Let us mention the following two known properties of m-norms.

Remark 5. Assume that $(Y, \|\cdot, \ldots, \cdot\|)$ is an m-normed space. Then:

(i) the mapping $\|\cdot, \ldots, \cdot\|$ is non-negative;

(ii) for any $k \in \mathbb{N}, x_2, \ldots, x_m \in Y$ and $y_i \in Y$ for $i \in \{1, \ldots, k\}$ we have

$$\Big\| \sum_{i=1}^{k} y_i, x_2, \ldots, x_m \Big\| \le \sum_{i=1}^{k} \|y_i, x_2, \ldots, x_m\|.$$

Let $(y_k)_{k\in\mathbb{N}}$ be a sequence of elements of an m-normed space $(Y, \|\cdot, \ldots, \cdot\|)$. We say that it is *Cauchy sequence* provided

$$\lim_{k,l\to\infty} \|y_k - y_l, x_2, \ldots, x_m\| = 0, \qquad x_2, \ldots, x_m \in Y.$$

On the other hand, the sequence $(y_k)_{k\in\mathbb{N}}$ is called *convergent* if there is a $y \in Y$ such that

$$\lim_{k\to\infty} \|y_k - y, x_2, \ldots, x_m\| = 0, \qquad x_2, \ldots, x_m \in Y.$$

Then the element y is said to be the *limit* of $(y_k)_{k\in\mathbb{N}}$ and it is denoted by $\lim_{k\to\infty} y_k$. Obviously each convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product hold true.

By an *m-Banach space* we mean an m-normed space such that each its Cauchy sequence is convergent.

We will also use the following known facts.

Remark 6. Assume that $(Y, \|\cdot, \ldots, \cdot\|)$ is an m-normed space. Then:

(i) if $x_1, \ldots, x_m \in Y, \alpha \in \mathbb{R}$, $i, j \in \{1, \ldots, m\}$ and $i < j$, then

$$\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m\| = \|x_1, \ldots, x_i, \ldots, x_j + \alpha x_i, \ldots, x_m\|;$$

(ii) if $x, y, y_2, \ldots, y_m \in Y$, then

$$\big| \|x, y_2, \ldots, y_m\| - \|y, y_2, \ldots, y_m\| \big| \le \|x - y, y_2, \ldots, y_m\|;$$

(iii) if $x \in Y$ and

$$\|x, y_2, \ldots, y_m\| = 0, \qquad y_2, \ldots, y_m \in Y,$$

then $x = 0$;

(iv) if $(x_k)_{k\in\mathbb{N}}$ is a convergent sequence of elements of Y, then

$$\lim_{k\to\infty} \|x_k, y_2, \ldots, y_m\| = \Big\| \lim_{k\to\infty} x_k, y_2, \ldots, y_m \Big\|, \qquad y_2, \ldots, y_m \in Y.$$

Let us finally mention that more information on m-normed spaces as well as on some problems investigated in them can be found for example in [5,8,16,17,22].

3.2. Main Result

Now, we prove the Ulam stability of equation (4).

Theorem 7. *Assume that $m \in \mathbb{N}$, $\varepsilon > 0$, Y is an $(m+1)$-Banach space, and (8) holds true. If $f : X^n \to Y$ is a function satisfying*

$$\begin{aligned} &\|f(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) - \\ &\textstyle\sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n} f(x_{1i_1}, \ldots, x_{ni_n}), z\| \le \varepsilon \end{aligned} \tag{14}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ *and* $z \in Y^m$, *then there is a unique mapping* $F : X^n \to Y$ *fulfilling equation* (4) *and*

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n), z\| \leq \frac{\varepsilon}{|\sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n}| - 1} \tag{15}$$

for $x_1, \ldots, x_n \in X$ *and* $z \in Y^m$.

Proof. Let A and a_i for $i \in \{1, \ldots, n\}$ be as in the proof of Theorem 2, and fix $l, p \in \mathbb{N}_0$ such that $l < p$. It is easy to see that

$$\begin{aligned} &\left\| \tfrac{f(a_1^p x_{11},\ldots,a_n^p x_{n1})}{A^p} - \tfrac{f(a_1^l x_{11},\ldots,a_n^l x_{n1})}{A^l}, z \right\| \leq \textstyle\sum_{j=l}^{p-1} \tfrac{\varepsilon}{|A|^{j+1}}, \\ &(x_{11}, \ldots, x_{n1}) \in X^n,\ z \in Y^m, \end{aligned} \tag{16}$$

which shows that for each $(x_{11}, \ldots, x_{n1}) \in X^n$, $\left(\frac{f(a_1^k x_{11},\ldots,a_n^k x_{n1})}{A^k}\right)_{k \in \mathbb{N}_0}$ is a Cauchy sequence. Using now the fact that Y is an $(m+1)$-Banach space we conclude that this sequence is convergent, which allows us to define the mapping $F : X^n \to Y$ by (13).

Next, putting $l = 0$ and letting $p \to \infty$ in (16), and using Remark 6 we see that

$$\begin{aligned} &\|f(x_{11}, \ldots, x_{n1}) - F(x_{11}, \ldots, x_{n1}), z\| \leq \tfrac{\varepsilon}{|A|-1}, \\ &(x_{11}, \ldots, x_{n1}) \in X^n,\ z \in Y^m, \end{aligned}$$

which means that condition (15) is satisfied.

Let us also note that from (14) we get

$$\begin{aligned} &\Big\| \tfrac{f(a_1^k(a_{11}x_{11}+a_{12}x_{12}),\ldots,a_n^k(a_{n1}x_{n1}+a_{n2}x_{n2}))}{A^k} - \\ &\textstyle\sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \tfrac{f(a_1^k x_{1i_1},\ldots,a_n^k x_{ni_n})}{A^k}, z \Big\| \leq \tfrac{\varepsilon}{|A|^k} \end{aligned}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$, $z \in Y^m$ and $k \in \mathbb{N}_0$. Thus, letting $k \to \infty$ and applying definition (13) and Remark 6 we finally conclude that the mapping $F : X^n \to Y$ is a solution of equation (4).

Let us finally assume that $G : X^n \to Y$ is another mapping fulfilling equation (4) and inequality (15) for $x_1, \ldots, x_n \in X$ and $z \in Y^m$. Then for any $l \in \mathbb{N}$, $(x_{11}, \ldots, x_{n1}) \in X^n$ and $z \in Y^m$ we have

$$\begin{aligned} &\|F(x_{11}, \ldots, x_{n1}) - G(x_{11}, \ldots, x_{n1}), z\| \leq \\ &\tfrac{1}{|A|^l} \big(\|F(a_1^l x_{11}, \ldots, a_n^l x_{n1}) - f(a_1^l x_{11}, \ldots, a_n^l x_{n1}), z\| + \\ &\|f(a_1^l x_{11}, \ldots, a_n^l x_{n1}) - G(a_1^l x_{11}, \ldots, a_n^l x_{n1}), z\|\big) \leq \tfrac{1}{|A|^l} \tfrac{2\varepsilon}{|A|-1}. \end{aligned}$$

Letting now $l \to \infty$ and using (8) together with Remarks 5 and 6 we conclude that $G = F$. $\square$

3.3. Corollaries

Now, we present three consequences of the main result of this section.

First, consider the case $a_{11} = a_{12} = \ldots = a_{n1} = a_{n2} = 1$ and $A_{i_1,\ldots,i_n} = 1$ for $i_1, \ldots, i_n \in \{1, 2\}$. Then from Theorem 7 we get the following outcome on the stability of equation (5).

Corollary 8. *Assume that $m \in \mathbb{N}$, $\varepsilon > 0$ and Y is an $(m+1)$-Banach space. If $f : X^n \to Y$ is a function satisfying*

$$\|f(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) - \sum_{i_1,\ldots,i_n \in \{1,2\}} f(x_{1i_1}, \ldots, x_{ni_n}), z\| \leq \varepsilon$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $z \in Y^m$, then there is a unique solution $F : X^n \to Y$ of equation (5) such that

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n), z\| \leq \frac{\varepsilon}{2^n - 1}$$

for $(x_1, \ldots, x_n) \in X^n$ and $z \in Y^m$.

Another consequence of Theorem 7 is the following result on the Ulam stability of equation (7).

Corollary 9. *Assume that $m \in \mathbb{N}$, $\varepsilon > 0$ and Y is an $(m+1)$-Banach space. If $f : X^n \to Y$ is a function satisfying*

$$\Big\|f(x_{11} + x_{12}, \ldots, x_{k1} + x_{k2}, \tfrac{1}{2}x_{k+11} + \tfrac{1}{2}x_{k+12}, \ldots, \tfrac{1}{2}x_{n1} + \tfrac{1}{2}x_{n2}) -$$

$$\textstyle\sum_{i_1,\ldots,i_n \in \{1,2\}} \frac{1}{2^{n-k}} f(x_{1i_1}, \ldots, x_{ni_n}), z\Big\| \leq \varepsilon$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $z \in Y^m$, then there is a unique solution $F : X^n \to Y$ of equation (7) such that

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n), z\| \leq \frac{\varepsilon}{2^k - 1}$$

for $(x_1, \ldots, x_n) \in X^n$ and $z \in Y^m$.

Let us finally mention that Theorem 7 generalizes Theorem 3.3 in [10].

3.4. Final Remark

In the paper we consider, among others due to historical reasons, m-normed and m-Banach spaces for $m \geq 2$. However, one can also admit the case $m = 1$ and then get the results of the previous section as corollaries of the ones proved in Sect. 3.

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