Cardinal Functions of Purely Atomic Measures

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Abstract. Let μ be a purely atomic finite measure. Without loss of generality we may assume that μ is defined on $\mathbb{N}$, and the atoms with smaller indexes have larger masses, that is $\mu(\{k\}) \geq \mu(\{k+1\})$ for $k \in \mathbb{N}$. By $f_\mu : [0, \infty) \rightarrow \{0, 1, 2, \dots, \omega, \mathfrak{c}\}$ we denote its cardinal function $f_\mu(t) = |\{A \subset \mathbb{N} : \mu(A) = t\}|$. We study the problem for which sets $R \subset \{0, 1, 2, \dots, \omega, \mathfrak{c}\}$ there is a measure μ such that $R = \text{rng}(f_\mu)$. We are also interested in the set-theoretic and topological properties of the set of μ -values which are obtained uniquely.

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1. Introduction

Let μ be a purely atomic finite measure. Following [5] we may assume that μ is defined on $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mu(\{k\}) \geq \mu(\{k+1\}) > 0$. Throughout the paper we assume that measures are always of this type. By the range of μ we understand the set $\text{rng}(\mu) := \{\mu(E) : E \subset \mathbb{N}\}$. To simplify the notation let $x_n = \mu(\{n\})$ be a measure of the n -th largest atom of μ . Note that

$$\text{rng}(\mu) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0, 1\}^{\mathbb{N}} \right\}.$$

The latter set is also denoted by $A(x_n)$ and it is called the achievement set of (x_n) (see [12]). We can identify a measure μ with the function $\{0, 1\}^{\mathbb{N}} \ni (\varepsilon_n) \mapsto \sum_{n=1}^{\infty} \varepsilon_n x_n$. This is the continuous mapping from the Cantor space $\{0, 1\}^{\mathbb{N}}$ to the real line, and therefore $\mu^{-1}(t)$ is a closed subset of the Cantor

space $\{0, 1\}^{\mathbb{N}}$. Thus the cardinality $|\mu^{-1}(t)|$ belongs to the set $\{0, 1, 2, \dots, \omega, \mathfrak{c}\}$ where ω stands for the first infinite cardinal while $\mathfrak{c}$ for the continuum [14, Section 6]. By f_μ we denote the cardinal function $f_\mu(t) = |\mu^{-1}(t)| = |\{A \subset \mathbb{N} : \mu(A) = t\}|$; we will write f instead of f_μ if it is clear which measure μ or series $\sum_{n=1}^\infty x_n$ is considered. By $R(f_\mu)$ (or $R(f)$) we denote the range of the cardinal function f_μ , that is the set of all cardinals $|\mu^{-1}(t)|$ where $t \in \text{rng}(\mu)$. Hence $R(f) \subset \{1, 2, \dots, \omega, \mathfrak{c}\}$. For example, $n \in R(f_\mu)$ means that there are n distinct sets $A_1, \dots, A_n \subset \mathbb{N}$ such that $\mu(A_i) = \mu(A_1)$ for $i = 2, \dots, n$ and $\mu(A) \neq \mu(A_1)$ for any $A \in \mathcal{P}(\mathbb{N}) \setminus \{A_1, \dots, A_n\}$, where $\mathcal{P}(\mathbb{N})$ stands for the family of all subsets of $\mathbb{N}$. If $f_\mu(t) = 1$, then we say that t is uniquely obtained, (or it has unique representation or unique expansion). The set $f_\mu^{-1}(1)$ will be called a set of uniqueness for μ . By Fin we denote the collection of all finite subsets of $\mathbb{N}$. Identifying subset of $\mathbb{N}$ with its characteristic functions, we consider on $\mathcal{P}(\mathbb{N})$ the topology from the Cantor space $\{0, 1\}^{\mathbb{N}}$. It turns out that Fin is dense in $\mathcal{P}(\mathbb{N})$, and therefore the set $\{\mu(A) : A \in Fin\}$ is dense in $\text{rng}(\mu)$. We will use the notation $[1, n] := \{1, 2, \dots, n\}$. A subset of a complete metric space is called a Cantor set if it is non-empty compact and dense-in-itself.

We are interested in the following problems:

- for which subsets R of $\{1, 2, \dots, \omega, \mathfrak{c}\}$ there is a measure μ such that $R(f_\mu) = R$?
- what are topological and set-theoretical properties of $f_\mu^{-1}(\alpha)$ for $\alpha \in \{1, 2, \dots, \omega, \mathfrak{c}\}$?

This kind of problems was considered by Bielas et al. [6] for particular achievement set $\mathbf{T}$, prescribed below in Theorem 1.2, which is of a great importance in this field. The Authors showed that its cardinal function f has only two values—1 and 2. Moreover $f^{-1}(i)$ has the cardinality continuum and is dense in $\mathbf{T}$ for $i = 1, 2$.

Achievement sets of sequences have been considered by many authors; some results have been rediscovered several times. Let us list basic properties of $A(x_n)$ (some of them discovered by Takeya [13]):

Theorem 1.1. *Let (x_n) be an absolutely summable sequence of real numbers. Then*

- $A(x_n)$ is a compact perfect or finite set;*
- if $|x_n| > \sum_{i>n} |x_i|$ for all sufficiently large n 's, then $A(x_n)$ is homeomorphic to the Cantor set C ;*
- if $|x_n| \leq \sum_{i>n} |x_i|$ for all sufficiently large n 's, then $A(x_n)$ is a finite union of closed intervals. Moreover, if $|x_n| \geq |x_{n+1}|$ for all but finitely many n 's and $A(x_n)$ is a finite union of closed intervals, then $|x_n| \leq \sum_{i>n} |x_i|$ for all but finitely many n 's.*

One can see that $A(x_n)$ is finite if and only if $x_n = 0$ for all but finite number of n 's, i.e. $(x_n) \in c_{00}$. Takeya conjectured that if $(x_n) \in \ell_1 \setminus c_{00}$, then $A(x_n)$

is always homeomorphic to the Cantor set C or it is a finite union of intervals. In [11] Guthrie and Nymann gave a very simple example of a sequence whose achievement set is not a finite union of closed intervals but it has nonempty interior. They used the following sequence $(t_n) = (\frac{3}{4}, \frac{2}{4}, \frac{3}{16}, \frac{2}{16}, \dots)$. Moreover, they formulated the following:

Theorem 1.2. *For any $(x_n) \in \ell_1 \setminus c_{00}$, the set $A(x_n)$ is one of the following types:*

- (i) *a finite union of closed intervals,*
- (ii) *homeomorphic to the Cantor set C ,*
- (iii) *homeomorphic to the set $\mathbf{T} = A(t_n) = A(\frac{3}{4}, \frac{2}{4}, \frac{3}{16}, \frac{2}{16}, \frac{3}{64}, \dots)$.*

Although their proof had a gap, the theorem is true and the correct proof was given by Nymann and Saenz [19]. Guthrie, Nymann and Saenz have observed that the set $\mathbf{T}$ is homeomorphic to the set N described by the formula

$$N = [0, 1] \setminus \bigcup_{n \in \mathbb{N}} U_{2n},$$

where U_n denotes the union of 2^{n-1} open intervals removed from the interval $[0, 1]$ at the n -th step of the classic Cantor ternary set construction. Homeomorphs of N (or $\mathbf{T}$) are called Cantorvals. This name emphasizes their similarity to the Cantor set and the fact that their are regularly closed, that is they are closures of their subintervals unions. Topological characterizations of Cantorvals can be found in [3, 20]. The set $\mathbf{T}$ is known as the Guthrie-Nymann Cantorval. Most known examples of sequences whose achievement sets are Cantorvals belong to the class of so-called multigeometric sequences. This class was deeply investigated in [1, 4, 12]. For example, the achievement sets of multigeometric series are the attractors of the affine iterated function systems, see [1]. For more information on achievement sets we refer readers to the survey papers [3, 16, 17].

Section 2 was inspired by the Franciszek Prus-Wiśniowski's question from 2017: is it possible to obtain some points of achievement set of absolutely convergent series for precisely countably many subseries? Or in our notation— is it possible to construct an absolutely convergent series such that $f^{-1}(\omega) \neq \emptyset$? We have found that this problem had been already solved by Erdős et al. [10], where the authors considered geometric series $\sum_{n=1}^{\infty} q^n$. The authors study the question for which q the number 1 has exactly one, countably or continuum many expansions. We show that it is possible to construct a bi-geometric series such that there exist at least countably many points, obtained for exactly countably many expansions each. Indeed, combine Theorem 2.2 (4) and Proposition 2.3. Our construction differs from that [10].

The Erdős et al. [10] initiated the large number of studies during last 30 years on non-integer bases. In the most general approach by an expansion of a real number x in base $q > 1$ on the alphabet $\Sigma = \{0 < \sigma_1 < \dots < \sigma_n\}$ we mean a sequence (c_n) of elements of Σ satisfying the equality

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \dots$$

In [10] the authors consider the set of all numbers $1 < q < 2$ such that 1 is in a set of uniqueness of $A(q^n)$. If there is a sequence $(a_1, \dots, a_k)$ such that $\Sigma = \{\sum_{i=1}^k \varepsilon_i a_i : \varepsilon_i = 0, 1\}$, then the set of numbers $x \geq 0$ which have at least one expansion in base q corresponds to the achievement set $A(a_1, \dots, a_k; q)$ of a multigeometric series $(a_1, \dots, a_k; 1/q)$ given by

$$a_1 + \dots + a_k + \frac{a_1}{q} + \dots + \frac{a_k}{q} + \frac{a_1}{q^2} + \dots + \frac{a_k}{q^2} + \dots$$

The set, say E , of numbers which have at least one expansion is not necessarily equal to an achievement set of any series. To see this consider the alphabet $\Sigma = \{0, 1, 2, 9\}$ and base $q = 3$; then E is neither finite union of intervals nor Cantor set nor Cantorval, see [18].

This shows that the theory of non-integer bases and that we present are different.

2. Measure μ with $f_\mu^{-1}(\omega) \neq \emptyset$

A function $\chi : \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}$ is called a finite signed measure if it is countably additive, $\chi(\emptyset) = 0$, $\sup\{\chi(A) : A \subset \mathbb{N}\} < \infty$ and $\inf\{\chi(A) : A \subset \mathbb{N}\} > -\infty$. If $x_n := \chi(\{n\})$, then the series $\sum_{n=1}^\infty x_n$ is absolutely convergent. A measure $|\chi|(A) = \sum_{n \in A} |x_n|$ is called the variation of χ . One can define a cardinal function f_χ for a signed measure χ .

Note that for every absolutely convergent series $\sum_{n=1}^\infty x_n$ we have $A(|x_n|) = A(x_n) + \sum_{n=1}^\infty x_n^-$, where for a real number x we consider $x^- := \max\{-x, 0\}$ and $x^+ := \max\{x, 0\}$. Moreover, if $x = \sum_{n \in A} |x_n|$, then

$$x - \sum_{n=1}^\infty x_n^- = \sum_{n \in A} x_n^+ + \sum_{n \in A} x_n^- - \sum_{n=1}^\infty x_n^- = \sum_{n \in A} x_n^+ - \sum_{n \in \mathbb{N} \setminus A} x_n^-.$$

Let $E = \{n : x_n > 0\}$. Since $\psi : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N})$ defined as $\psi(A) := (E \cap A) \cup ((\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus E))$ is bijective, we get $f(x - \sum_{n=1}^\infty x_n^-) = g(x)$, where f and g are the cardinal functions for $\sum_{n=1}^\infty x_n$ and $\sum_{n=1}^\infty |x_n|$ respectively. This shows that f_χ and $f_{|\chi|}$ are equal up to shift of the domain. Therefore we will consider only measures or series with positive terms.

We start by giving some basic properties of cardinal function f .

Proposition 2.1. *Let $x_n = \mu(\{n\})$ for some measure μ . Then*

- (i) *there are at least two uniquely obtained values in $\text{rng}(\mu)$;*
- (ii) *$f(x) = f(\sum_{n=1}^\infty x_n - x)$ for each $x \in \mathbb{R}$;*

Proof. (i). Clearly $\sum_{n \in A} x_n = \sum_{n=1}^\infty x_n$ if and only if $A = \mathbb{N}$. Since $x_n > 0$, $\sum_{n \in B} x_n = 0$ if and only if $B = \emptyset$. Hence 0 and $\sum_{n=1}^\infty x_n$ have unique representations.

(ii). Let $\mathcal{A}$ be the family of all sets A such that $\sum_{n \in A} x_n = x$. Then $\mathcal{A}^c = \{\mathbb{N} \setminus A : A \in \mathcal{A}\}$ is the family of all sets $\mathbb{N} \setminus A$ such that $\sum_{n \in \mathbb{N} \setminus A} x_n = \sum_{n=1}^{\infty} x_n - x$. Since $A \mapsto \mathbb{N} \setminus A$ is bijection, we obtain $|\mathcal{A}| = |\mathcal{A}^c|$. $\square$

Before we answer the Prus-Wisniowski's question, we give some definitions. For a convergent series $\sum_{n=1}^{\infty} x_n$ of positive terms, by $r_n = \sum_{k=n+1}^{\infty} x_k$ we denote its n -th remainder. We say that $\sum_{n=1}^{\infty} x_n$ is quickly convergent if $x_n > r_n$ for each $n \in \mathbb{N}$. A series $\sum_{n=1}^{\infty} x_n$ has property

- (A) if $x_i > \sum_{n=i+1}^{\infty} (x_n + r_{n-1})$ for each $i \in \mathbb{N}$;
- (B) if $r_{i-1} > \sum_{n=i+1}^{\infty} (x_n + r_{n-1})$ for each $i \in \mathbb{N}$.

Since $r_{i-1} > x_i$, it is easy to see that each (A) series is a (B) series. Furthermore every (B) series is quickly convergent. Indeed, if $r_{i-1} > \sum_{n=i+1}^{\infty} (x_n + r_{n-1})$, then from $r_{i-1} = x_i + r_i$ we get $x_i > \sum_{n=i+1}^{\infty} (x_n + r_n) > \sum_{n=i+1}^{\infty} x_n$.

Theorem 2.2. *Let $\sum_{n=1}^{\infty} x_n$ be a (B) series. We define $y_{2n-1} = r_{n-1}$, $y_{2n} = x_n$ for each $n \in \mathbb{N}$. Let f be the cardinal function of $\sum_{n=1}^{\infty} y_n$. Then:*

- (1) $f(r_n) = \omega$ for every $n \in \mathbb{N}$;
- Moreover if $\sum_{n=1}^{\infty} x_n$ is (A) series then:
- (2) $f(x_n) = 1$ for every $n \in \mathbb{N}$;
- (3) $f(\sum_{n \in F} x_n) = 1$ for each $F \subset \mathbb{N}$ such that $|\mathbb{N} \setminus F| = \infty$;
- (4) $f(\sum_{n \in F} x_n) = \omega$ for each $F \subset \mathbb{N}$ such that $1 \leq |\mathbb{N} \setminus F| < \infty$;
- (5) there are continuum many points t such that $f(t) = \mathfrak{c}$;
- (6) $R(f) = \{1, 2, \dots, k, \dots, \omega, \mathfrak{c}\}$.

Proof. (1) Let $W(t) := \{A \subset \mathbb{N} : \sum_{n \in A} y_n = t\}$. Fix $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Then $r_k = x_{k+1} + r_{k+1} = x_{k+1} + x_{k+2} + r_{k+2} = \dots = \sum_{n=k+1}^{\infty} x_n + r_m$ for each $m \geq k$. Thus $W(r_k) \supset \{\{2k+2, 2k+4, \dots, 2k+2m, 2k+2m+1\} : m \in \mathbb{N}_0\}$. Hence $f(r_k) \geq \omega$. Note that if $A \in W(r_k)$ then $A \subset [2k+1, \infty)$, since for $n \leq 2k$ we have $y_n > r_k$. Given a $k \in \mathbb{N}_0$, let $W_m^k = \{2k+2m-1\} \cup \{2k+2j : 1 \leq j \leq m-1\}$ for $m \in \mathbb{N}$ and $W_{\infty}^k = \{2j : j \in \mathbb{N}, j > k\}$.

We will prove that $W(r_k) = \{W_m^k : m = 1, 2, \dots, \infty\}$ by showing that W_m^k is the only element A of $W(r_k)$ with $|A| = m$. For $m = 1$ the assertion follows from (B). Assume that we have already proved that for each $k \in \mathbb{N}_0$ and for some $p \in \mathbb{N}$ every p -elements set $A \in W(r_k)$ equals to W_p^k . Let $B \in W(r_k)$, $|B| = p+1$. Note that $2k+1 \in B$ implies $|B| = 1$, which yields a contradiction. Thus $2k+1 \notin B$. Suppose that $2k+2 \notin B$. Then $r_k > \sum_{n=k+2}^{\infty} (x_n + r_{n-1}) > \sum_{n \in B} y_n = r_k$ and again we get a contradiction. Then $2k+2 \in B$, so $\sum_{n \in B \setminus \{2k+2\}} y_n = \sum_{n \in B} y_n - y_{2k+2} = r_k - x_{k+1} = r_{k+1}$. Since $|B \setminus \{2k+2\}| = p$ by the inductive assumption we obtain $B \setminus \{2k+2\} = W_p^{k+1}$. Thus $B = W_{p+1}^k$.

To finish the proof of the first part we need to show that if $A \in W(r_k)$ is infinite, then $A = W_{\infty}^k$. Note that $2k+1 \notin A$ and $2k+2 \in A$. Assume that we have already proved that $2k+2i-1 \notin A$ and $2k+2i \in A$ for each

$i \leq p$. We need to show that $2k + 2p + 1 \notin A$ and $2k + 2p + 2 \in A$. Since $\sum_{i=1}^p y_{2k+2i} + y_{2k+2p+1} = r_k$ and $|A| > p + 1$ we obtain that $2k + 2p + 1 \notin A$. Moreover $r_k - \sum_{i=1}^p y_{2k+2i} = r_{k+p} > \sum_{i \geq 2k+2p+3} y_i$, so $2k + 2p + 2 \in A$. By the induction we obtain $A = W_\infty^k$. Thus for each $k \in \mathbb{N}$ we have $f(r_k) = \omega$.

- (2) Fix $k \in \mathbb{N}$ and assume that $\sum_{n \in A} y_n = x_k$. Note that for $n < 2k$ we have $y_n > x_k$, so $A \cap [1, 2k - 1] = \emptyset$. If $2k \notin A$, the inequality $x_k > \sum_{n > 2k} y_n$ leads to a contradiction. Hence $A = \{2k\}$. Thus $f(x_k) = 1$.
- (3) Let $F = (f_n)_{n=1}^k$ be an increasing sequence of indices with $|\mathbb{N} \setminus F| = \infty$, where $k \in \mathbb{N} \cup \{\infty\}$ and $x := \sum_{n=1}^k x_{f_n} = \sum_{n=1}^k y_{2f_n}$. Let $G \subset \mathbb{N}$ be such that $x = \sum_{n \in G} y_n$. Since $|\mathbb{N} \setminus F| = \infty$, then $\sum_{n \geq i} x_{f_n} < r_{f_i-1}$. Therefore $y_{2f_1-1} > x$ and consequently $G \cap [1, 2f_1) = \emptyset$. Note that $2f_1 \in G$. Otherwise

$$x = \sum_{n \in G} y_n \leq \sum_{n > 2f_1} y_n = \sum_{n=f_1+1}^{\infty} x_n + \sum_{n=f_1+1}^{\infty} r_{n-1} \stackrel{(A)}{<} x_{f_1} = y_{2f_1} \leq x$$

which yields a contradiction. Next $x - y_{2f_1} < r_{f_2-1}$, so $G \cap (2f_1, 2f_2) = \emptyset$. As before $y_{2f_2} \in G$; otherwise $x - y_{2f_1} \leq \sum_{n > 2f_2} y_n < y_{2f_2} \leq x - y_{2f_1}$. By a simple induction we obtain $2f_n \in G$ for each $n \leq k$. Since $y = \sum_{n=1}^k y_{2f_n}$ we obtain $G = 2F$. Hence $f(x) = 1$.

- (4) Now let $F = (f_n)_{n=1}^\infty$ be an increasing sequence of indices and $|\mathbb{N} \setminus F| < \infty$. Thus $F = (f_n)_{n=1}^m \cup [p, \infty)$ for some $p > f_m + 1$ and $m \in \mathbb{N}$. Let $x = \sum_{n \in F} x_n = \sum_{n=1}^m x_n + r_{p-1}$. Assume that $x = \sum_{n \in G} x_n$. Using the same argument as before we show that $G \supset (2f_n)_{n=1}^m$. Thus $\sum_{n \in G \setminus (2f_n)_{n=1}^m} y_n = r_{p-1}$ and hence $G = \{2f_n : n = 1, \dots, m\} \cup A$ for some $A \in W(r_{p-1})$. Therefore $f(\sum_{n \in F} x_n) \leq \omega$ by (1). On the other hand, for any $A \in W(r_{p-1})$ we get $\sum_{n \in \{2f_n : n=1, \dots, m\} \cup A} y_n = \sum_{n \in F} x_n$ and thus $f(\sum_{n \in F} x_n) \geq \omega$ by (1) again.
- (5) We will show that $f(x_H) = \mathfrak{c}$ for $x_H := \sum_{n \in H} x_{2n} + \sum_{n=1}^\infty r_{2n}$ where $H \subset \mathbb{N}$. Since $r_{2n} = x_{2n+1} + r_{2n+1}$ for each $n \in \mathbb{N}$, we obtain $x_H = \sum_{n \in H} x_{2n} + \sum_{n=1}^\infty v_n$, where by every v_n we can take r_{2n} either $x_{2n+1} + r_{2n+1}$.
- (6) Fix $k \in \mathbb{N}$. We will show that $f(x) = k$ for $x := \sum_{n=1}^\infty x_{kn} + r_0$. Since $r_0 = x_1 + r_1 = x_1 + x_2 + r_2 = \dots = x_1 + x_2 + \dots + x_{n-1} + r_{n-1}$, it is clear that $f(x) \geq k$. Moreover by (A) series property, if we omit r_0 in our sum, we must take x_1 , if we omit r_1 , we must take x_2 , and so on. Therefore each representation $\sum_{n \in M} x_n$ of x need to contain r_0 or $r_1 + x_1$ or one of the $k - 2$ remaining representations of r_0 . Now let us assume that $x = \sum_{n \in M} x_n + r_0$. Note that $p \notin M$ for each $p < k$, since otherwise $\sum x_{m_n} + r_0 \geq x_p + r_0 > x$. On the other hand if $k \notin M$, then $\sum_{n \in M} x_n + r_0 < x_k + r_0 < x$ and we reach a contradiction as well. Hence $k \in M$. In the same way for consecutive $m \in \mathbb{N}$, we inductively eliminates the elements x_p for $p \in [mk + 1, (m + 1)k)$ and show that $(m + 1)k \in M$.

Hence $M = \{kn : n \in \mathbb{N}\}$ and there are no new subseries which sum up to x . That is $f(x) = k$. $\square$

Now we give the characterization of (A) series and (B) series within the class of geometric series.

Proposition 2.3. *Let us consider a geometric sequence $x_n := q^n$. Then the series $\sum_{n=1}^{\infty} x_n$ has*

- (i) *property (A) for $q \in (0, \frac{2-\sqrt{2}}{2})$;*
- (ii) *property (B) for $q \in (0, \frac{3-\sqrt{5}}{2})$.*

Proof. (i) Fix $k \in \mathbb{N}$. We have $r_{m-1} = \sum_{n=m}^{\infty} q^n = \frac{q^m}{1-q}$ for every $m \in \mathbb{N}$. The series has (A) property if and only if $q^k = x_k > \sum_{n=k+1}^{\infty} (x_n + r_{n-1}) = \sum_{n=k+1}^{\infty} (q^n + \frac{q^n}{1-q}) = \frac{(2-q)q^{k+1}}{(1-q)^2}$. Thus

$$1 > \frac{(2-q)q}{(1-q)^2} \Leftrightarrow 1 - 2q + q^2 > 2q - q^2 \Leftrightarrow 2q^2 - 4q + 1 > 0 \Leftrightarrow q \in \left(-\infty, \frac{2-\sqrt{2}}{2}\right) \cup \left(\frac{2+\sqrt{2}}{2}, \infty\right).$$

Hence the geometric series has (A) property for $q \in (0, \frac{2-\sqrt{2}}{2})$.

(ii) The series has (B) property if and only if $r_{k-1} = \frac{q^k}{1-q} > \frac{(2-q)q^{k+1}}{(1-q)^2}$. Hence

$$1 > \frac{(2-q)q}{(1-q)^2} \Leftrightarrow 1 - q > 2q - q^2 \Leftrightarrow q^2 - 3q + 1 > 0 \Leftrightarrow q \in \left(-\infty, \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}, \infty\right).$$

It shows that the geometric series has (B) property for $q \in (0, \frac{3-\sqrt{5}}{2})$. $\square$

Remark 2.4. Let $x_n = q^n$ for some $q \in (\frac{2-\sqrt{2}}{2}, \frac{3-\sqrt{5}}{2})$, that is by the Proposition 2.3 the series $\sum_{n=1}^{\infty} x_n$ is (B), but it is not (A). We define the terms of the series $\sum_{n=1}^{\infty} y_n$ in the same way as in the Theorem 2.2. Fix $n \in \mathbb{N}$. We have:

$$y_{2n-1} = r_{n-1} = \sum_{k=n}^{\infty} x_k = \sum_{k=n}^{\infty} y_{2k} \leq \sum_{k=2n}^{\infty} y_k$$

and

$$y_{2n} = x_n \leq \sum_{k=n}^{\infty} (x_{k+1} + r_k) = \sum_{k=2n+1}^{\infty} y_k,$$

since $\sum_{n=1}^{\infty} x_n$ is not (A) series. Hence $y_n \leq \sum_{k=n+1}^{\infty} y_k$ for each $n \in \mathbb{N}$. Then $A(y_n) = [0, \sum_{n=1}^{\infty} y_n]$. By Theorem 2.2 we get that $f^{-1}(\{\omega\}) \neq \emptyset$. Hence it

is possible to construct a measure, which range is a closed interval and there exists a value, which is obtained for exactly countably infinite many distinct sets.

Proposition 2.3 shows us that we can construct a series in Theorem 2.2 in the way it is bigeometric. It is a subclass of so-called multigeometric series. A multigeometric series is a series, whose terms are the following:

$$(k_0, k_1, k_2, \dots, k_m, k_0q, k_1q, k_2q, \dots, k_mq, k_0q^2, k_1q^2, k_2q^2, \dots, k_mq^2, \dots)$$

for some finite set $\{k_0, k_1, k_2, \dots, k_m\}$ of real numbers and $q \in (0, 1)$. A bigeometric series is a multigeometric series with $m = 1$. Using Theorem 2.2 for $x_n = q^n$ we obtain the bigeometric series $\sum_{n=1}^{\infty} y_n$, where $k_0 = \frac{q}{1-q}$, $k_1 = q$. Moreover, by using the methods of Theorem 2.2 we cannot obtain (y_n) to be a geometric sequence; by a simple calculations (y_n) is geometric provided $q = \frac{3-\sqrt{5}}{2}$, which leads to a series, which does not have the property (B).

Proposition 2.5. *There exists a measure μ for which $R(f_\mu) = \mathbb{N} \cup \{\mathfrak{c}\}$, that is the cardinal function f_μ has all possible values but ω .*

Proof. Let $A_n = [\frac{n^2-n+2}{2}, \frac{n^2+n}{2}]$ for each $n \in \mathbb{N}$. Clearly $A_1, A_2, \dots$ is a partition of $\mathbb{N}$ with $|A_n| = n$. Let $\sum_{n=1}^{\infty} x_n$ be any series satisfying the following two conditions:

- (i) for each $n \in \mathbb{N}$ and any $j, l \in A_n$ we have $x_j = x_l$;
- (ii) for each $n \in \mathbb{N}$ and $k \in A_n$ the inequality $x_k > \sum_{m>n} \sum_{i \in A_m} x_i$ holds.

For example $x_k = \frac{1}{10^{n^2}}$ for $k \in A_n$ and $n \in \mathbb{N}$ satisfies conditions (i) and (ii). Note that $\sum_{i \in B_j} x_i = \sum_{i \in C_j} x_i$, where $B_j \subset A_j$, $C_j \subset A_j$ for $j \in \mathbb{N}$, implies, by (i) and (ii), that $|B_j| = |C_j|$ for all j . Let $x = \sum_{i \in B_j} x_i$. Firstly, consider the case when $\emptyset \neq B_j \neq A_j$ holds for finitely many j 's, and denote them by $j_1, \dots, j_r$. Then $f(x) = \prod_{p=1}^r \binom{j_p}{|B_{j_p}|}$. In particular for an element x such that $r = 1$, $|B_{j_1}| = 1$, $j_1 = n$ we obtain $f(x) = n$. Hence $\mathbb{N} \subset R(f)$. Secondly, let us assume that $\emptyset \neq B_j \neq A_j$ holds for infinitely many j 's. Then clearly $f(x) = \mathfrak{c}$. $\square$

The next example shows that there exists a measure μ such that $R(f_\mu) = \{1, \mathfrak{c}\}$.

Example 2.6. Let $x_{2n-1} = x_{2n} = \frac{1}{2^n}$. Then $A(x_n) = [0, 2]$. By Proposition 2.1 we know that $f(0) = f(2) = 1$. Assume that $x \in (0, 1]$. Then there exists unique, infinite subset $F = (k_n)_{n=1}^{\infty}$ of odd indices such that $x = \sum_{n \in F} x_n$. Let us consider $\mathcal{F} = \{(k_n + j_n)_{n=1}^{\infty} : (j_n) \in \{0, 1\}^{\mathbb{N}}\}$. Since for every $A \in \mathcal{F}$ we have $x = \sum_{n \in A} x_n$ and $|\mathcal{F}| = \mathfrak{c}$. From Proposition 2.1(2) we obtain $f(x) = \mathfrak{c}$ for every $x \in (0, 2)$.

One can prove a more general statement. Assume that for an absolutely convergent series $\sum_{n=1}^{\infty} x_n$ there exists two sequences of pairwise disjoint sets

of indices $(F_n), (G_n)$ such that for all $i, j \in \mathbb{N}$ we have $F_i \cap G_j = \emptyset$ and $\sum_{n \in G_i} x_n = \sum_{n \in F_i} x_n$. Then $f^{-1}(\mathfrak{c})$ contains a Cantor set. Indeed, if $t = \sum_{i=1}^{\infty} \sum_{n \in F_i} x_n$, then

$$t = \sum_{i=1}^{\infty} \sum_{n \in F_i^{\varepsilon_i}} x_n,$$

where $(\varepsilon_i) \in \{0, 1\}^{\mathbb{N}}$, $F_i^0 = F_i$ and $F_i^1 = G_i$. Therefore $f(t) = \mathfrak{c}$. The same holds for $t_X = \sum_{i \in X} \sum_{n \in F_i} x_n$ for every infinite set $X \subset \mathbb{N}$. Let $t_i = \sum_{n \in F_i} x_n$. Then

$$A(t_i) = \{t_X : X \text{ is infinite}\} \cup \{t_X : X \text{ is finite}\}$$

Note that $\{t_X : X \text{ is finite}\}$ is countable and $A(t_i)$ is uncountable compact. Therefore $f^{-1}(\mathfrak{c})$ contains an uncountable G_δ set $\{t_X : X \text{ is infinite}\}$; in particular $f^{-1}(\mathfrak{c})$ contains a Cantor set [14].

Every example of a measure μ with $f_\mu^{-1}(\mathfrak{c}) \neq \emptyset$ we have examined had the property that $f_\mu^{-1}(\mathfrak{c})$ contains a Cantor set. However we are not able to determine whether such phenomenon holds for every measure.

Problem 2.7. Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series and $f^{-1}(\mathfrak{c}) \neq \emptyset$. Is it true that $f^{-1}(\mathfrak{c})$ contains a Cantor set?

In general, if there is value $t \in \text{rng}(\mu)$ which is achieved infinitely many times, then there are infinitely many such values. More precisely the following holds.

Proposition 2.8. Assume that there is $t \in \text{rng}(\mu)$ with $f_\mu(t) \geq \omega$. Then the set $f_\mu^{-1}(\{\omega, \mathfrak{c}\})$ is dense in $\text{rng}(\mu)$.

Proof. Assume that $f(t) = \omega$, that is $|\{A \subset \mathbb{N} : \sum_{n \in A} x_n = t\}| = \omega$. Hence at least one set of the form $\{A \subset \mathbb{N} : 1 \in A, \sum_{n \in A} x_n = t\}$ or $\{A \subset \mathbb{N} : 1 \notin A, \sum_{n \in A} x_n = t\}$ is of cardinality ω . Inductively we define a 0-1 sequence $\alpha \in \{0, 1\}^{\mathbb{N}}$ such that for any k the family

$$\left\{ A \subset \mathbb{N} : \sum_{n \in A} x_n = t \text{ and } \forall n \leq k (x_n \in A \iff \alpha(n) = 1) \right\}$$

has cardinality ω . Let $\{n_1 < n_2 < \dots\}$ be an increasing enumeration of $\{i \in \mathbb{N} : \alpha(i) = 1\}$. Then for any k

$$\left| \left\{ A \subset \mathbb{N} : n_1, n_2, \dots, n_k \in A \text{ and } \sum_{n \in A} x_n = t \right\} \right| = \omega \iff$$

$$\left| \left\{ A \subset \mathbb{N} \setminus \{n_1, \dots, n_k\} : \sum_{n \in A} x_n = t - \sum_{i=1}^k x_{n_i} \right\} \right| = \omega \iff f\left(t - \sum_{i=1}^k x_{n_i}\right) = \omega.$$

Let $(\varepsilon_i)_{i=1}^m \in \{0, 1\}^m$. There is k such that $n_k \geq m$ and $t - \sum_{i=1}^k x_{n_i} < \varepsilon$. For any $A \subset \mathbb{N} \setminus \{n_1, \dots, n_k\}$ with $\sum_{n \in A} x_n = t - \sum_{i=1}^k x_{n_i}$ the element

$\sum_{i=1}^m \varepsilon_i x_i + \sum_{n \in A} x_n$ is ε -close to $\sum_{i=1}^m \varepsilon_i x_i$ and $f(\sum_{i=1}^m \varepsilon_i x_i + \sum_{n \in A} x_n) \geq \omega$. Since the set of elements of the form $\sum_{i=1}^m \varepsilon_i x_i$ is dense in $\text{rng}(\mu)$, we obtain the assertion. Clearly, the same argument works if $f(t) = \mathfrak{c}$. $\square$

We do not know if $f(t) = \omega$ implies that there is t' with $f(t') = \mathfrak{c}$. We also do not know if $f^{-1}(\omega) \neq \emptyset$ implies that $f^{-1}(\omega)$ is dense in $\text{rng}(\mu)$.

Problem 2.9. Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series and $f^{-1}(\omega) \neq \emptyset$. Is it true that $f^{-1}(\mathfrak{c}) \neq \emptyset$? Is it true that $f^{-1}(\omega)$ is dense in $\text{rng}(\mu)$?

In paper [10], the authors considered geometric series for which some value is obtained by exactly ω subseries. All but one of the subseries was finite. In the next proposition we show that infinite subseries always appear.

Proposition 2.10. Let μ be a measure. If $f(t) = \omega$, then there exists an infinite $A \subset \mathbb{N}$ with $\mu(A) = t$.

Proof. Let $F : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ be given by $F(\varepsilon_n) = \sum_{n=1}^{\infty} \varepsilon_n x_n$ where $x_n := \mu(\{n\})$. As we have mentioned in the Introduction the function F is continuous. Therefore $F^{-1}(t)$ is closed in the compact Cantor space $\{0, 1\}^{\mathbb{N}}$. Since $F^{-1}(t)$ is infinite compact, it has an accumulation point A , that is there is a one-to-one sequence $(A_n) \subset \{0, 1\}^{\mathbb{N}}$ such that $F(A_n) = t$ and A_n tends to A in $\{0, 1\}^{\mathbb{N}}$. If there exists n with $|A_n| = \omega$, then we are done. Otherwise $|A_n| < \omega$ for every $n \in \mathbb{N}$. We will show that $|A| = \omega$. Note that there are 2^m subsets of $\mathbb{N}$ with cardinality $\leq m$. Since the sequence (A_n) is one-to-one, all but finitely many A_n 's have cardinality $> m$. Passing to a subsequence we may assume that $|A_n| \geq n$ for every $n \in \mathbb{N}$. For a fixed m there is k such that $A \cap [1, m] = A_n \cap [1, m]$ for every $n \geq k$. Suppose that there is m with $A \subset [1, m]$. We can find $n > m$ such that $A = A_n \cap [1, m]$. Then there is $l > m$ with $l \in A_n$. We have

$$t = \mu(A) = \mu(A_n \cap [1, m]) < \mu(A_n \cap [1, m]) + \mu(\{l\}) \leq \mu(A_n) = t$$

which yields a contradiction. Hence A is infinite. $\square$

By Proposition 2.1 we know that $\mu(\emptyset)$ and $\mu(\mathbb{N})$ are uniquely obtained. As shows the Example 2.6 there is a measure μ with $\mu(\emptyset)$ and $\mu(\mathbb{N})$ are the only such values. Sequences (x_n) for which $A(x_n)$ is an interval (so called interval filling sequence) and its set of uniqueness consists of $\{0, \sum x_n\}$ only, have already been studied in [7–9]; such sequences are called plentiful in [9]. The authors also considered a sequence (x_n) (or a series $\sum_{n=1}^{\infty} x_n$) which is a locker, that is $A((x_n)_{n \neq N})$ is an interval for each $N \in \mathbb{N}$. In [7] it is proved that each locker is plentiful. Note that the series defined in the Example 2.6 is a locker, so that it is also plentiful. Using simple observations about quasiregular expansions from [15] we are able to show some deeper idea staying behind Example 2.6. An expansion $x = \sum_{n=1}^{\infty} \varepsilon_n x_n$ is quasiregular if and only if (ε_n) is defined in the following way: having defined ε_i for $i \in [1, n-1]$ we put $\varepsilon_n = 1$ if and only if $\sum_{k=1}^{n-1} \varepsilon_k x_k + x_n < x$. The authors of [7] observed that for every

interval filling sequence of positive terms (x_n) every $x \in (0, \sum_{n=1}^{\infty} x_n]$ has an infinite quasiregular expansion. It gives a universal way of transforming any interval filling sequence into locker with all but two points obtained for $\mathfrak{c}$ -many expansions.

Corollary 2.11. *Let (x_n) be an interval filling, positive sequence. Then (y_n) defined as $y_{2n} = y_{2n-1} = x_n$ for each $n \in \mathbb{N}$ is locker and for the cardinal function of (y_n) we have $f(y) = \mathfrak{c}$ for all $y \in (0, 2\sum_{n=1}^{\infty} x_n)$.*

Problem 2.12. *Is it true that the existence of a sequence $(n_k) \rightarrow \infty$ such that $f^{-1}(n_k) \neq \emptyset$ for all $k \in \mathbb{N}$ implies that $f^{-1}(\{\omega, \mathfrak{c}\}) \neq \emptyset$? Note that by Proposition 2.5 we know that it may happen that $f^{-1}(\omega) = \emptyset$.*

Problem 2.13. *Does $f^{-1}(\omega) \neq \emptyset$ (or the conjunction $f^{-1}(\omega) \neq \emptyset$ and $f^{-1}(\mathfrak{c}) \neq \emptyset$) imply the existence of a sequence $(n_k) \rightarrow \infty$ such that $f^{-1}(n_k) \neq \emptyset$ for all $k \in \mathbb{N}$. Note that by Example 2.6 the condition $f^{-1}(\mathfrak{c}) \neq \emptyset$ does not imply the assertion.*

3. Kakeya Like Condition

In Theorem 1.1, attributed to Kakeya, the properties of $\text{rng}(\mu)(= A(x_n))$ depend on the relation between terms $x_n = \mu(\{n\})$ and remainders $r_n = \sum_{k>n} x_k = \mu((n, \infty))$. We show that the quick convergence, that is $x_n > r_n$ for every n , implies that each point of $\text{rng}(\mu)$ is uniquely obtained; slow convergence, that is $x_n \leq r_n$ for all but finitely many n 's, implies that the set of uniqueness has an empty interior while $\text{rng}(\mu)$ is a union of finitely many closed intervals.

Let us start from prescribing a class of measures for which every values is obtained in the unique way. The following proposition can be found in [21]; we present its proof for completeness.

Proposition 3.1. *Let μ be a measure and let $x_n := \mu(\{n\})$. If a series $\sum_{n=1}^{\infty} x_n$ is quickly convergent, i.e. $\mu(\{n\}) > \mu(\{n+1, n+2, \dots\})$ for every $n \in \mathbb{N}$, then each point of $\text{rng}(\mu)$ has unique representation.*

Proof. Suppose that $x = \sum_{n=1}^{\infty} \varepsilon_n x_n = \sum_{n=1}^{\infty} \delta_n x_n$ for two distinct sequences $(\varepsilon_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ and $(\delta_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$. Let k be the smallest natural number such that $\varepsilon_k \neq \delta_k$. Without losing generality let $\varepsilon_k = 1$ and $\delta_k = 0$. Then $0 = x - x = \sum_{n=1}^{\infty} \varepsilon_n x_n - \sum_{n=1}^{\infty} \delta_n x_n = \sum_{n=1}^{\infty} (\varepsilon_n - \delta_n) x_n = x_k - \sum_{n=k+1}^{\infty} (\varepsilon_n - \delta_n) x_n \geq x_k - \sum_{n=k+1}^{\infty} x_n > 0$, which yields a contradiction. $\square$

As shows the next example, there exists a series $\sum_{n=1}^{\infty} x_n$, which is not quickly convergent, but the assertion of Proposition 3.1 remains true. It means that one cannot reverse Proposition 3.1.

Example 3.2. Let $x_1 := \frac{27}{32}$ and $x_n := \frac{3}{4^{n-1}}$ for $n \geq 2$. The series is not quickly convergent, since $x_1 = \frac{27}{32} < 1 = \sum_{n=2}^{\infty} x_n$. On the other hand the series $\sum_{n=2}^{\infty} x_n$ is quickly convergent so for $y_n = x_{n+1}$ every point of the achievement set $A(y_n)$ is obtained in the unique way. Note that $A(y_n) \subset [0, \frac{2}{32}] \cup [\frac{6}{32}, \frac{8}{32}] \cup [\frac{24}{32}, \frac{26}{32}] \cup [\frac{30}{32}, \frac{32}{32}]$. Thus $x_1 + A(y_n) \subset [\frac{27}{32}, \frac{29}{32}] \cup [\frac{33}{32}, \frac{35}{32}] \cup [\frac{51}{32}, \frac{53}{32}] \cup [\frac{57}{32}, \frac{59}{32}]$, so $A(y_n) \cap (x_1 + A(y_n)) = \emptyset$ (see the picture). Hence $f^{-1}(\{1\}) = A(y_n) \cup (x_1 + A(y_n)) = A(x_n)$.

Now we consider the case when $\text{rng}(\mu)$ is a closed interval or a finite union of closed intervals. It turns out that the set of uniqueness is topologically small.

Theorem 3.3. Let μ be a measure and let $x_n := \mu(\{n\})$. Assume that $\text{rng}(\mu)$ is a closed interval. Then $f_\mu(x_{k_1} + x_{k_2} + \cdots + x_{k_m}) \geq 2$ for every finite set $k_1 < k_2 < \cdots < k_m$ of indexes. In particular the set of uniqueness of μ has an empty interior.

Proof. Since $r_n \geq x_n$ for all $n \in \mathbb{N}$, there is a set $A_n \subset [n+1, \infty) \cap \mathbb{N}$ such that $x_n = \sum_{i \in A_n} x_i$. Thus, $\sum_{i=1}^m x_{k_i} = \sum_{i=1}^{m-1} x_{k_i} + \sum_{i \in A_{k_m}} x_i$ are two distinct representations and hence $f_\mu(x_{k_1} + x_{k_2} + \cdots + x_{k_m}) \geq 2$. Note that $\{\sum_{k \in F} x_k : F \in \text{Fin}\}$ is a dense subset of $A(x_n)$. Hence the set of uniqueness has an empty interior. $\square$

Corollary 3.4. Assume that $\text{rng}(\mu)$ is a finite union of closed intervals, equivalently $x_n \leq r_n$ for all but finitely many n 's. Then the set of uniqueness has an empty interior.

Proof. Let $x_n := \mu(\{n\})$. By Theorem 1.1 there exists $m \in \mathbb{N}$ such that $x_n \leq \sum_{k=n+1}^{\infty} x_k$ for each $n \geq m$. By Theorem 3.3 used to the series $\sum_{n=m}^{\infty} x_n$ we obtain $f(\sum_{k \in F} x_k) \geq 2$ provided F is finite and $F \cap [m, \infty) \neq \emptyset$. Since $\{\sum_{k \in F} x_k : F \in \text{Fin}, F \cap [m, \infty) \neq \emptyset\}$ is dense in $\text{rng}(\mu)$, we obtain that the set of uniqueness has an empty interior. $\square$

Although the set of uniqueness has an empty interior, it may be quite large as shows the following.

Example 3.5. Let us consider a sequence (y_n) given by $y_n := \frac{1}{2^n}$. Then $A(y_n) = [0, 1]$. Note that

$$f(t) = 2 \iff \exists m \exists (\varepsilon_i)_{i=1}^m \in \{0, 1\}^m \left(\varepsilon_m = 1 \text{ and } t \right.$$

$$= \sum_{i=1}^m \frac{\varepsilon_i}{2^i} = \sum_{i=1}^{m-1} \frac{\varepsilon_i}{2^i} + \sum_{i=m+1}^{\infty} \frac{1}{2^i} \Bigg) \iff \\ \iff t \in (0, 1) \text{ has finite dyadic expansion.}$$

Since other points in $(0, 1)$ have unique dyadic expansions, the set of uniqueness is cocountable, and therefore it is comeager and conull in $[0, 1]$.

Lemma 3.6. *Let μ be a measure such that $\mu(\{n\}) = \mu((n, \infty))$ for every $n \in \mathbb{N}$. Then*

- (i) $R(f_\mu) = \{1, 2\}$;
- (ii) $f_\mu(y) = 2$ if and only if $y = \sum_{n \in A} x_n$ for some finite set A .

Proof. Fix A such that $|A| < \infty$. Let $A := \{n_i : i \in [1, k]\}$ and consider $y := \mu(A)$. As always put $x_n := \mu(\{n\})$ and $r_n := \mu((n, \infty))$. Then

$$\begin{aligned} y &= x_{n_1} + x_{n_2} + \dots + x_{n_{k-1}} + x_{n_k} = x_{n_1} + x_{n_2} + \dots + x_{n_{k-1}} + r_{n_k} \\ &= x_{n_1} + x_{n_2} + \dots + x_{n_{k-1}} + \sum_{n=n_k+1} x_n, \end{aligned}$$

and consequently $f(y) \geq 2$.

Suppose that $f(y) > 2$. Then there would be $C \subset \mathbb{N}$ such that $\mu(A) = \mu(B) = \mu(C)$, $C \neq A$ and $C \neq B$ where $B = (A \setminus \{n_k\}) \cup (n_k, \infty)$. Let $C = \{t_i : i \in I\}$ where $I = \mathbb{N}$ or $I = [1, p]$ for some $p \in \mathbb{N}$. If $t_1 < n_1$, then

$$\sum_{n \in C} x_n \geq x_{t_1} \geq \sum_{i > t_1} x_i > \sum_{n \in A} x_n$$

which is a contradiction. Similarly $t_1 > n_1$ yields a contradiction. Thus $t_1 = n_1$. Inductively one can show that $t_1 = n_1, \dots, t_{k-1} = n_{k-1}$ and either $t_k = n_k$ or $t_{k+i} = n_k + i + 1$ for every $i \geq 0$. Hence $C = A$ or $C = B$. This is a contradiction. Note that $f(\mu(A)) = 2$ also if A is a cofinite set of indexes.

Let $A \subset \mathbb{N}$ and put $y := \mu(A)$. We will show that $f(A) \geq 2$ implies that A is either finite or cofinite. Assume that there exists $B \neq A$ such that $\sum_{n \in A} x_n = \sum_{n \in B} x_n$. Denote $k = \min(A \Delta B)$ where $A \Delta B$ stands for the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of sets A and B . We will use the following notations $A_{k+1} = A \setminus [1, k]$ and $B_{k+1} = B \setminus [1, k]$. Hence

$$0 = \left| \sum_{n \in A} x_n - \sum_{n \in B} x_n \right| \geq x_k - \left| \sum_{n \in A_{k+1}} x_n - \sum_{n \in B_{k+1}} x_n \right| \geq x_k - r_k = 0,$$

which means that $|\sum_{n \in A_{k+1}} x_n - \sum_{n \in B_{k+1}} x_n| = r_k$, so $A_{k+1} = \mathbb{N} \setminus [1, k]$ and $B_{k+1} = \emptyset$ or vice-versa. But if A_{k+1} is cofinite, so is A , and if A_{k+1} is empty, then A is finite. $\square$

Lemma 3.7. *Let $\sum_{n=1}^{\infty} x_n$ be a convergent series of positive terms and let f be its cardinal function. Moreover let f_k stand for the cardinal function of $\sum_{n=k}^{\infty} x_n$. The following assertions are equivalent:*

- (i) f is bounded;

- (ii) f_k is bounded for all $k \in \mathbb{N}$;
 (iii) f_k is bounded for some $k \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). Since $A(x_n) \supset A((x_n)_{n \geq k})$ and $f_k \leq f$ on $A((x_n)_{n \geq k})$ we are done.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Assume that f_k is bounded. By the already proved implication (i) $\Rightarrow$ (ii) it is enough to show that f is bounded. First, we show that f_{k-1} is bounded. Fix $y \in A((x_n)_{n \geq k})$. Note that

$$\begin{aligned} \left\{ A \subset [k-1, \infty) : \sum_{n \in A} x_n = y \right\} &= \left\{ A \subset [k, \infty) : \sum_{n \in A} x_n = y \right\} \\ &\cup \left\{ A \subset [k, \infty) : \sum_{n \in A} x_n = y - x_{k-1} \right\} \end{aligned}$$

is finite as a union of two finite sets. Hence f_{k-1} is bounded. Using a simple induction, we show that $f_{k-2}, \dots, f_2, f_1 = f$ are bounded. $\square$

Theorem 3.8. Let μ be a measure with $\mu(\{n\}) \leq \mu((n, \infty))$. Then

- (i) if $\mu(\{n\}) < \mu((n, \infty))$ for finitely many $n \in \mathbb{N}$, then f_μ is bounded;
 (ii) if $\mu(\{n\}) < \mu((n, \infty))$ for infinitely many $n \in \mathbb{N}$, then $f_\mu^{-1}(\{\omega, \mathfrak{c}\}) \neq \emptyset$;

Proof. Let $x_n := \mu(\{n\})$ and $r_n := \mu((n, \infty))$. (i). Let k be the largest natural number n for which the inequality $x_n < r_n$ holds. Therefore $x_n = r_n$ for $n \geq k+1$. By Lemma 3.6 the cardinal function f_{k+1} is bounded. Hence by Lemma 3.7 we obtain the boundedness of f .

(ii). Firstly we inductively define a sequence of indexes $k_0 < k_1 < \dots$ such that

$$x_{k_j} < r_{k_j} \text{ for every } j \geq 0 \text{ and } \sum_{j=l}^i x_{k_j} < r_{k_l} \text{ for every } 0 \leq l \leq i. \quad (1)$$

Find k_0 such that $x_{k_0} < r_{k_0}$. Then (1) holds for $l = i = 0$. Assume that we have already constructed $k_0 < k_1 < \dots < k_{p-1}$. We find $k_p > k_{p-1}$ such that $x_{k_p} < r_{k_p}$ and $x_{k_p} < \min_{l < p} (r_{k_l} - \sum_{j=l}^{p-1} x_{k_j})$. Then (1) holds for all $0 \leq l \leq i \leq p$.

Now, consider an element $t := x_{k_0} + x_{k_1} + x_{k_2} + \dots$. Note that

$$t - \sum_{j=0}^{l-1} x_{k_j} = \sum_{j=l}^{\infty} x_{k_j} \leq r_{k_l}.$$

Thus $t - \sum_{j=0}^{l-1} x_{k_j} \in [0, r_{k_l}]$ and the slow convergence implies that $[0, r_{k_l}] = A((x_n)_{n > k_l})$. Therefore there is $A_l \subset [k_l + 1, \infty) \cap \mathbb{N}$ such that

$$t - \sum_{j=0}^{l-1} x_{k_j} = \sum_{n \in A_l} x_n.$$

Hence $t = \sum_{n \in B_l} x_n$ where $B_l = \{k_0, k_1, \dots, k_{l-1}\} \cup A_l$. Since $k_0, k_1, \dots, k_{l-1} \in B_l$ and $k_l \notin B_l$, then $B_1, B_2, \dots$ are pairwise distinct, and therefore $f(t) \geq \omega$. $\square$

Immediately we obtain the following.

Corollary 3.9. *Let μ be a measure such that $f_\mu^{-1}(\{\omega, \mathfrak{c}\}) = \emptyset$ and f_μ is unbounded in $\mathbb{N}$. Then $\mu(\{n\}) > \mu((n, \infty))$ for infinitely many n 's.*

We do not know if there is a measure which fulfills the assumptions of Corollary 3.9.

Remark 3.10. Note that the fact that inequality $x_n < r_n$ holds for infinitely many n 's does not imply that $f^{-1}(\{\omega, \mathfrak{c}\}) \neq \emptyset$. Consider the following sequence $(x_n) := (\frac{2}{4}, \frac{3}{4}, \frac{2}{4^2}, \frac{3}{4^2}, \dots)$. As we have mentioned in the Introduction the set $\mathbf{T} = A(x_n)$ is known as the Guthrie-Nymann Cantorval. Bielas, Plewik and Walczyńska showed in [6] that $R(f) = \{1, 2\}$ where f is the cardinal function of $A(x_n)$. This shows that the slow convergence assumption in Theorem 3.8 is essential.

By a gap in E we understand any such interval (a, b) that $a, b \in E$ and $(a, b) \cap E = \emptyset$. The following lemma can be found in [2].

Lemma 3.11. *Suppose that (a, b) is a gap in $E = A(x_n)$ such that for any gap (a_1, b_1) with $b_1 < a$ we have $b - a > b_1 - a_1$ (in other words (a, b) is the longest gap from the left). Then $b = x_k$ for some $k \in \mathbb{N}$ and $a = r_k$.*

Proposition 3.12. *Let μ be a measure which set of uniqueness equals $\{0, \mu(\mathbb{N})\}$. Then $\text{rng}(\mu)$ is an interval.*

Proof. Suppose to the contrary that $\text{rng}(\mu)$ is not an interval. Then it contains a gap. Take the longest gap from the left in $\text{rng}(\mu)$, say (a, b) . By Lemma 3.11 a and b have unique representations. This yields a contradiction. $\square$

Remark 3.13. The examples considered so far may suggest that the set of uniqueness is either small (meager) in $A(x_n)$ or large (comeager) in $A(x_n)$. The following simple example shows that this is not the case. Consider the sequence $x_1 = x_2 := 1/2$ and $x_n := 1/2^{n-1}$ for $n \geq 3$. Then $A(x) = [0, 3/2]$; all but countably many points in $[0, 1/2)$ and in $(1, 3/2]$ have unique expansion, while all points in $[1/2, 1]$ can be obtained for at least two ways (for the expansion of any $x \in [1/2, 1]$ one need to use x_1 or x_2).

Let us go back to the Guthrie-Nymann Cantorval $\mathbf{T}$. As we have mentioned in the Introduction and in Remark 3.10 Bielas et al. [6] determined the cardinal function for $\mathbf{T}$. They have characterized the set of all points of the Cantorval, which are not obtained in the unique way and have shown that such points have exactly two expansions. It turns out that the set of uniqueness is dense in $\mathbf{T}$, has cardinality continuum, but it does not contain any interval (its complement in $\mathbf{T}$ is dense as well).

Problem 3.14. *Does there exists a measure μ such that its set of uniqueness contains an interval? Note that $\text{rng}(\mu)$ cannot be a finite union of intervals by Corollary 3.4.*

Note that the properties of the set of uniqueness strongly depends on the direction of the inequality between elements x_n and the tails r_n . Namely if the series $\sum_{n=1}^{\infty} x_n$ is quickly convergent, that is $x_n > r_n$ for each $n \in \mathbb{N}$, then each point of the measure range is obtained uniquely by Proposition 3.1. Example 3.2 shows that the latter equality may also hold for series, which is not quickly convergent. On the other hand the equality $x_n \leq r_n$ holds for large enough n if and only if $A(x_n)$ is a finite union of closed intervals. In that case the set of uniqueness has an empty interior in $\text{rng}(\mu)$, by Corollary 3.4. In the next Example we show that there exists a series for which $x_n \leq r_n$ holds for infinitely many n , but each point of the measure range is obtained uniquely. It strengthens the Example 3.2 and shows that the assumption in Corollary 3.4 is optimal in sense of relation between x_n 's and remainders r_n 's.

Example 3.15. We define $x_{2n-1} := \frac{1}{10^{n^2}} + \frac{1}{2 \cdot 10^{(n+1)^2}}$, $x_{2n} := \frac{1}{10^{n^2}}$ for all natural n . Note that $x_{2n-1} \leq x_{2n} + x_{2n+1} < r_{2n-1}$ for all $n \in \mathbb{N}$. We will show that each point of $A(x_n)$ has unique representation. Here we give a list of properties of the series $\sum_{n=1}^{\infty} x_n$ that we will use in the sequel; their proofs are tedious and we will left them to the readers. All of the inequalities hold for all $k \in \mathbb{N}$.

- (i) $x_{2k} > r_{2k}$;
- (ii) $x_{2k+1} > r_{2k+2}$;
- (iii) $x_{2k+1} - x_{2k+2} > r_{2k+5}$;
- (iv) $x_{2k+2} + x_{2k+4} - x_{2k+1} > r_{2k+4}$;
- (v) $x_{2k+1} + x_{2k+3} - x_{2k+2} - x_{2k+4} > r_{2k+4}$;
- (vi) $x_{2k+1} + x_{2k+4} - x_{2k+2} - x_{2k+3} > r_{2k+4}$.

Let us assume that there exists $x \in A(x_n)$ such that $f(x) \geq 2$, that is $x = \sum_{n \in A_0} x_n + \sum_{n \in A_1} x_n = \sum_{n \in B_0} x_n + \sum_{n \in B_1} x_n$ for some $A_0, B_0 \subset 2\mathbb{N}$, $A_1, B_1 \subset 2\mathbb{N} - 1$, where $A_0 \neq B_0$ or $A_1 \neq B_1$. Note that both $(x_{2n})_{n=1}^{\infty}$ and $(x_{2n-1})_{n=1}^{\infty}$ are quickly convergent. So if $A_i = B_i$, then $\sum_{n \in A_{1-i}} x_n = \sum_{n \in B_{1-i}} x_n$, and by the quick convergence of the sequence $(x_{2n+1-i})_{n=1}^{\infty}$, we obtain $A_{1-i} = B_{1-i}$ which is impossible. Thus $A_0 \neq B_0$ and $A_1 \neq B_1$.

Let $2k+1$ be the minimal index belonging to $A_1 \setminus B_1$. By (i) we obtain $A_0 \cap \{1, \dots, 2k\} = B_0 \cap \{1, \dots, 2k\}$. By (ii) we have $2k+2 \in B_0 \setminus A_0$. By (iii), we get $2k+4 \in B_0 \setminus A_0$ or $2k+3 \in B_1 \setminus A_1$. If $2k+4 \in B_0 \setminus A_0$, then by the inequality (iv), we get $2k+3 \in A_1 \setminus B_1$. If $2k+3 \in B_1 \setminus A_1$, then we have $x_{2k+2} + x_{2k+3} - x_{2k+1} > r_{2k+4}$, so $2k+4 \in A_0 \setminus B_0$. Therefore:

either $2k+4 \in B_0 \setminus A_0$ and $2k+3 \in A_1 \setminus B_1$ **or** $2k+3 \in B_0 \setminus A_0$ and
 $2k+4 \in A_1 \setminus B_1$.

For shortening a notation for any set A we denote $A^k = A \cap \{1, \dots, k\}$. Hence

$$\begin{aligned}
 0 &= \left| \sum_{n \in A_0} x_n + \sum_{n \in A_1} x_n - \sum_{n \in B_0} x_n - \sum_{n \in B_1} x_n \right| = \\
 &\left| \sum_{n \in A_0^{2k+4}} x_n + \sum_{n \in A_1^{2k+4}} x_n - \sum_{n \in B_0^{2k+4}} x_n - \sum_{n \in B_1^{2k+4}} x_n + \sum_{n \in A_0 \setminus A_0^{2k+4}} x_n \right. \\
 &\quad \left. + \sum_{n \in A_1 \setminus A_1^{2k+4}} x_n - \sum_{n \in B_0 \setminus B_0^{2k+4}} x_n - \sum_{n \in B_1 \setminus B_1^{2k+4}} x_n \right| \geq \\
 &\left| \sum_{n \in A_0^{2k+4}} x_n + \sum_{n \in A_1^{2k+4}} x_n - \sum_{n \in B_0^{2k+4}} x_n - \sum_{n \in B_1^{2k+4}} x_n \right| - \left| \sum_{n \in A_0 \setminus A_0^{2k+4}} x_n \right. \\
 &\quad \left. + \sum_{n \in A_1 \setminus A_1^{2k+4}} x_n - \sum_{n \in B_0 \setminus B_0^{2k+4}} x_n - \sum_{n \in B_1 \setminus B_1^{2k+4}} x_n \right| \\
 &\geq \left| \sum_{n \in A_0^{2k+4}} x_n + \sum_{n \in A_1^{2k+4}} x_n - \sum_{n \in B_0^{2k+4}} x_n - \sum_{n \in B_1^{2k+4}} x_n \right| - r_{2k+4} > 0.
 \end{aligned}$$

The last strict inequality, which leads to a contradiction, follows from (v) in the case (A), and from (vi) in the case (B).

4. Measures with Prescribed Cardinal Functions

In this section we show how to produce a measure with given range of its cardinal function. We also give examples of sets $R \subset \mathbb{N}$ such that there is no measure μ with $R = R(f_\mu)$. Let us start from the case when the set of uniqueness is the whole range $\text{rng}(\mu)$ up to some finite set.

Proposition 4.1. *Assume that the set of uniqueness for μ equals $\text{rng}(\mu) \setminus \{x\}$ for some $x \in \text{rng}(\mu)$. Then $f_\mu(x) = 2$ and there are $A, B \subset \mathbb{N}$ with $\mu(A) = \mu(B)$, $A \cap B = \emptyset$ and $A \cup B = \mathbb{N}$.*

Proof. Suppose to the contrary that $f(x) \geq 3$. Since f is symmetric it is clear that $x = \frac{1}{2}\mu(\mathbb{N})$. Otherwise we would have $f(x) \geq 3$ and $f(\sum_{n=1}^{\infty} x_n - x) \geq 3$ by Proposition 2.1(2). There exist three distinct sets A, B, C such that $x = \mu(A) = \mu(B) = \mu(C)$. Note that the sets A, B, C cannot be pairwise disjoint, since then $\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) = 3x > \mu(\mathbb{N})$. Without loss of generality we may assume that $A \cap B \neq \emptyset$. Hence for $y := \mu(A \setminus B) = \mu(B \setminus A) < x$ we have $f(y) \geq 2$. But this contradicts the assumption.

Therefore $f(x) = 2$. Hence there are $A, B \subset \mathbb{N}$ with $\mu(A) = \mu(B) = \frac{1}{2}\mu(\mathbb{N})$. As before we show that $A \cap B = \emptyset$. Since $\mu(A \cup B) = \mu(\mathbb{N})$ and the support of μ equals $\mathbb{N}$, we obtain that $A \cup B = \mathbb{N}$. $\square$

The following example shows that the assumption of Proposition 4.1 is fulfilled for some measure.

Example 4.2. Let $\sum_{n=1}^{\infty} y_n$ be any convergent series with positive terms such that each of its subsums has unique representation. Let us define $x_1 := \sum_{n=1}^{\infty} y_n$ and $x_{n+1} := y_n$ for $n \in \mathbb{N}$. Then $A(x_n) = A(y_n) \cup (A(y_n) + x_1)$ and $A(y_n) \cap (A(y_n) + x_1) = \{x_1\}$. Consequently the set of uniqueness equals $A(x_n) \setminus \{x_1\}$ and $f(x_1) = 2$. Here the sets from the assertion of Proposition 4.1 are $A := [2, \infty)$ and $B := \{1\}$.

We can also construct an example in which both sets A and B are infinite. We drop out the terms monotonicity assumption for better readability of the computations in the following example.

Example 4.3. Let $x_1 := \frac{8}{9}$, $x_2 := \frac{2}{3}$, $x_{2n+1} := \frac{1}{10^n}$, $x_{2n+2} := \frac{3}{10^n}$ for $n \in \mathbb{N}$. Note that $\sum_{n=1}^{\infty} x_n = 2$ and $\sum_{n=1}^{\infty} x_{2n-1} = \sum_{n=1}^{\infty} x_{2n} = 1$. The sets from the assertion of Proposition 4.1 are $A := 2\mathbb{N}$ and $B := 2\mathbb{N} - 1$. We will show that other points of $\text{rng}(\mu)$ are uniquely obtained where μ is the measure given by the series $\sum_{n=1}^{\infty} x_n$. Let $x \in \text{rng}(\mu)$. There is $E \subset \mathbb{N}$ with $x = \mu(E)$. Using the fact that $0 \leq x \leq \mu(\mathbb{N}) = 2$ and the symmetry of $\text{rng}(\mu)$, it is enough to prove the uniqueness of E for $x < 1$. Consider the following cases.

Case 1. $0 < x < x_2$. Then $1, 2 \notin E$. Thus $x \in \{\sum_{n=1}^{\infty} \frac{\delta_n}{10^n} : (\delta_n) \in \{0, 1, 3, 4\}^{\omega}\}$. For every $n \in \mathbb{N}$ we have

- $\delta_n = 0$ if and only if $2n+1 \notin E$ and $2n+2 \notin E$;
- $\delta_n = 1$ if and only if $2n+1 \in E$ and $2n+2 \notin E$;
- $\delta_n = 3$ if and only if $2n+1 \notin E$ and $2n+2 \in E$;
- $\delta_n = 4$ if and only if $2n+1 \in E$ and $2n+2 \in E$.

Hence the set E is a unique representation of x .

Case 2. $x_2 < x < x_1$. Since $\mu([3, \infty)) < \mu(\{2\}) = x_2 < x < x_1 = \mu(\{1\})$, then $2 \in E$ and $1 \notin E$. Note that $x - x_2 = \mu(E) \setminus \mu(\{2\}) = \mu(E \setminus \{2\})$. Since $x - x_2 < x_1 - x_2 < x_2$, then by already proved Case 1 we obtain that $E \setminus \{2\}$ is a unique representation of $x - x_2$. Since every representation of x contains x_2 , the set E is unique.

Case 3. $x_1 < x \leq 1$. Assume that there is $D \subset \mathbb{N}$, $D \neq E$ with $x = \mu(D)$. We will show that $x = 1$. If $1 \in D \cap E$ then $x - x_1 < 1 - x_1 < x_2$ and by the already proved Case 1 the representation $x - x_1 = \mu(E \setminus \{1\})$ is unique. Thus $D \setminus \{1\} = E \setminus \{1\}$, and consequently $D = E$ which is a contradiction. Hence $1 \notin D \cap E$. In the same way one can prove that $2 \notin D \cap E$. Since $x > \mu([3, \infty))$, then $1 \in D$ and $2 \in E$ or vice-versa. Let assume that $1 \in D$ and $2 \in E$. We will prove inductively that D and E are the sets of all odd and even numbers, respectively. Let us assume that for some $n \in \mathbb{N}$ we have already prove that $D \supset \{1, 3, \dots, 2n-1\}$, $D \cap \{2, 4, \dots, 2n\} = \emptyset$ and $E \supset \{2, 4, \dots, 2n\}$,

$E \cap \{1, 3, \dots, 2n-1\} = \emptyset$. Since

$$\begin{aligned} \sum_{k=1}^n x_{2k} + x_{2n+1} + \sum_{k=2n+3}^{\infty} x_k &= \left[\frac{2}{3} + \frac{1}{3} \left(1 - \frac{1}{10^{n-1}} \right) \right] + \frac{1}{10^n} + \frac{4}{9 \cdot 10^n} \\ &< \frac{8}{9} + \frac{1}{9} \left(1 - \frac{1}{10^{n-1}} \right) = \sum_{k=1}^n x_{2k-1} \leq \mu(D), \end{aligned}$$

we obtain $2n+2 \in E$. Note that $\sum_{k=1}^{n+1} x_{2k} + x_{2n+1} > 1$, so $2n+1 \notin E$. Since $\sum_{k=1}^n x_{2k-1} + x_{2n+2} > 1$, we obtain $2n+2 \notin D$. Moreover $\sum_{k=1}^n x_{2k-1} + \sum_{k=2n+3}^{\infty} x_k < \sum_{k=1}^{n+1} x_{2k} \leq \mu(E)$, so $2n+1 \in D$. By the induction we obtain that $D = 2\mathbb{N} - 1$ and $E = 2\mathbb{N}$. Thus $x = 1$. Note that we have also shown that $\mu(C) = 1$ if and only if $C = 2\mathbb{N} - 1$ or $C = 2\mathbb{N}$. Hence $f(1) = 2$.

Case 4. $x = x_2$. It is clear that we must have $E = \{2\}$.

Case 5. $x = x_1$. Suppose $1 \notin E$. Since $x_1 > \mu([3, \infty))$, then $2 \in E$. Therefore $x - x_2 \in \text{rng}(\mu)$. Note that

$$x - x_2 = \frac{2}{9} = \sum_{n=1}^{\infty} \frac{2}{10^n} \notin \left\{ \sum_{n=1}^{\infty} \frac{\delta_n}{10^n} : (\delta_n) \in \{0, 1, 3, 4\}^\omega \right\} = \text{rng}(\mu) \cap [0, x_2).$$

This is a contradiction. Thus $E = \{1\}$ is a unique representation of x_1 .

Now we will focus on the more general situation when the set of uniqueness of a measure μ equals $\text{rng}(\mu)$ up to some finite set F . The following shows that for any k is is possible to define μ such that F has exactly k points. Let us start from $k = 2$.

Example 4.4. Let $\sum_{n=1}^{\infty} y_n$ be any convergent series with positive terms such that $f_y^{-1}(\{1\}) = A(y_n)$. Let us define $x_1 := 3 \sum_{n=1}^{\infty} y_n$, $x_2 := \sum_{n=1}^{\infty} y_n$ and $x_{n+2} := y_n$ for each $n \in \mathbb{N}$. Clearly $A(x_n) = A(y_n) \cup (A(y_n) + x_2) \cup (A(y_n) + x_1) \cup (A(y_n) + x_1 + x_2)$. Moreover $f(x_2) = f(x_1 + x_2) = 2$ and $f^{-1}(\{1\}) = A(x_n) \setminus \{x_1, x_1 + x_2\}$.

For arbitrary k we have the following.

Proposition 4.5. *Fix $k \in \mathbb{N}$. Then there exists a measure μ such that its set of uniqueness equals $\text{rng}(\mu)$ up to a set F with $|F| = k$, and such that $f_\mu(t) = 2$ for $t \in F$.*

Proof. We will proceed inductively. For $k = 1$ it is constructed in Example 4.2 (or in Example 4.3). Assume that $\sum_{n=1}^{\infty} y_n$ is a series such that $A(y_n) = f^{-1}(\{1\}) \cup f^{-1}(\{2\})$ and $|f^{-1}(\{2\})| = k$, where f is a cardinal function for $\sum_{n=1}^{\infty} y_n$. We will show that it is possible to construct two series $\sum_{n=1}^{\infty} x_n^1$ and $\sum_{n=1}^{\infty} x_n^2$ such that $A(x_n^i) = f_i^{-1}(\{1\}) \cup f_i^{-1}(\{2\})$ and:

- (1) $|f_1^{-1}(\{2\})| = 2k$;
- (2) $|f_2^{-1}(\{2\})| = 2k + 1$

where f_i is a cardinal function for $\sum_{n=1}^{\infty} x_n^i$.

(1). Let x_1^1 be any real number larger than $\sum_{n=1}^{\infty} y_n$, $x_{n+1}^1 := y_n$ for $n \in \mathbb{N}$. Then $A(x_n^1) = A(y_n) \cup (x_1^1 + A(y_n))$ and we have $f_1(x) = f(x)$ for all $x \in A(y_n)$ and $f_1(x) = f(x - x_1)$ for each $x \in x_1 + A(y_n)$.

(2). Let $x_1^2 := \sum_{n=1}^{\infty} y_n$, $x_{n+1}^2 := y_n$ for $n \in \mathbb{N}$. Then $A(x_n^2) = A(y_n) \cup (x_1^2 + A(y_n))$. We have $f_2(x) = f(x)$ for all $x \in A(y_n) \setminus \{x_1^2\}$, $f(x_1^2) = 2$ and $f_2(x) = f(x - x_1^2)$ for each $x \in (x_1^2 + A(y_n)) \setminus \{x_1^2\}$. $\square$

The construction in the Proof of Proposition 4.5 non-unique points are obtained in exactly two ways. Now we will prove that this is not a special situation but general phenomenon.

Proposition 4.6. *Let μ be a measure such that its set of uniqueness equals $\text{rng}(\mu)$ up to a finite set F . Then any value from F has exactly two representations. In other word, if there is a value $x \in \text{rng}(\mu)$ with $f_\mu(x) \geq 3$, then there are infinitely many non-unique values in $\text{rng}(\mu)$.*

Proof. Suppose that there exist three distinct sets $A, B, C \subset \mathbb{N}$ with $\mu(A) = \mu(B) = \mu(C)$. Note that $\mu(A \setminus \{k\}) = \mu(B \setminus \{k\})$ and $A \setminus \{k\} \neq B \setminus \{k\}$ for $k \in A \cap B$. Therefore $A \cap B$ has to be a finite set. Similarly $A \cap C$, $C \cap B$ are finite. Further, $\mu(A \cup \{k\}) = \mu(B \cup \{k\})$ and $A \cup \{k\} \neq B \cup \{k\}$ for $k \in \mathbb{N} \setminus (A \cup B)$. Therefore the sets $A \cup B$, $A \cup C$, $B \cup C$ have to be cofinite. Hence $C \setminus (A \cup B)$ is finite as a subset of finite set $\mathbb{N} \setminus (A \cup B)$; similarly $B \setminus (A \cup C)$ is finite. Thus

$$B \cup C = [C \setminus (A \cup B)] \cup [B \setminus (A \cup C)] \cup (B \cap C) \cup (B \cap A) \cup (C \cap A)$$

is finite as a finite union of finite sets, which is a contradiction. $\square$

Let us give a couple of examples which show how to construct a measure μ (or a series) with prescribed cardinal function f_μ or with a given range $R(f_\mu)$. The first example in the row shows that there is a measure μ (or series) which has exactly two points $t \neq v$ with $f_\mu(t) = f_\mu(v) = 3$.

Example 4.7. Let $\sum_{n=1}^{\infty} y_n = 1$ be a series such that each point of $A(y_n)$ has a unique representation. Let us define $x_1 = x_2 := \sum_{n=1}^{\infty} y_n$ and $x_{n+2} := y_n$ for each $n \in \mathbb{N}$. Clearly $A(x_n) = A(y_n) \cup (A(y_n) + x_1) \cup (A(y_n) + 2x_1)$. We have $f^{-1}(\{1\}) = (A(y_n) \cup (A(y_n) + 2x_1)) \setminus \{x_1, 2x_1\}$, $f^{-1}(\{2\}) = (A(y_n) + x_1) \setminus \{x_1, 2x_1\}$ and $f^{-1}(\{3\}) = \{x_1, 2x_1\}$.

The following illustration describe the cardinal function for Example 4.7 with $A = A(y_n)$.

The picture should be read as follows. Point of $A \cap [0, 1)$ and $(2 + A) \cap (2, 3]$ have unique representations; points of $(1 + A) \cap (1, 2)$ have two representations, while points 1 and 2 have three representations.

The following shows that it is possible to find μ such that $\mu(\mathbb{N})/2$ has exactly 4 representation, and there are two points with 3 representations.

Example 4.8. Let $\sum_{n=1}^{\infty} y_n = 1$ be a series such that each point of $A(y_n)$ has a unique representation. Let us define $x_1 = x_2 := 2 \sum_{n=1}^{\infty} y_n$, $x_3 := \sum_{n=1}^{\infty} y_n$ and $x_{n+3} := y_n$ for each $n \in \mathbb{N}$. The cardinal function is illustrated below.

It is possible to have $R(f_\mu) = \{1, 2, 4\}$.

Example 4.9. Let us consider the following series, a slight modification of Example 4.8. A series $\sum_{n=1}^{\infty} y_n = 1$ is such that each point of $A(y_n)$ has a unique representation and $x_1 = x_2 := 3 \sum_{n=1}^{\infty} y_n$, $x_3 := \sum_{n=1}^{\infty} y_n$, $x_{n+3} := y_n$. Its cardinal function is as follows.

It is possible also to have $R(f_\mu) = \{1, 3\}$.

Example 4.10. Let $\sum_{n=1}^{\infty} y_n = 1$ be a series such that each point of $A(y_n)$ has a unique representation. Put $x_1 = x_2 = x_3 := 2 \sum_{n=1}^{\infty} y_n$ and $x_{n+3} := y_n$ for each $n \in \mathbb{N}$. Its cardinal function is described below.

An it is possible to have $R(f_\mu) = \{1, 3, 4\}$.

Example 4.11. Let $\sum_{n=1}^{\infty} y_n = 1$ be a series such that each point of $A(y_n)$ has a unique representation. Define $x_1 := 4 \sum_{n=1}^{\infty} y_n$, $x_2 = x_3 = x_4 := 2 \sum_{n=1}^{\infty} y_n$ and $x_{n+4} := y_n$ for each $n \in \mathbb{N}$. We have:

The next two propositions give methods to produce a measure μ with prescribed cardinal properties having other measure ν with given cardinal properties.

Let ν be a measure. Put $E := R(f_\nu)$ and $F := f_\nu[\text{rng}(\nu) \setminus \{0, \nu(\mathbb{N})\}]$. Note that $F = E \setminus \{1\}$ if and only if the series $\sum_{n=1}^{\infty} x_n$ is plentiful where $x_n := \nu(\{n\})$. As usual $tA := \{ta : a \in A\}$ for any $t \in \mathbb{R}$ and $A \subset \mathbb{R}$.

Proposition 4.12. *Let ν be a measure and $m \in \mathbb{N}$. There exist measures μ_i with cardinal functions g_i , $i = 1, 2, 3, 4$, such that*

- (1) $R(g_1) = E \cup \{2\}$;

- (2) $R(g_2) = E \cup \{3\} \cup 2F$;
 (3) $R(g_3) = \bigcup_{k=0}^m \binom{m}{k} F \cup \{ \binom{m+1}{k} : k = 0, 1, \dots, m+1 \}$;
 (4) $R(g_4) = \bigcup_{k=0}^m \binom{m}{k} E$.

Proof. Let x_n be the value of n -th ν -atom.

- (1) The construction is the same as in Example 4.2—now the measure ν is arbitrary. Define $y_1 := \sum_{n=1}^{\infty} x_n$ and $y_{n+1} := x_n$ for each natural number n . Then $\text{rng}(\mu_1) = A(y_n) = A(x_n) \cup (\sum_{n=1}^{\infty} x_n + A(x_n))$ and we have:

$$g_1(x) = \begin{cases} f(x), & x \in A(x_n) \setminus \{\sum_{n=1}^{\infty} x_n\} \\ f(x - \sum_{n=1}^{\infty} x_n), & x \in (\sum_{n=1}^{\infty} x_n + A(x_n)) \setminus \{\sum_{n=1}^{\infty} x_n\} \\ 2, & x = \sum_{n=1}^{\infty} x_n \end{cases}$$

- (2) Here we have the same construction as in Example 4.7. Define $y_1 = y_2 := \sum_{n=1}^{\infty} x_n$ and $y_{n+2} := x_n$ for all $n \in \mathbb{N}$. Thus $\text{rng}(\mu_2) = A(y_n) = A(x_n) \cup (\sum_{n=1}^{\infty} x_n + A(x_n)) \cup (2 \sum_{n=1}^{\infty} x_n + A(x_n))$. The cardinal function is as follows:

$$g_2(x) = \begin{cases} f(x), & x \in A(x_n) \setminus \{\sum_{n=1}^{\infty} x_n\} \\ 2f(x - \sum_{n=1}^{\infty} x_n), & x \in (\sum_{n=1}^{\infty} x_n + A(x_n)) \setminus \{\sum_{n=1}^{\infty} x_n, 2 \sum_{n=1}^{\infty} x_n\} \\ 2f(x - 2 \sum_{n=1}^{\infty} x_n), & x \in (2 \sum_{n=1}^{\infty} x_n + A(x_n)) \setminus \{2 \sum_{n=1}^{\infty} x_n\} \\ 3, & x \in \{\sum_{n=1}^{\infty} x_n, 2 \sum_{n=1}^{\infty} x_n\} \end{cases}$$

- (3) Define $y_n := \sum_{n=1}^{\infty} x_n$ for all $n \leq m$ and $y_n := x_{n-m}$ for all $n > m$. Then $\text{rng}(\mu_3) = A(y_n) = \bigcup_{k=0}^m k \sum_{n=1}^{\infty} x_n + A(x_n)$. We have $g_3(k \sum_{n=1}^{\infty} x_n) = \binom{m+1}{k}$ for every $k \in \{0, 1, \dots, m+1\}$ and $g_3(k \sum_{n=1}^{\infty} x_n + x) = \binom{m}{k} f(x)$ for all $x \in A(x_n) \setminus \{0, \sum_{n=1}^{\infty} x_n\}$ and $k \in \{0, 1, \dots, m\}$. Hence $R(g_3) = \bigcup_{k=0}^m \binom{m}{k} F \cup \{ \binom{m+1}{k} : k = 0, 1, \dots, m+1 \}$.
 (4) Fix $m \in \mathbb{N}$. Define $y_n := 2 \sum_{n=1}^{\infty} x_n$ for all $n \leq m$ and $y_n := x_{n-m}$ for all $n > m$. Then $\text{rng}(\mu_4) = A(y_n) = \bigcup_{k=0}^m (2k \sum_{n=1}^{\infty} x_n + A(x_n))$. We have $g_3(2k \sum_{n=1}^{\infty} x_n + x) = \binom{m}{k} f(x)$ for all $x \in A(x_n)$ and $k \in \{0, 1, \dots, m\}$. Hence $R(g_3) = \bigcup_{k=0}^m \binom{m}{k} E$. $\square$

Let us recall one basic fact about arithmetic of cardinal numbers. If κ is infinite cardinal number and $k \in \mathbb{N}$, then $k \cdot \kappa = \kappa \cdot k = \kappa$. If κ is a natural number, then $\kappa \cdot k$ is a usual product of two natural numbers. Therefore $k \cdot \omega = \omega$ and $k \cdot \mathfrak{c} = \mathfrak{c}$.

Proposition 4.13. *Let τ be a measure on $[1, j]$, and let ν be a measure on $\mathbb{N}$. Then there is a measure μ such that $R(f_\mu) = R(f_\tau) \cdot R(f_\nu) = \{k \cdot \kappa : k \in \text{rng}(\tau), \kappa \in \text{rng}(\nu)\}$ holds.*

Proof. Let $k_i := \tau(\{i\})$ for $i \in [1, j]$ and $y_n := \nu(\{n\})$ for $n \in \mathbb{N}$. Let $\text{rng}(\tau) := \{\sigma_1, \dots, \sigma_m\}$. If $(a\nu)(E) := a \cdot \nu(E)$, then $R(f_{a\nu}) = R(f_\nu)$. Hence we may assume that $\sum_{n=1}^{\infty} y_n < \min\{\sigma_i - \sigma_l : i, l \in \{1, \dots, m\}, i \neq l\}$. Let us define $x_i := k_i$ for $i \in [1, j]$ and $x_{j+n} := y_n$ for all $n \in \mathbb{N}$. Thus $\text{rng}(\mu) = \text{rng}(\tau) + \text{rng}(\nu) = \bigcup_{p=1}^m (\sigma_p + \text{rng}(\nu))$ and $(\sigma_i + \text{rng}(\nu)) \cap (\sigma_l + \text{rng}(\nu)) = \emptyset$ for

all $i, l \in \{1, \dots, m\}$, $i \neq l$. As a result we have $f(\sigma_p + y) = f_\sigma(\sigma_p) \cdot f_y(y)$ for all $p \in \{1, \dots, m\}$, $y \in \text{rng}(\nu)$. Hence $R(f_\mu) = R(f_\tau) \cdot R(f_\nu)$. $\square$

Note that to produce a measure μ on $\mathbb{N}$ with given range $R = R(f_\mu) \subset \mathbb{N}$, it is enough to find a measure τ on some finite set $[1, n]$ with $R(f_\tau) = R$. To see it take a measure ν on $\mathbb{N}$ such that any value of $\text{rng}(\nu)$ has a unique representation. Then by Proposition 4.12 we find μ with $R(f_\mu) = R(f_\tau) \cdot 1 = R$. On the other hand if ν is such that $R(f_\nu) = \{1, \mathfrak{c}\}$, see Example 2.6, then there is μ with $R(f_\mu) = R(f_\tau) \cdot \{1, \mathfrak{c}\} = R \cup \{\mathfrak{c}\}$.

The following Table summarizes our searching of measures μ with $R(f_\mu) \subset [1, 6]$. Some of them can be found by Proposition 4.12, while for the others we need to produce new examples.

Range $R(f_\mu)$	Method
1,2	Proposition 4.12 (1)
1,2,3	Proposition 4.12 (2)
1,3	Proposition 4.12 (4) for $m = 3$
1,2,3,4	Proposition 4.12 (2) for ν with $R(f_\nu) = \{1, 2\}$
1,2,4	Proposition 4.12 (4) for $m = 2$ and ν with $R(f_\nu) = \{1, 2\}$
1,3,4	Example 4.11
1,2,3,5	Example 4.17
1,2,4,5	Example 4.18
1,4,6	Proposition 4.12 (4) for $m = 4$
1,3,4,6	Proposition 4.12 (3) for $m = 3$
1,4,5,6	Example 4.15
1,3,4,5,6	Example 4.16
1,3,6	Example 4.14

Examples 4.14–4.19 are constructed in a similar way that in Proposition 4.13. We assume that $\sum_{n=1}^{\infty} y_n$ is a series such that any its point has unique representation, and f is the cardinal function for the final series $\sum_{n=1}^{\infty} x_n$.

Example 4.14. Let $x_1 = x_2 := 6 \sum_{n=1}^{\infty} y_n$, $x_3 = x_4 = x_5 := 2 \sum_{n=1}^{\infty} y_n$ and $x_{n+5} := y_n$ for each $n \in \mathbb{N}$. Then $R(f) = \{1, 3, 6\}$.

Example 4.15. Let $x_1 := 12 \sum_{n=1}^{\infty} y_n$, $x_2 := 6 \sum_{n=1}^{\infty} y_n$, $x_3 = x_4 = x_5 = x_6 := 2 \sum_{n=1}^{\infty} y_n$ and $x_{n+6} := y_n$ for each $n \in \mathbb{N}$. Then $R(f) = \{1, 4, 5, 6\}$.

Example 4.16. Let $x_1 := 6 \sum_{n=1}^{\infty} y_n$, $x_2 := 4 \sum_{n=1}^{\infty} y_n$, $x_3 = x_4 = x_5 := 2 \sum_{n=1}^{\infty} y_n$ and $x_{n+5} := y_n$ for each $n \in \mathbb{N}$. Then $R(f) = \{1, 3, 4, 5, 6\}$.

Example 4.17. Let $x_1 := 6 \sum_{n=1}^{\infty} y_n$, $x_2 = x_3 := 4 \sum_{n=1}^{\infty} y_n$, $x_4 = x_5 := 2 \sum_{n=1}^{\infty} y_n$ and $x_{n+5} := y_n$ for each $n \in \mathbb{N}$. Then $R(f) = \{1, 2, 3, 5\}$.

Example 4.18. Let $x_1 := 10 \sum_{n=1}^{\infty} y_n$, $x_2 = x_3 := 6 \sum_{n=1}^{\infty} y_n$, $x_4 = x_5 := 4 \sum_{n=1}^{\infty} y_n$ and $x_{n+5} := y_n$ for each $n \in \mathbb{N}$. Then $R(f) = \{1, 2, 4, 5\}$.

Example 4.19. Let $x_1 = x_2 := 4 \sum_{n=1}^{\infty} y_n$, $x_3 = x_4 = x_5 := 2 \sum_{n=1}^{\infty} y_n$ and $x_{n+5} := y_n$ for each $n \in \mathbb{N}$. Then $R(f) = \{1, 3, 5, 7\}$.

The above table does not contain some subsets of $[1, 6]$ (clearly each range has to contain 1). For example we were not able to produce a measure μ with $R(f_\mu) = \{1, 4\}$. We have found that there is no measure ν defined on a finite set with $R(f_\nu) = \{1, 4\}$, but we do not know whether there is a measure μ on $\mathbb{N}$ with $R(f_\mu) = \{1, 4\}$. On the other hand we found that if such measure exists, then the mapping $n \mapsto \mu(\{n\})$ has to be one-to-one.

Proposition 4.20. *Suppose that μ is a measure with $R(f_\mu) = \{1, 4\}$. Then $\mu(\{n\}) \neq \mu(\{m\})$ for distinct $n, m \in \mathbb{N}$*

Proof. Let $x_n := \mu(\{n\})$. Assume that $x_k = x_m$. Since $f(x_k) \geq 2$, we have $f(x_k) = 4$, that is $x_k = x_m = \mu(A) = \mu(B)$, where $A \neq B$, $\{k\} \neq A \neq \{m\}$ and $\{k\} \neq B \neq \{m\}$. Hence $x_k + x_m = \mu(A \cup \{k\}) = \mu(A \cup \{m\}) = \mu(B \cup \{k\}) = \mu(B \cup \{m\})$, so $f(x_k + x_m) \geq 5$, which gives a contradiction. $\square$

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