Harmonic $SU(3)$ - and G_2 -Structures via Spinors

Kamil Niedziałomski

Abstract. In this note, using the spinorial description of $SU(3)$ and G_2 -structures obtained recently by other authors, we give necessary and sufficient conditions for harmonicity of the above mentioned structures. We describe the results on appropriate homogeneous spaces. Here, harmonicity means harmonicity of a certain unique section induced by the G -structure in consideration.

Mathematics Subject Classification. 53C10, 53C25, 53C27, 53C43.

Keywords. $SU(3)$ -structure, G_2 -structure, intrinsic torsion, harmonic G -structure, spin structure, homogeneous space.

1. Introduction

Let M be a spin manifold equipped with a $SU(3)$ (then $n = \dim M = 6$) or G_2 (then $n = \dim M = 7$) structure. This means, that the special orthonormal frame bundle $SO(M)$ has a reduction of the structure group $SO(n)$ to $SU(3)$ or G_2 , respectively. Recently, the authors of [2] have studied such structures via spinorial approach. A global unit spinor φ in the spinor bundle defines, depending on the dimension of a manifold, the structures mentioned above. Thus it is natural to study the geometry of a defining spinor φ . The crucial observation in [2] is the existence of an endomorphism $S:TM \rightarrow TM$ and a 1-form η (which vanishes in the G_2 case for dimensional reasons) which describe the covariant derivative of φ ,

$$\nabla_X \varphi = S(X) \cdot \varphi + \eta(X)j \cdot \varphi,$$

where j is a certain almost complex structure on the spinor bundle.

On the other hand, C. M. Wood [14, 15] and later J. C. González-Dávila and F. Martín Cabrera [7] introduced and studied the so called harmonic G -structures. Each G -structure $P \subset SO(M)$ defines a section σ_P of the associated bundle $SO(M) \times_G (SO(n)/G)$ as follows

$$\sigma(x) = [p, eG], \quad \pi_{SO(M)}(p) = x.$$

In the compact case, if σ is a harmonic section, then we say that the corresponding G -structure is harmonic. This condition is a differential equation $\sum_i (\nabla_{e_i} \xi)_{e_i} = 0$ involving the intrinsic torsion ξ , the main ingredient of all considerations. We treat it as a harmonicity condition also in the non-compact case. The intrinsic torsion ξ , shortly speaking, is a $(2, 1)$ -tensor field which is the difference of the Levi-Civita connection ∇ and the G -connection ∇^G induced by the $\mathfrak{g}$ -component of the connection form of ∇ (here $\mathfrak{g}$ is the Lie algebra of G), $\xi_X Y = \nabla_X Y - \nabla_X^G Y$. Thus the intrinsic torsion measures the defect of the G -structure to have holonomy in G .

In this note, the author tries to combine these two approaches for $G = SU(3)$ and $G = G_2$. The condition for harmonicity becomes a differential condition on S and η . In some cases, for example when $\eta = 0$, it takes a really simple form. Among others, we conclude that if a $SU(3)$ -structure is in its $\mathcal{W}_1$ or $\mathcal{W}_3$ class from the Gray–Hervella classification of possible intrinsic torsion modules, then it is harmonic. Analogously, if a G_2 -structure is in $\mathcal{W}_1$ class with the defining function λ being constant or in $\mathcal{W}_3$ class, then it is harmonic.

Finally, we present some examples on homogeneous spaces. We begin with general considerations, which lead to the following conclusion—if a unit spinor defining a G -structure is induced by a fixed point of an isotropy representation and the minimal G -connection is induced by the zero map $\Lambda_{\mathfrak{g}}$ (see the last section for details), then the structure is harmonic. We justify these results by presenting appropriate examples. Although, these examples have been already considered in the literature, they have not been studied from this point of view. In other words, we find “new” examples of harmonic $SU(3)$ and G_2 -structures.

2. The Intrinsic Torsion and Spinors—A General Approach

Let (M, g) be an oriented Riemannian manifold equipped with a G -structure, $G \subset SO(n)$, where $n = \dim M$. Let ∇^g be the Levi-Civita connection of g and ω be its connection form on $SO(M)$. The splitting on the level of Lie algebras

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}, \quad (1)$$

where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(n)$, defines the splitting $\omega = \omega_{\mathfrak{g}} + \omega_{\mathfrak{m}}$. By the fact that (1) is $\text{ad}(G)$ -invariant, it follows that the $\mathfrak{g}$ -component $\omega_{\mathfrak{g}}$ is a connection form on the G -reduction $P \subset SO(M)$ and hence defines a connection ∇^G on M . The difference

$$\xi_X Y = \nabla_X^G Y - \nabla_X^g Y, \quad X, Y \in \Gamma(TM),$$

defines a tensor $\xi \in T^*M \otimes \mathfrak{m}(TM)$ called the *intrinsic torsion*. It follows immediately that its alternation is, up to a sign, the torsion T^G of ∇^G .

Denote by N the associated bundle $SO(M) \times_{SO(n)} (SO(n)/G)$. There is a one to one correspondence between the G -structures and the sections of N . Thus, $P \subset SO(M)$ defines a unique section $\sigma_P \in \Gamma(N)$. We say that the G -structure P is harmonic if σ_P is a harmonic section [7] (if M is compact). It can be shown [7] that harmonicity is equivalent to vanishing of the following tensor

$$L = \sum_i (\nabla_{e_i}^g \xi)_{e_i}, \quad (2)$$

where the summation is taken with respect to any orthonormal basis. We treat this condition as the harmonicity condition also in the non-compact case. Since, informally speaking, ∇^G preserves the decomposition (1) and

$$L = \sum_i ((\nabla_{e_i}^G \xi)_{e_i} + \xi_{\xi_{e_i} e_i}),$$

we see that $L \in \mathfrak{m}(TM)$ [7].

Assume M is equipped with a spin structure. Let $\rho : \text{Spin}(n) \rightarrow \Delta_n$ be a spin representation. We will denote the Clifford multiplication, and all induced actions, by a “dot”. In particular acting on spinors, we have

$$X^\flat \cdot \omega - \omega \cdot X^\flat = 2X \lrcorner \omega. \quad (3)$$

Denote by $S(M)$ the induced spinor bundle, $S(M) = \text{Spin}(M) \times_\rho \Delta_n$, where $\text{Spin}(M)$ is a spin structure. Assume G is the stabilizer of some unit spinor $\varphi_0 \in \Delta_n$ and let $\varphi \in S$ be the corresponding unit spinor, which defines a G -structure. Hence, with the usual identification of $\mathfrak{so}(TM)$ with the space of 2-forms on M , the action of $\mathfrak{m}(TM)$ on φ is injective. Therefore, vanishing of $L \in \mathfrak{m}(TM)$ is equivalent to the relation

$$L \cdot \varphi = 0, \quad (4)$$

where $\cdot$ denotes the action of skew-forms on spinors.

Let us describe the above condition with the use of spinorial laplacian. Denote by the same symbol ∇^g the connection on $S(M)$ induced from the Levi-Civita connection ∇^g . Then we put [6]

$$\Delta\psi = - \sum_i (\nabla_{e_i}^g \nabla_{e_i}^g \psi - \nabla_{\nabla_{e_i}^g e_i}^g \psi).$$

We have [2]

$$\nabla_X^g \varphi = \frac{1}{2} \xi_X \cdot \varphi \quad (5)$$

which follows from the fact that $\nabla^G \varphi = 0$. Differentiating (5) we get

$$\Delta\varphi = -\frac{1}{2} L \cdot \varphi - \frac{1}{4} \sum_i \xi_{e_i} \cdot \xi_{e_i} \cdot \varphi.$$

The second component on the right hand side can be interpreted in the following way. For any tensor $T \in T^*M \otimes \mathfrak{so}(TM)$ define

$$c_T = \frac{1}{2} \sum_i T_{e_i} \cdot T_{e_i} \quad \text{and} \quad \sigma_T = \frac{1}{2} \sum_i T_{e_i} \wedge T_{e_i}$$

as elements of the Clifford bundle acting on spinors. These elements, defined for totally skew tensors, have been already considered and their important role has been established [1]. Then (we do not require T to be totally skew-symmetric)

$$c_T = \sigma_T - \frac{3}{2}|T|^2. \quad (6)$$

The above equation shows that c_T acting on spinors contains element of second order, namely σ_T , and the scalar $|T|^2$. We have proven the following general characterization of harmonic G -structures defined by a spinor.

Proposition 1. *A G -structure on a spin manifold (M, g, φ) defined by a unit spinor field $\varphi \in S$ is harmonic if and only if*

$$\Delta\varphi = -\frac{1}{2}c_\xi \cdot \varphi = -\frac{1}{2}\sigma_\xi \cdot \varphi + \frac{3}{4}|\xi|^2\varphi,$$

where ξ is the intrinsic torsion.

By the Lichnerowicz formula we have an immediate corollary.

Corollary 1. *Assume that the defining unit spinor φ is harmonic, i.e., φ is in the kernel of the Dirac operator. Then, a G -structure is harmonic if and only if σ_ξ , equivalently c_ξ , acts on φ as a scalar. In this in the case, the scalar corresponding to σ_ξ is*

$$-\frac{1}{2}\text{Scal}^g + \frac{3}{2}|\xi|^2,$$

where Scal^g is the scalar curvature of g .

Proof. Assume that a given G -structure is harmonic. Then, by the discussion above, $\Delta\varphi = -\frac{1}{2}c_\xi \cdot \varphi$ and by (6) the second equality holds. Moreover, by the Lichnerowicz formula, $\Delta\varphi = -\frac{1}{2}\text{Scal}^g\varphi$. Thus the second part follows.

Conversely, if c_ξ acts as a scalar, say C , then by the Lichnerowicz formula and the above considerations

$$-\frac{1}{2}\text{Scal}^g\varphi = \Delta\varphi = -\frac{1}{2}L \cdot \varphi - \frac{1}{2}C\varphi.$$

Hence, $L \cdot \varphi = (\text{Scal}^g - C)\varphi$. Since $L \cdot \varphi$ is either orthogonal to φ or 0, it must be 0, so $L = 0$. $\square$

Let us provide a useful sufficient condition for harmonicity in the case of a homogeneous reductive space. We rely on [4]. Consider a homogeneous space $M = K/H$, where K is a compact, connected Lie group and H its closed subgroup. Denote by $\mathfrak{k}$ and $\mathfrak{h}$ the Lie algebras of K and H , respectively. Assume we have a decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the orthogonal complement of

$\mathfrak{h}$ with respect to and $\text{ad}(H)$ -invariant positive bilinear form $\mathbf{B}$ on $\mathfrak{k}$. Then $\mathbf{B}$ induces a Riemannian metric g on M . By a well known theorem by Wang the Levi-Civita connection ∇^g is identified with an invariant linear map $\Lambda : \mathfrak{n} \rightarrow \mathfrak{so}(\mathfrak{n})$.

Denote by $\lambda : H \rightarrow \text{SO}(\mathfrak{n})$ the isotropy representation. Notice, that the tangent bundle of M may be described as $TM = K \times_\lambda \mathfrak{n}$ and hence any tensor bundle $T^{\otimes k}M$ equals $K \times_\lambda \mathfrak{n}^{\otimes k}$, etc. Consider additionally a G -structure on M . Then we have a splitting $\mathfrak{so}(\mathfrak{n}) = \mathfrak{g} \oplus \mathfrak{m}$. The intrinsic torsion ξ is a section of the bundle $K \times_\lambda (\mathfrak{n}^* \otimes \mathfrak{m})$. Since, there is a bijection between sections of the associated bundle $K \times_\tau V$, for a representation $\tau : H \rightarrow \text{End}(V)$, and τ -invariant functions $f : K \rightarrow V$, it follows that ξ may be considered as an invariant function $f_\xi : K \rightarrow \mathfrak{n}^* \otimes \mathfrak{m}$,

$$f_\xi(g) = \text{ad}(g^{-1})\Lambda_{\mathfrak{m}}, \quad (7)$$

where $\Lambda_{\mathfrak{m}}$ is the $\mathfrak{m}$ -component of Λ .

Following [4], let us introduce a spin structure on M . Assume that there is a lift $\tilde{\lambda} : H \rightarrow \text{Spin}(\mathfrak{n})$, i.e., $\pi \circ \tilde{\lambda} = \lambda$, where $\pi : \text{Spin}(\mathfrak{n}) \rightarrow \text{SO}(\mathfrak{n})$ is the double covering. Then M admits a spin structure, namely $\text{Spin}(M) = K \times_{\tilde{\lambda}} \text{Spin}(\mathfrak{n})$. The connection on the spinor bundle $S = K \times_{\rho_{\tilde{\lambda}}} \Delta$, where $\rho : \text{Spin}(\mathfrak{n}) \rightarrow \text{End}(\Delta)$ is a spin representation, induced from the Levi-Civita connection is therefore identified with an invariant linear map $\tilde{\Lambda} : \mathfrak{n} \rightarrow \mathfrak{spin}(\mathfrak{n})$ via the correspondence

$$\pi_* \circ \tilde{\Lambda} = \Lambda.$$

Assume there is an isotropy invariant unit spinor φ_0 . Thus, as a constant function $f_\varphi(k) = \varphi_0$, $k \in K$, it induces a global unit spinor φ on the spinor bundle S over M . Then, by the result of Ikeda [9], $\nabla_X \varphi$ corresponds to

$$f_X(f_\varphi) + \rho_*(\tilde{\Lambda}(f_X))f_\varphi.$$

Since f_φ is constant, the first element vanishes. Therefore, the spinorial laplacian of φ corresponds to

$$-\sum_i \rho_*(\tilde{\Lambda}(E_i))\rho_*(\tilde{\Lambda}(E_i))\varphi_0, \quad (8)$$

where (E_i) is an orthonormal basis of $\mathfrak{n}$. Moreover, the element c_ξ acting on φ by the definition equals

$$\frac{1}{2} \sum_i \rho_*(\tilde{\Lambda}_{\mathfrak{m}}(E_i))\rho_*(\tilde{\Lambda}_{\mathfrak{m}}(E_i))\varphi_0. \quad (9)$$

Proposition 2. *Let $M = K/H$ be a reductive homogeneous space with a spin structure induced by a lift $\tilde{\lambda}$ of the isotropy representation λ . Assume that M has a G -structure is induced by a spinor φ_0 , which is a fixed point of λ . Then, the G -structure is harmonic if and only if*

$$\sum_i (\rho_*(\tilde{\Lambda}_{\mathfrak{g}}(E_i))\rho_*(\tilde{\Lambda}_{\mathfrak{m}}(E_i))\varphi_0 + \rho_*(\tilde{\Lambda}_{\mathfrak{m}}(E_i))\rho_*(\tilde{\Lambda}_{\mathfrak{g}}(E_i))\varphi_0 + \rho_*(\tilde{\Lambda}_{\mathfrak{g}}(E_i))\rho_*(\tilde{\Lambda}_{\mathfrak{g}}(E_i))\varphi_0) = 0. \quad (10)$$

Proof. Follows immediately by Proposition 1, relations (8) and (9). $\square$

As an immediate consequence we have the following useful fact.

Corollary 2. *If the minimal connection ∇^G is induced by the zero map $\Lambda_{\mathfrak{g}} : \mathfrak{n} \rightarrow \mathfrak{g}$, where $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{n}$ is a reductive decomposition (i.e., ∇^G is the canonical connection), then the G -structure is harmonic.*

The converse implication is not true as can be seen by appropriate examples (see the last section).

In the following sections we obtain a characterization of harmonicity of $SU(3)$ and G_2 -structures using the spinorial approach in [2]. We show relations with the above general approach and state appropriate examples.

3. $SU(3)$ -Structures

Let $\rho : \text{Spin}(6) \rightarrow \text{End}(\Delta)$, $\Delta = \Delta_6 = \mathbb{R}^8$, be a spin representation. It can be realized via the Clifford multiplication defined in the following way [2]

$$\begin{aligned} e_1 &= +E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46}, \\ e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48}, \\ e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= +E_{14} - E_{23} - E_{58} + E_{67}, \end{aligned} \quad (11)$$

where E_{ij} is a skew-symmetric matrix such that $E_{ij}e_j = -e_i$. Moreover, let $j = e_1 \cdot e_2 \cdot \dots \cdot e_6$. Acting on spinors, $j : \Delta \rightarrow \Delta$ is a complex structure anti-commuting with the Clifford multiplication by vectors. The crucial observation in [2] is that for a fixed unit spinor $\varphi \in \Delta$ we have the following orthogonal decomposition

$$\Delta = \text{span}\{\varphi\} \oplus \text{span}\{j \cdot \varphi\} \oplus \{X \cdot \varphi \mid X \in \mathbb{R}^6\}.$$

Such a spinor defines the group $SU(3) \subset SO(6)$ in a sense that $SU(3)$ is the stabilizer of φ , or equivalently, the Lie algebra that annihilates φ is $\mathfrak{su}(3)$. Moreover,

$$\begin{aligned} \mathfrak{su}(3)^\perp \cdot \varphi &= \text{span}\{\varphi\}^\perp, \\ \mathfrak{u}(3)^\perp \cdot \varphi &= \text{span}\{\varphi, j \cdot \varphi\}^\perp = \{X \cdot \varphi \mid X \in \mathbb{R}^6\}. \end{aligned} \quad (12)$$

Example 1. Let us demonstrate the above formulas by an appropriate example. Choose $\varphi = (0, 0, 0, 0, 1, 0, 0, 0) = s_5 \in \Delta$. Such a choice is determined by Examples 3 and 4 from the last section. Simple calculations show, by the realization (11), that

$$\{X \cdot \varphi \mid X \in \mathbb{R}^6\} = \text{span}\{s_1, s_2, s_3, s_4, s_7, s_8\}, \quad j \cdot \varphi = s_6$$

and the Lie algebra $\mathfrak{su}(3)$ of the annihilator of the spinor φ is generated by

$$e_{13} - e_{24}, e_{14} + e_{23}, e_{15} + e_{26}, e_{16} - e_{25}, e_{35} - e_{46}, e_{36} + e_{45}, e_{12} + e_{34}, e_{34} + e_{56}.$$

Hence, $\mathfrak{m} = \mathfrak{su}(3)^\perp$ is the span of

$$e_{35} + e_{46}, e_{36} - e_{45}, e_{15} - e_{26}, e_{16} + e_{25}, e_{13} + e_{24}, e_{14} - e_{23}, e_{12} - e_{34} + e_{56}.$$

Let (M, g) be a 6-dimensional spin manifold with a unit spinor (field) φ . Since the stabilizer in $SO(6)$ of a unit spinor is $SU(3)$ we get the existence of $SU(3)$ -structure on M [2]. This induces the splitting of the real spinor bundle S and implies the existence of an endomorphism $S \in \text{End}(TM)$ and a 1-form η such that [2]

$$\nabla_X^g \varphi = S(X) \cdot \varphi + \eta(X)j \cdot \varphi. \quad (13)$$

By (13) and (5) we have

$$\begin{aligned} \frac{1}{2} \nabla_Y^g (\xi_X \cdot \varphi) &= \frac{1}{2} ((\nabla_Y^g \xi_X) \cdot \varphi + \xi_X \cdot \nabla_Y^g \varphi) \\ &= \frac{1}{2} ((\nabla_Y^g \xi_X) \cdot \varphi + \xi_X \cdot S(Y) \cdot \varphi + \eta(Y) \xi_X \cdot j \cdot \varphi) \end{aligned}$$

and, on the other hand, since j is ∇^g -parallel,

$$\begin{aligned} \frac{1}{2} \nabla_Y^g (\xi_X \cdot \varphi) &= (\nabla_Y^g S)(X) \cdot \varphi + S(\nabla_Y^g X) \cdot \varphi + S(X) \cdot S(Y) \cdot \varphi \\ &\quad + \eta(Y)S(X) \cdot j \cdot \varphi + (\nabla_Y^g \eta)(X)j \cdot \varphi + \eta(\nabla_Y^g X)j \cdot \varphi \\ &\quad + \eta(X)j \cdot S(Y) \cdot \varphi - \eta(X)\eta(Y)\varphi. \end{aligned}$$

Hence, comparing both sides with $X = Y = e_i$ and taking into account the equality

$$\frac{1}{2} \xi_{\nabla_{e_i}^g e_i} \cdot \varphi = S(\nabla_{e_i}^g e_i) \cdot \varphi + \eta(\nabla_{e_i}^g e_i)\varphi.$$

we get

$$\begin{aligned} \frac{1}{2} L \cdot \varphi &= -\frac{1}{2} \sum_i \xi_{e_i} \cdot S(e_i) \cdot \varphi - \frac{1}{2} \xi_{\eta^\sharp} \cdot j \cdot \varphi + (\text{div} S) \cdot \varphi - |S|^2 \varphi + S(\eta^\sharp) \cdot j \cdot \varphi \\ &\quad + \text{div}(\eta^\sharp)j \cdot \varphi + j \cdot S(\eta^\sharp) \cdot \varphi - |\eta|^2 \varphi. \end{aligned}$$

Applying (3) we obtain

$$\frac{1}{2} L \cdot \varphi = \chi^S \cdot \varphi - \frac{1}{2} \xi_{\eta^\sharp} \cdot j \cdot \varphi + (\text{div} S) \cdot \varphi + \text{div}(\eta^\sharp)j \cdot \varphi + j \cdot S(\eta^\sharp) \cdot \varphi - |\eta|^2 \varphi,$$

where χ^S is a vector field given by

$$\chi^S = \sum_i \xi_{e_i} S(e_i).$$

We may state and prove the main theorem of this section. Before we do that, let us say a few words about the Gray-Hervella classes of possible $SU(3)$ -structures and state some additional simple observations.

In general, for any $U(n)$ -structure the intrinsic torsion belongs to the space $T^*M \otimes \mathfrak{u}(n)^\perp(TM)$. Under the natural action of $U(n)$ it splits into four

modules, so called Gray-Hervella classes [8], $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. For $SU(n)$ we have one additional class $\mathcal{W}_5$, which corresponds to the 1-form η . The case $n = 3$ is special. Each module $\mathcal{W}_1$ and $\mathcal{W}_2$ splits into two modules $\mathcal{W}_i^\pm$, $i = 1, 2$. Therefore, we have the following splitting

$$T^*M \otimes \mathfrak{su}(3)^\perp(TM) = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5.$$

Each class has a nice interpretation in terms of S and η (see [2]):

$$\begin{aligned} \mathcal{W}_1^+ : \quad & S = \lambda J_\varphi, \quad \eta = 0, \\ \mathcal{W}_1^- : \quad & S = \mu \text{Id}, \quad \eta = 0, \\ \mathcal{W}_2^+ : \quad & S \in \mathfrak{su}(3), \quad \eta = 0, \\ \mathcal{W}_2^- : \quad & S \in \text{Sym}_0^2(T^*M), SJ_\varphi = J_\varphi S, \quad \eta = 0, \\ \mathcal{W}_3 : \quad & S \in \text{Sym}_0^2(T^*M), SJ_\varphi = -J_\varphi S, \quad \eta = 0, \\ \mathcal{W}_4 : \quad & S \in \Lambda^2(T^*M), SJ_\varphi = -J_\varphi S, \quad \eta = 0, \\ \mathcal{W}_5 : \quad & S = 0, \end{aligned}$$

where λ, μ are constants and J_φ is the almost complex structure induced by φ [2],

$$J_\varphi(X) \cdot \varphi = j \cdot X \cdot \varphi, \quad X \in TM.$$

Assume for a while that $\eta = 0$. From the definition of J_φ we immediately get

$$(\nabla_Y^g J_\varphi)(X) \cdot \varphi = 2S(Y) \cdot J_\varphi(X) \cdot \varphi + 2g(J(X), S(Y))\varphi - 2g(X, S(Y))j \cdot \varphi, \quad (14)$$

which implies

$$(\text{div} J_\varphi) \cdot \varphi = 2 \sum_i S(e_i) \cdot J_\varphi(e_i) \cdot \varphi - 2(\text{tr} S)j \cdot \varphi - 2\text{tr}(J_\varphi S)\varphi. \quad (15)$$

The following lemma shows that in many cases the vector field χ^S vanishes.

Lemma 1. *If $\eta = 0$, then $\chi^S = 0$.*

Proof (First proof). Since $\eta = 0$, the intrinsic torsion ξ may be described as follows $g(\xi_X Y, Z) = \psi_\varphi(S(X), Y, Z)$, where ψ_φ is a 3-form induced by φ , $\psi_\varphi(X, Y, Z) = -\langle X \cdot Y \cdot Z \cdot \varphi, \varphi \rangle$ [2]. Thus $\xi_X S(X) = 0$ for any $X \in TM$. In particular, χ^S vanishes. $\square$

Proof (Second proof). By the assumption $\eta = 0$, the intrinsic torsion is in fact the intrinsic torsion of the corresponding $U(3)$ -structure. It is well known that in this case

$$\xi_X Y = -\frac{1}{2} J_\varphi(\nabla_X^g J_\varphi)Y.$$

Thus, by (14) and the definition of J_φ

$$\begin{aligned}
2J_\varphi(\chi^S) &= \sum_i (\nabla_{e_i}^g J_\varphi) S(e_i) \\
&= 2 \sum_i S(e_i) \cdot J_\varphi(S(e_i)) - 2|S|^2 j \cdot \varphi \\
&= 2 \sum_i S(e_i) \cdot j \cdot S(e_i) \cdot \varphi - 2|S|^2 j \cdot \varphi \\
&= 0.
\end{aligned}$$

Hence $\chi^S = 0$. □

The main theorem of this section reads as follows.

Theorem 1. *An $SU(3)$ -structure on a 6-dimensional spin manifold induced by a unit spinor φ is harmonic if and only if the following condition holds*

$$\chi^S \cdot \varphi - \frac{1}{2} \xi_{\eta^\#} \cdot j \cdot \varphi + (\operatorname{div} S) \cdot \varphi + \operatorname{div}(\eta^\#) j \cdot \varphi + j \cdot S(\eta^\#) \cdot \varphi - |\eta|^2 \varphi = 0. \quad (16)$$

If $\eta = 0$, then harmonicity is equivalent to $\operatorname{div} S = 0$. In particular, if the intrinsic torsion ξ belongs to $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$ class or to the pure class $\mathcal{W}_3$ or defines a locally conformally Kähler structure (contained in $\mathcal{W}_4$), then the $SU(3)$ -structure is harmonic.

Proof. The only thing which is left to prove is harmonicity of the $SU(3)$ -structure belonging to the mentioned classes.

$\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$ case : In this case, $S = \lambda J_\varphi + \mu \operatorname{Id}$ for two constants λ, μ . Then, $\operatorname{div}(S) = \lambda \operatorname{div} J_\varphi$, thus it suffices to show that the divergence of J_φ vanishes, but this follows immediately by (15).

$\mathcal{W}_3$ case : Since S is symmetric and traceless, it follows that $\sum_i e_i \cdot S(e_i) \cdot \varphi = 0$. Thus $D\varphi = 0$, where D is the Dirac operator. By Corollary 1, it suffices to prove that c_ξ acts on φ by a scalar. Using (3) we obtain

$$\begin{aligned}
2c_\xi \cdot \varphi &= \sum_i \xi_{e_i} \cdot \xi_{e_i} \cdot \varphi = 2 \sum_i \xi_{e_i} \cdot S(e_i) \cdot \varphi \\
&= 2 \sum_i (S(e_i) \cdot \xi_{e_i} \cdot \varphi + 2\chi^S \cdot \varphi) = -4|S|^2 \varphi,
\end{aligned}$$

since by Lemma 1, χ^S vanishes.

Locally conformally Kähler case: This means that ξ is in $\mathcal{W}_4$ module and the Lee form θ is closed [13]. We have that S is skew-symmetric and $SJ_\varphi = -J_\varphi S$. In other words, $S \in \mathfrak{u}(3)^\perp(TM)$. Hence, see (12), there is a unique vector field Z_φ such that

$$S \cdot \varphi = Z_\varphi \cdot \varphi. \quad (17)$$

Let us first describe Z_φ . In this case, (15) reduces to

$$(\operatorname{div} J_\varphi) \cdot \varphi = 2 \sum_i S(e_i) \cdot J_\varphi(e_i) \cdot \varphi. \quad (18)$$

Moreover, $\sum_i S(e_i) \cdot e_i = 2S$ as acting on spinors. Applying j to both sides and using (17), we have

$$\sum_i S(e_i) \cdot J_\varphi(e_i) \cdot \varphi = -2j \cdot S \cdot \varphi = -2J_\varphi(Z_\varphi) \cdot \varphi,$$

which by (18) implies

$$J_\varphi(\operatorname{div} J_\varphi) = 4Z_\varphi.$$

Therefore, $4Z_\varphi$ is dual to the Lee form θ of the locally conformally Kähler structure.

Now we wish to determine S in terms of Z_φ . Choose a local coordinate system such that $\varphi = s_5$, as in Example 1. Writing $Z_\varphi = \sum_i z_i e_i$, we see that S is represented by the following matrix

$$S = \frac{1}{2} \begin{pmatrix} 0 & 0 & -z_5 & -z_6 & z_3 & z_4 \\ 0 & 0 & z_6 & -z_5 & z_4 & -z_3 \\ z_5 & -z_6 & 0 & 0 & -z_1 & z_2 \\ z_6 & z_5 & 0 & 0 & -z_2 & -z_1 \\ -z_3 & -z_4 & z_1 & z_2 & 0 & 0 \\ -z_4 & z_3 & -z_2 & z_1 & 0 & 0 \end{pmatrix}.$$

Computing the divergence of S we get

$$2\operatorname{div} S = \sum_{\text{some}} i, j, k dZ_\varphi^b(e_i, e_j)e_k,$$

where $\sum_{\text{some}} i, j, k$ denotes the summation over certain (twelve) permutations of pairwise different indices $i, j, k = 1, \dots, 6$. Since the Lee form is closed, it follows that $\operatorname{div} S = 0$. $\square$

Remark 1. The harmonicity in Theorem 1 can be stated, maybe in more elegant but less applicable way for our further considerations, as follows. Introduce an operation (commutator) $[\omega, \tau]$ defined for any 2-forms ω and τ by (compare [11] and [1])

$$[\omega, \tau] = \sum_i (e_i \lrcorner \omega) \wedge (e_i \lrcorner \tau).$$

Then $[\omega, \tau]$ is a 2-form with the following action on spinors

$$\omega \cdot \tau - \tau \cdot \omega = 2[\omega, \tau].$$

Since the element j corresponds to J_φ treated as a Kähler form, we have

$$2[\xi_{\eta^\sharp}, J_\varphi] \cdot \varphi = \xi_{\eta^\sharp} \cdot j \cdot \varphi - 2j \cdot S(\eta^\sharp) \cdot \varphi + 2|\eta|^2 \varphi.$$

Hence, condition (16) reads as

$$\chi^S \cdot \varphi - [\xi_{\eta^\sharp}, J_\varphi] \cdot \varphi + (\operatorname{div} S) \cdot \varphi + \operatorname{div}(\eta^\sharp) j \cdot \varphi = 0.$$

Remark 2. The fact that, in general, the $U(n)$ -structure of Gray-Hervella pure classes $\mathcal{W}_1$ or $\mathcal{W}_3$ or locally conformally Kähler structure is harmonic was proved, without spinorial approach, in [7]. Our approach is based only on the definitions of J_φ and S by spinorial approach in [2].

4. G_2 -Structures

Analogously as in dimension 6, a spin representation $\rho : \text{Spin}(7) \rightarrow \text{End}(\Delta)$, $\Delta = \Delta_7 = \mathbb{R}^8$, is realized via Clifford multiplication defined identically as in the 6-dimensional case with additional action of e_7 given by

$$e_7 = +E_{12} - E_{34} - E_{56} + E_{78}.$$

Fix a unit spinor $\varphi \in \Delta$. By dimensional reasons we have

$$\Delta = \text{span}\{\varphi\} \oplus \{X \cdot \varphi \mid X \in \mathbb{R}^7\}.$$

Example 2. Analogously, as in $SU(3)$ case, let us perform some calculations on a concrete example. Choose a unit spinor $\varphi = s_5 \in \Delta$. Then, the Lie algebra, which annihilates φ via Clifford multiplication on 2-forms is spanned by the elements

$$\begin{aligned} e_{16} + e_{37}, \quad e_{16} - e_{25}, \quad e_{15} + e_{26}, \quad e_{26} + e_{47}, \quad e_{17} - e_{36}, \quad e_{17} + e_{45}, \quad e_{27} - e_{35}, \\ e_{27} - e_{46}, \quad e_{12} + e_{34}, \quad e_{12} - e_{56}, \quad e_{13} - e_{24}, \quad e_{13} - e_{67}, \quad e_{14} + e_{23}, \quad e_{14} + e_{57}. \end{aligned}$$

Hence it is $\mathfrak{g}_2$. Moreover, its orthogonal complement $\mathfrak{m} = \mathfrak{g}_2^\perp$ in $\mathfrak{so}(7)$ is spanned by

$$\begin{aligned} e_{16} - e_{37} + e_{25}, \quad e_{15} - e_{26} + e_{47}, \quad e_{17} + e_{36} - e_{45}, \quad e_{27} + e_{35} + e_{46}, \\ e_{12} - e_{34} + e_{56}, \quad e_{13} + e_{24} + e_{67}, \quad e_{14} - e_{23} - e_{57}. \end{aligned}$$

Let (M, g) be a 7-dimensional spin manifold with the spinor bundle S and a unit spinor (field) φ . The above decomposition induces a splitting of the spinor bundle and implies existence of an endomorphism $S \in \text{End}(TM)$ such that

$$\nabla_X \varphi = S(X) \cdot \varphi. \quad (19)$$

Since the stabilizer in $SO(7)$ of a unit spinor is G_2 , (M, g, φ) becomes a G_2 -structure [2]. The harmonicity condition, or more generally, the formula for the tensor L becomes, just putting $\eta = 0$ in the $SU(3)$ case,

$$\frac{1}{2}L \cdot \varphi = (\text{div}S) \cdot \varphi.$$

Notice, that in the G_2 case the vector field χ^S vanishes, since the intrinsic torsion may be described as follows $g(\xi_X Y, Z) = \frac{2}{3}\psi_\varphi(S(X), Y, Z)$ with a 3-form ψ_φ defined (up to a sign) in the same way as in the $SU(3)$ case, i.e., $\psi_\varphi(X, Y, Z) = \langle X \cdot Y \cdot Z \cdot \varphi, \varphi \rangle$. Thus we have the following observation.

Theorem 2. *A G_2 -structure on a spin 7-dimensional manifold is harmonic if and only if $\operatorname{div} S = 0$.*

For a G_2 -structure the space of all possible intrinsic torsions $T^*M \otimes \mathfrak{g}_2^\perp(TM)$ splits into four irreducible modules $\mathcal{W}_1, \dots, \mathcal{W}_4$ [2]:

$$\begin{aligned}\mathcal{W}_1 : \quad & S = \lambda \operatorname{Id}, \\ \mathcal{W}_2 : \quad & S \in \mathfrak{g}_2, \\ \mathcal{W}_3 : \quad & S \in \operatorname{Sym}_0^2(T^*M), \\ \mathcal{W}_4 : \quad & S = V \lrcorner \psi_\varphi, \quad V \in TM.\end{aligned}$$

The class for which the condition of harmonicity may be explicitly described is $\mathcal{W}_1$ defined by the condition $S = \lambda \operatorname{Id}$, where λ is a smooth function. Then $\operatorname{div} S = \operatorname{grad}(\lambda)$. Hence, a G_2 -structure in $\mathcal{W}_1$ class is harmonic if and only if λ is constant. Moreover, by the same argument as in the proof of Theorem 1, we see that a G_2 -structure belonging to the $\mathcal{W}_3$ class is harmonic. Hence we may state the following fact.

Corollary 3. *A G_2 -structure belonging to the pure class*

1. $\mathcal{W}_1$ *is harmonic if and only if λ is a constant, i.e., a G_2 -structure is nearly parallel,*
2. $\mathcal{W}_3$ *is harmonic.*

Remark 3. Some of the results, especially the second part of Theorem 1 and Corollary 3(1), may be obtained with an alternative approach communicated to the author by I. Agricola. Namely, assuming that an $SU(3)$ or a G_2 -structure admits a characteristic connection ∇^c (see [1, p. 45] for a definition), which holds for the considered structures excluding $\mathcal{W}_2$ cases [6, 12], harmonicity condition by [7, Theorem 3.7] is equivalent to $\delta T^c = 0$, where T^c is a torsion of a characteristic connection (characteristic torsion). In particular, if $\nabla^c T^c = 0$, then the considered structure is harmonic. It is known that nearly Kähler and nearly parallel G_2 -structures admit parallel characteristic connections [5, 10], thus are harmonic as $U(3)$ - and G_2 -structures, respectively.

5. Examples

We present examples that justify our results. We begin with a suitable introduction.

Consider a homogeneous space $M = K/H$, where K is a compact, connected Lie group and H its closed subgroup, as in the second section, i.e., assume that on the level of Lie algebras we have a decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the orthogonal complement of $\mathfrak{h}$ with respect to an $\operatorname{ad}(H)$ -invariant positive bilinear form $\mathbf{B}$ on $\mathfrak{k}$. Now, we deform $\mathbf{B}$ to $\mathbf{B}_t$, $t > 0$, in the following way (see [4]): we assume that there is a $\mathbf{B}$ -orthogonal splitting $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$. Then we put

$$\mathbf{B}_t = \mathbf{B}|_{\mathfrak{n}_0 \times \mathfrak{n}_0} + 2t\mathbf{B}|_{\mathfrak{n}_1 \times \mathfrak{n}_1}.$$

$\mathbf{B}_t$ induces a one parameter family of Riemannian metrics g_t on M . The Levi-Civita connection of g_t is identified with the invariant linear map $\Lambda_t : \mathfrak{n} \rightarrow \mathfrak{so}(\mathfrak{n})$.

Below, we discuss a behavior of (M, g_t, φ) in three cases.

Example 3. Consider the complex projective space $M = \mathbb{CP}^3$, which was studied in detail in [4]. Here, we review all necessary facts and develop these which are indispensable for our purposes. Recall, that $\mathbb{CP}^3$ is a homogeneous space of the form $SO(5)/U(2)$. On the level of Lie algebras, $\mathfrak{so}(5) = \mathfrak{u}(2) \oplus \mathfrak{n}$, where

$$\begin{aligned}\mathfrak{u}(2) &= \text{span}\{E_{12}, E_{34}, E_{13} - E_{24}, E_{14} + E_{23}\}, \\ \mathfrak{n} &= \text{span}\{E_{15}, E_{25}, E_{35}, E_{45}, E_{13} + E_{24}, E_{14} - E_{23}\}.\end{aligned}$$

Moreover, decompose $\mathfrak{n}$ into $\mathfrak{n}_0 \oplus \mathfrak{n}_1$, where

$$\mathfrak{n}_0 = \text{span}\{E_{15}, E_{25}, E_{35}, E_{45}\}, \quad \mathfrak{n}_1 = \text{span}\{E_{13} + E_{24}, E_{14} - E_{23}\}.$$

An orthonormal basis of $\mathfrak{n}$ with respect to $\mathbf{B}_t$ (here $\mathbf{B}$ is (the negative of) the Killing form) can be chosen in the following way

$$\begin{aligned}X_1 &= E_{15}, & X_2 &= E_{25}, & X_3 &= E_{35}, \\ X_4 &= E_{45}, & X_5 &= \frac{1}{2\sqrt{t}}(E_{13} + E_{24}), & X_6 &= \frac{1}{2\sqrt{t}}(E_{14} - E_{23}).\end{aligned}$$

Thus we have an identification $\mathfrak{n} = \mathbb{R}^6$. The Levi-Civita connection Λ_t takes the form

$$\begin{aligned}\Lambda_t(X_1) &= \frac{\sqrt{t}}{2}(E_{35} + E_{46}), & \Lambda_t(X_2) &= \frac{\sqrt{t}}{2}(E_{45} - E_{36}), \\ \Lambda_t(X_3) &= \frac{\sqrt{t}}{2}(E_{26} - E_{15}), & \Lambda_t(X_4) &= \frac{\sqrt{t}}{2}(-E_{16} - E_{25}), \\ \Lambda_t(X_5) &= \frac{1-t}{2\sqrt{t}}(E_{13} + E_{24}), & \Lambda_t(X_6) &= \frac{1-t}{2\sqrt{t}}(E_{14} - E_{23}).\end{aligned}$$

The (differential of the) isotropy representation $\lambda_* : \mathfrak{u}(2) \rightarrow \mathfrak{so}(6)$ is of the form

$$\begin{aligned}\lambda_*(E_{12}) &= E_{12} - E_{56}, & \lambda_*(E_{34}) &= E_{34} + E_{56}, \\ \lambda_*(E_{13} - E_{24}) &= E_{13} - E_{24}, & \lambda_*(E_{14} + E_{23}) &= E_{14} + E_{23}.\end{aligned}$$

It can be shown [4] that λ has a lift to a map $\tilde{\lambda} : U(2) \rightarrow \text{Spin}(6)$, which implies existence of a spin structure on M . Via the Clifford representation (11) we see that $\tilde{\lambda}_*$ has the following form

$$\begin{aligned}\tilde{\lambda}_*(E_{12}) &= E_{34} + E_{78}, & \tilde{\lambda}_*(E_{34}) &= E_{12} - E_{78}, \\ \tilde{\lambda}_*(E_{13} - E_{24}) &= -E_{13} + E_{24}, & \tilde{\lambda}_*(E_{14} + E_{23}) &= -E_{14} - E_{23}.\end{aligned}$$

Thus, spinors $\varphi = s_5$ and $\varphi = s_6$ are annihilated by the above isotropy representation. Thus, each spinor in the span of s_5 and s_6 defines a global spinor field. Fix the spinor $\varphi = s_5$ defining an $SU(3)$ -structure on M .

From Example 1 we see that Λ_t has values in $\mathfrak{m}$, hence corresponds to the intrinsic torsion, whereas, the $SU(3)$ -connection corresponds to the zero map, $\Lambda_{\text{su}(3)} \equiv 0$. Thus, by Corollary 2, the structure is harmonic for all $t > 0$.

Let us consider a different approach. By the definition of $\tilde{\Lambda}$, which corresponds to the induced connection on the spinor bundle, we have $\tilde{\Lambda}(X)\varphi = S(X) \cdot \varphi$, where S is diagonal of the form (compare [2])

$$S = -\text{diag} \left(\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2} \frac{1-t}{2\sqrt{t}}, \frac{1-t}{2\sqrt{t}} \right),$$

Hence the 1-form η vanishes. Since

$$S = -\frac{t+1}{6\sqrt{t}}\text{Id} - \frac{2t-1}{6\sqrt{t}}\text{diag}(1, 1, 1, 1, -2, -2),$$

the considered $SU(3)$ -structure is of class $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ for $t \neq \frac{1}{2}$ and $\mathcal{W}_1^-$ for $t = \frac{1}{2}$. In both cases, $\text{div} S$, which corresponds to $\sum_i (\Lambda(X_i)S)X_i$, vanishes, hence the considered $SU(3)$ -structures are harmonic.

Example 4. Consider the Lie group $M = \text{Spin}(4) \equiv \mathbb{S}^3 \times \mathbb{S}^3$. Then its Lie algebra is isomorphic to $\mathfrak{so}(4)$. Consider the following decomposition $\mathfrak{so}(4) = \mathfrak{n}_0 \oplus \mathfrak{n}_1$, where $\mathfrak{n}_0 = \text{span}\{E_{14}, E_{24}, E_{34}\}$ and $\mathfrak{n}_1 = \text{span}\{E_{12}, E_{13}, E_{23}\}$ [4]. Computing the Lie brackets of generators, we see that $\mathfrak{n}_1$ is a Lie subalgebra. Choose the following orthonormal basis of $\mathfrak{so}(4)$ with respect to $\mathbf{B}_t$ (here $\mathbf{B}$ is again the negative of the Killing form):

$$\begin{aligned} X_1 &= E_{14}, & X_2 &= E_{24}, & X_3 &= E_{34}, \\ X_4 &= \frac{1}{\sqrt{2t}}E_{12}, & X_5 &= \frac{1}{\sqrt{2t}}E_{13}, & X_6 &= \frac{1}{\sqrt{2t}}E_{23}. \end{aligned}$$

With this choice, $\mathfrak{so}(4) = \mathbb{R}^6$. The Levi-Civita connection of $\mathbf{B}_t$ is represented by the map $\Lambda_t : \mathbb{R}^6 \rightarrow \mathfrak{so}(6)$ (compare [4])

$$\begin{aligned} \Lambda_t(X_1) &= \frac{1}{2}\sqrt{2t}E_{24} + \frac{1}{2}\sqrt{2t}E_{35}, & \Lambda_t(X_2) &= -\frac{1}{2}\sqrt{2t}E_{14} + \frac{1}{2}\sqrt{2t}E_{36}, \\ \Lambda_t(X_3) &= -\frac{1}{2}\sqrt{2t}E_{15} - \frac{1}{2}\sqrt{2t}E_{26}, & \Lambda_t(X_4) &= \frac{1-t}{\sqrt{2t}}E_{12} + \frac{1}{2\sqrt{2t}}E_{56}, \\ \Lambda_t(X_5) &= \frac{1-t}{\sqrt{2t}}E_{13} - \frac{1}{2\sqrt{2t}}E_{46}, & \Lambda_t(X_6) &= \frac{1-t}{\sqrt{2t}}E_{23} + \frac{1}{2\sqrt{2t}}E_{45} \end{aligned}$$

The spin structure is the trivial one $M \times \text{Spin}(4)$ and the spinor bundle is, again, the trivial bundle $M \times \Delta$ [4]. Hence, each smooth function $f_\varphi : M \rightarrow \Delta$ defines a global spinor field. Choose the defining spinor to be the constant function equal to $s_5 \in \Delta$. Then, the equality $\tilde{\Lambda}(X)s_5 = S(X) \cdot s_5 + \eta(X)j \cdot s_5$

is satisfied by

$$S = \begin{pmatrix} -\frac{1}{2}\sqrt{2t} & 0 & 0 & 0 & \frac{1}{2\sqrt{2t}} & 0 \\ 0 & \frac{1}{2}\sqrt{2t} & 0 & 0 & 0 & -\frac{1}{2\sqrt{2t}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{2t} & 0 & 0 & -\frac{1-t}{\sqrt{2t}} & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2t} & 0 & 0 & 0 & \frac{1-t}{\sqrt{2t}} \end{pmatrix}, \quad \eta = \left(\frac{3}{2} - t\right) \frac{1}{\sqrt{2t}} X_4^\flat.$$

Hence the considered $SU(3)$ -structure is of type $\mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$. Notice that $j \cdot s_5 = s_6$ (see Example 1). Moreover, it is not hard to check that $\operatorname{div} S = 0$, $S(\eta^\sharp) = 0$, $\operatorname{div}(\eta^\sharp) = 0$ and $|\eta^\sharp|^2 = \frac{1}{2t} \left(\frac{3}{2} - t\right)^2$. Thus the harmonicity condition (Theorem 1) has the following form

$$\chi^S \cdot s_5 - \frac{1}{2} \xi_{\eta^\sharp} \cdot s_6 - \frac{1}{2t} \left(\frac{3}{2} - t\right)^2 s_5 = 0. \quad (20)$$

We need to compute $\xi_{\eta^\sharp} \cdot s_5$, which corresponds to $\left(\frac{3}{2} - t\right) \frac{1}{\sqrt{2t}} \Lambda_{\mathfrak{su}(3)^\perp}(X_4)$. Since

$$\Lambda_{\mathfrak{su}(3)^\perp}(X_4) = \frac{\sqrt{t}}{2} (-E_{16} - E_{25})_{\mathfrak{su}(3)^\perp} = \frac{3-2t}{6\sqrt{2t}} (E_{12} - E_{34} + E_{56}).$$

we see that

$$\xi_{\eta^\sharp} \cdot s_5 = \left(\frac{3}{2} - t\right)^2 \frac{1}{6t} s_6.$$

Thus, the harmonicity condition (20) simplifies to

$$\chi^S \cdot s_5 - \frac{1}{12t} \left(\frac{3}{2} - t\right)^2 s_6 - \frac{1}{2t} \left(\frac{3}{2} - t\right)^2 s_5 = 0.$$

Since $\chi^S \cdot s_5$ is orthogonal to s_5 and s_6 we have $t = \frac{3}{2}$ and, in particular, η vanishes. Thus, by the fact that $\operatorname{div} S = 0$ and by Theorem 1, the considered $SU(3)$ -structure is harmonic only for $t = \frac{3}{2}$. In this case, it is of type $\mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.

Example 5. Let M be the Aloff-Wallach space $N(1, 1) = SU(3)/\mathbb{S}^1$, where the action $\mathbb{S}^1 \rightarrow SU(3)$ is by the diagonal matrices

$$\theta \mapsto \operatorname{diag}(e^{\theta i}, e^{\theta i}, e^{-2\theta i}).$$

Consider the splitting $\mathfrak{su}(3) = \mathbb{R} \oplus \mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$ is given by

$$\mathfrak{n}_0 = \operatorname{span}\{L, E_{12}, \tilde{E}_{12}\}, \quad \mathfrak{n}_1 = \operatorname{span}\{E_{13}, \tilde{E}_{13}, E_{23}, \tilde{E}_{23}\},$$

where

$$L = \operatorname{diag}(i, i, 0), \quad \tilde{E}_{kl} = iS_{kl}$$

and S_{kl} is a symmetric matrix with $S_{kl}e_l = e_k$. Then, an orthonormal basis of $\mathfrak{n}$ with respect to $\mathbf{B}_t$, induced from the Killing form, can be chosen as follows

$$\begin{aligned} X_1 &= E_{12}, & X_2 &= \tilde{E}_{12}, & X_3 &= \frac{1}{\sqrt{2t}}E_{13}, & X_4 &= \frac{1}{\sqrt{2t}}\tilde{E}_{13}, \\ X_5 &= \frac{1}{\sqrt{2t}}E_{23}, & X_6 &= \frac{1}{\sqrt{2t}}\tilde{E}_{23}, & X_7 &= L. \end{aligned}$$

The Levi-Civita connection of g_t defines the map $\Lambda_t : \mathfrak{n} \rightarrow \mathfrak{so}(\mathfrak{n})$ [3, 4]:

$$\begin{aligned} \Lambda_t(X_1) &= E_{27} - \left(1 - \frac{1}{4t}\right)(E_{35} + E_{46}), & \Lambda_t(X_2) &= -E_{17} - \left(1 - \frac{1}{4t}\right)(E_{45} - E_{36}), \\ \Lambda_t(X_3) &= \frac{1}{4t}E_{47} - \frac{1}{4t}(E_{26} - E_{15}), & \Lambda_t(X_4) &= -\frac{1}{4t}E_{37} + \frac{1}{4t}(E_{16} + E_{25}), \\ \Lambda_t(X_5) &= -\frac{1}{4t}E_{67} - \frac{1}{4t}(E_{13} + E_{24}), & \Lambda_t(X_6) &= \frac{1}{4t}E_{57} - \frac{1}{4t}(E_{14} - E_{23}), \\ \Lambda_t(X_7) &= E_{12} + \left(1 - \frac{1}{4t}\right)(E_{34} - E_{56}). \end{aligned}$$

The isotropy representation $\lambda : \mathbb{S}^1 \rightarrow \mathrm{SO}(\mathfrak{n})$ has a lift to a map $\tilde{\lambda} : \mathbb{S}^1 \rightarrow \mathrm{Spin}(\mathfrak{n})$, thus there is a spin structure on M [4]. Moreover, the spinor $\varphi_0 = s_5$ is a fixed point of this action, hence as a constant function from $\mathrm{SU}(3)$ to Δ defines a global spinor field φ . Consider the G_2 structure induced by φ . By Example 2 the map Λ_t takes values in $\mathfrak{m}$ if and only if $t = \frac{1}{8}$. In this case, by Corollary 2, the G_2 -structure is harmonic. Let us check harmonicity for remaining values of t . It is easy to see that an endomorphism S satisfying $\tilde{\Lambda}(X)\varphi_0 = S(X) \cdot \varphi_0$ equals

$$S = \frac{1}{2} \mathrm{diag} \left(\left(\frac{1}{2t} - 1 \right), \left(\frac{1}{2t} - 1 \right), -\frac{3}{4t}, -\frac{3}{4t}, -\frac{3}{4t}, -\frac{3}{4t}, \left(\frac{1}{2t} - 1 \right) \right).$$

In particular, the considered G_2 -structure is of type $\mathcal{W}_1 \oplus \mathcal{W}_3$ for $t \neq \frac{5}{4}$ and of pure type $\mathcal{W}_1$ for $t = \frac{5}{4}$. The divergence of S , corresponding to $\sum_i \Lambda(X_i)S(X_i)$, vanishes. Hence, for any $t > 0$ the considered G_2 -structure is harmonic.

Acknowledgements

The author wishes to thank Ilka Agricola for fruitful conversations during his short visits to Marburg University, in particular, for discussions concerning the geometry of homogeneous spaces. He is also grateful for notes concerning the geometry of the complex projective space $\mathbb{CP}^3$ and the remark about an alternative approach to some of the obtained results (see Remark 3).

The author is supported by the National Science Center, Poland- Grant Miniatura 2017/01/X/ST1/01724.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- [1] Agricola, I.: The Srni lectures on non-integrable geometries with torsion. Arch. Math. (Brno) **42**, 5–84 (2006)
- [2] Agricola, I., Chiossi, S.G., Friedrich, T., Höll, J.: Spinorial description of $SU(3)$ - and G_2 -manifolds. J. Geom. Phys. **98**, 535–555 (2015)
- [3] Agricola, I., Friedrich, T.: On the holonomy of connections with skew-symmetric torsion. Math. Ann. **328**(4), 711–748 (2004)
- [4] Baum, H., Friedrich, T., Grunewald, R., Kath, I.: Twistors and Killing spinors on Riemannian manifolds. Teubner Texts in Mathematics, Stuttgart (1991)
- [5] Friedrich, T.: G_2 -manifolds with parallel characteristic torsion. Differ. Geom. Appl. **25**(6), 632–648 (2007)
- [6] Friedrich, T., Ivanov, S.: Parallel spinors and connections with skew-symmetric torsion in string theory. Asian J. Math. **6**(2), 303–335 (2002)
- [7] González-Dávila, J.C., Martín Cabrera, F.: Harmonic G-structures. Math. Proc. Cambridge Philos. Soc. **146**(2), 435–459 (2009)
- [8] Gray, A., Hervella, L.: The sixteen classes of almost hermitian manifolds and their linear invariants. Ann. Mat. Pura Appl. **4**(123), 35–58 (1980)
- [9] Ikeda, A.: Formally self adjointness for the Dirac operator on homogeneous spaces. Osaka J. Math. **12**, 173–185 (1975)
- [10] Kirichenko, V.F.: K-spaces of maximal rank. Mat. Zametki **22**(4), 465–476 (1977)
- [11] Nagy, P.-A.: Skew-symmetric prolongations of Lie algebras and applications. J. Lie Theory **23**(1), 1–33 (2013)
- [12] Schoemann, N.: Almost Hermitian structures with parallel torsion. J. Geom. Phys. **57**(11), 2187–2212 (2007)
- [13] Vaisman, I.: On locally conformal almost Kähler manifolds. Israel J. Math. **24**(3–4), 338–351 (1976)
- [14] Wood, C.M.: Harmonic almost-complex structures. Compos. Math. **99**, 183–212 (1995)
- [15] Wood, C.M.: Harmonic sections of homogeneous fibre bundles. Differ. Geom. Appl. **19**, 193–210 (2003)

Kamil Niedziałomski
Department of Mathematics and Computer Science
University of Łódź
ul. Banacha 22
90-238 Łódź
Poland
e-mail: kamil.niedzialomski@wmii.uni.lodz.pl

Received: August 1, 2019.

Accepted: June 23, 2020.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.