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Results in Mathematics

Superlinear Perturbations of the Eigenvalue Problem for the Robin Laplacian Plus an Indefinite and Unbounded Potential

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Abstract. We consider a superlinear perturbation of the eigenvalue problem for the Robin Laplacian plus an indefinite and unbounded potential. Using variational tools and critical groups, we show that when λ is close to a nonprincipal eigenvalue, then the problem has seven nontrivial solutions. We provide sign information for six of them.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric semilinear Robin problem

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \lambda u(z) + f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega, \ \lambda \in \mathbb{R}. \end{cases} \tag{P_λ}$$

In this problem, $\xi \in L^s(\Omega)$ with $s > N$ and it is indefinite (that is, sign-changing). We assume that $\xi(\cdot)$ is bounded from above (that is, $\xi^+ \in L^\infty(\Omega)$). So, the differential operator (the left-hand side) of problem (P_λ) is not coercive. In the reaction (the right-hand side) of (P_λ), we have the parametric linear term $u \mapsto \lambda u$ and a perturbation $f(z,x)$ which is a measurable function such that $f(z,\cdot)$ is continuously differentiable. We assume that $f(z,\cdot)$ exhibits superlinear growth near $\pm\infty$, but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). Instead we employ a less restrictive condition that incorporates in our framework superlinear nonlinearities with slower growth near $\pm\infty$ which fail to satisfy the AR-condition. So, problem (P_λ) can be viewed as a perturbation of the classical eigenvalue problem for the operator $u \mapsto -\Delta u + \xi(z)u$ with Robin boundary condition.

In the past, such problems were studied primarily in the context of Dirichlet equations with no potential term. The first work is that of Mugnai [5], who used a general linking theorem of Marino & Saccon [4] to produce three nontrivial solutions. The work of Mugnai was extended by Rabinowitz, Su & Wang [18] who based their method of proof on bifurcation theory, variational techniques and critical groups in order to produce three nontrivial solutions. Analogous results for scalar periodic equations were proved by Su & Zeng [20]. All the aforementioned works use the AR-condition to express the superlinearity of the perturbation $f(z,\cdot)$. A more general superlinearity condition was employed by Ou & Li [7] who also produced three nontrivial solutions for $\lambda > 0$ near a nonprincipal eigenvalue. As we already mentioned earlier, in all the aforementioned works, there is no potential term and so the differential operator is coercive. This facilitates the analysis of the problem. Papageorgiou, Rădulescu & Repovš [13] went beyond Dirichlet problems and studied Robin problems with an indefinite potential. In [13] the emphasis is on the existence and multiplicity of positive solutions. So, the conditions on the perturbation $f(z,\cdot)$ are different, leading to a bifurcation-type result describing the change in the set of positive solutions as the parameter λ moves in $\overset{\circ}{\mathbb{R}}_+ = (0,+\infty)$. We also mention the works of Castro, Cassio & Velez [1], Papageorgiou & Papalini [8] (Dirichlet problems), and Hu & Papageorgiou [3] (Robin problems) who also produce seven nontrivial solutions. In Castro, Cassio & Velez [1] there is no potential term, while Papageorgiou & Papalini [8] and Hu & Papageorgiou [3] have an indefinite potential term and moreover, provide sign information for all solution they produce. For related results we refer to Papageorgiou & Rădulescu [10], Papageorgiou & Winkert [16], Papageorgiou & Zhang [17], and Rolando [19]. Finally, we mention the work of Papageorgiou & Rădulescu [12] who proved multiplicity results for nearly resonant Robin problems.

In the present paper, using variational tools from the critical point theory together with suitable truncation, perturbation and comparison techniques and using also critical groups (Morse theory), we show that when the parameter $\lambda > 0$ is close to an eigenvalue of $(-\Delta u + \xi u,\ H^1(\Omega))$ with Robin boundary condition, then the problem has seven nontrivial smooth solutions, also providing sign information for six of them.

2. Mathematical Background and Hypotheses

The main spaces in the analysis of problem (P_λ) are the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue spaces $L^p(\partial\Omega)$, $1 \leqslant p \leqslant \infty$.

The Sobolev space $H^1(\Omega)$ is a Hilbert space with the following inner product

$$(u,h) = \int_\Omega uhdz + \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u,h \in H^1(\Omega).$$

By $\|\cdot\|$ we denote the norm corresponding to this inner product. So

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{1/2} \text{ for all } u \in H^1(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered by the positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff measure (surface measure) $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary value spaces $L^p(\partial\Omega)$, where $1 \leqslant p \leqslant \infty$. From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. We know that

$$\operatorname{im}\gamma_0 = H^{1/2,2}(\partial\Omega) \text{ and } \ker\gamma_0 = H_0^1(\Omega).$$

The linear map $\gamma_0(\cdot)$ is compact from $H^1(\Omega)$ into $L^p(\partial\Omega)$ for all $p \in \left[1, \frac{2(N-1)}{N-2}\right)$ if $N \geqslant 3$ and into $L^p(\partial\Omega)$ for all $1 \leqslant p < \infty$, if $N = 2$.

In the sequel, for the sake of notational simplicity, we drop the use of the map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $x \in \mathbb{R}$. We set $x^\pm = \max\{\pm x, 0\}$ and for any given $u \in H^1(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in H^1(\Omega), \ u = u^+ - u^-, \ |u| = u^+ + u^-.$$

Given $u, v \in H^1(\Omega)$ with $u \leqslant v$, we set

$$[u,v] = \{h \in H^1(\Omega) : \ u(z) \leqslant h(z) \leqslant v(z) \text{ for a.a. } z \in \Omega\}.$$

By $\operatorname{int}_{C^1(\overline{\Omega})}[u,v]$ we denote the interior in the $C^1(\overline{\Omega})$-norm topology of $[u,v] \cap C^1(\overline{\Omega})$.

Let us introduce our hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$.

$H_0 : \xi \in L^s(\Omega)$ with $s > N$ if $N \geqslant 2$ and $s > 1$ if $N = 2$, $\xi^+ \in L^\infty(\Omega)$ and $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

As we mentioned in the introduction, our analysis of problem (P_λ) relies on the spectrum of $u \mapsto -\Delta u + \xi(z)u$ with Robin boundary condition. So, we consider the following linear eigenvalue problem

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \hat{\lambda} u(z) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue", if problem (1) admits a nontrivial solution $\hat{u} \in H^1(\Omega)$ known as an "eigenfunction" corresponding to the eigenvalue $\hat{\lambda}$. From hypotheses H_0 and the regularity theory of Wang [21], we know that $\hat{u} \in C^1(\overline{\Omega})$.

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\gamma(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma \text{ for all } u \in H^1(\Omega).$$

From D'Agui, Marano & Papageorgiou [2] (see also Papageorgiou & Rădulescu [11]), we know that there exists $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geqslant \hat{C}\|u\|^2 \text{ for some } \hat{C} > 0, \text{ all } u \in H^1(\Omega). \tag{2}$$

Using (2) and the spectral theorem for compact, self-adjoint operators on a Hilbert space, we show (see [2,11]) that the spectrum of (1) consists of a sequence $\{\hat{\lambda}_k\}_{k\in\mathbb{N}}$ of distinct eigenvalues such that $\hat{\lambda}_k \to +\infty$ as $k \to \infty$. There is also a corresponding sequence $\{\hat{u}_k\}_{k\in\mathbb{N}} \subseteq H^1(\Omega)$ of eigenfunctions which form an orthogonal basis for $H^1(\Omega)$ and an orthonormal basis for $L^2(\Omega)$. As we already mentioned, $\hat{u}_k \in C^1(\overline{\Omega})$ for all $k \in \mathbb{N}$. By $E(\hat{\lambda}_k)$ we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$. We have $E(\hat{\lambda}_k) \subseteq C^1(\overline{\Omega})$ for all $k \in \mathbb{N}$, this subspace is finite dimensional and

$$H^1(\Omega) = \overline{\mathop{\oplus}_{k\geqslant 1} E(\hat{\lambda}_k)}.$$

Moreover, each eigenspace $E(\hat{\lambda}_k)$ has the "unique continuation property" (the UCP for short) which says that

$$\text{"if } u \in E(\hat{\lambda}_k) \text{ and } u(\cdot) \text{ vanishes on a set of positive measure, then } u \equiv 0''.$$

The first (principal) eigenvalue $\hat{\lambda}_1$ is simple, that is, $\dim E(\hat{\lambda}_1) = 1$. All the eigenvalues admit variational characterizations in terms of the Rayleigh quotient $\frac{\gamma(u)}{\|u\|_2^2}$, $u \in H^1(\Omega)$, $u \neq 0$. We have

$$\hat{\lambda} = \inf\left\{\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right\}, \tag{3}$$

$$\begin{aligned}\hat{\lambda}_k &= \sup\left\{\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{H}_k = \overset{k}{\underset{m=1}{\oplus}} E(\hat{\lambda}_m), u \neq 0\right\}\\ &= \inf\left\{\frac{\gamma(u)}{\|u\|_2^2} : u \in \hat{H}_k = \overline{\underset{m\geqslant k}{\oplus} E(\hat{\lambda}_m)}, u \neq 0\right\},\ k \geqslant 2.\end{aligned} \tag{4}$$

In (3) the infimum is realized on $E(\hat{\lambda}_1)$, while in (4) both the supremum and the infimum are realized on $E(\hat{\lambda}_k)$.

From (3) it follows that the elements of $E(\hat{\lambda}_1)$ have fixed sign, while from (4) and the orthogonality of the eigenspaces, we see that the elements of $E(\hat{\lambda}_k)$ (for $k \geqslant 2$) are nodal (that is, sign-changing). By $\hat{u}_1$ we denote the positive, L^2-normalized (that is, $\|\hat{u}\|_2 = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. The regularity theory and the Hopf maximum principle imply that $\hat{u}_1 \in \operatorname{int} C_+$.

Let X be a Banach space, $c \in \mathbb{R}$ and $\varphi \in C^1(X, \mathbb{R})$. We introduce the following sets

$$\begin{aligned}K_\varphi &= \{u \in X : \varphi'(u) = 0\}(\text{ the critical set of } \varphi),\\ \varphi^c &= \{u \in X : \varphi(u) \leqslant c\}.\end{aligned}$$

We say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1}$ such that
$\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded
and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
admits a strongly convergent subsequence$''$.

This is a compactness-type condition on the functional $\varphi(\cdot)$. Since the ambient space is not in general locally compact (being infinite dimensional), the burden of compactness is passed to the functional $\varphi(\cdot)$. Using the C-condition one can prove a deformation theorem from which follows the minimax theory of the critical values of $\varphi(\cdot)$ (see, for example, Papageorgiou, Rădulescu & Repovš [14, Chapter 5]).

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. Given $k \in \mathbb{N}_0$, we denote by $H_k(Y_1, Y_2)$ the kth-relative singular homology group for the pair (Y_1, Y_2) with $\mathbb{Z}$-coefficients. If $\varphi \in C^1(X, \mathbb{R})$, $u \in K_\varphi$ is isolated and $c = \varphi(u)$, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \text{ for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the isolating neighborhood U.

We say that a Banach X has the "Kadec-Klee property" if the following is true

$$\text{“}u_n \xrightarrow{w} u \text{ in } X \text{ and } \|u_n\|_X \to \|u\|_X \Rightarrow u_n \to u \text{ in } X''.$$

A uniformly convex space has the Kadec-Klee property. In particular, Hilbert spaces have the Kadec-Klee property.

We denote by $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ the operator defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in H^1(\Omega).$$

Also, by $\delta_{k,m}$ we denote the Kronecker symbol defined by

$$\delta_{k,m} = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m. \end{cases}$$

Finally, let 2^* denote the Sobolev critical exponent corresponding to 2, that is,

$$2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geqslant 3 \\ +\infty, & \text{if } N = 2. \end{cases}$$

Now we introduce the hypotheses on the perturbation $f(z, x)$.

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(*i*) $|f'_x(z, x)| \leqslant a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $2 < r < 2^*$;

(*ii*) if $F(z, x) = \displaystyle\int_0^x f(z, s)ds$, then $\displaystyle\lim_{x\to\pm\infty} \frac{F(z, x)}{x^2} = +\infty$ uniformly for a.a. $z \in \Omega$;

(*iii*) there exists $\tau \in \left((r-2)\max\left\{1, \frac{N}{2}\right\}, 2^*\right)$ such that

$$0 < \hat{\beta}_0 \leqslant \liminf_{x\to\pm\infty} \frac{f(z, x)x - 2F(z, x)}{|x|^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(*iv*) $f'_x(z, 0) = \displaystyle\lim_{x\to 0} \frac{f(z, x)}{x} = 0$ uniformly for a.a. $z \in \Omega$;

(*v*) there exist $C^*, \delta > 0$ and $q > 2$ such that $F(z, x) \geqslant -C^*|x|^q$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ and $0 \leqslant f(z, x)x$ for a.a. $z \in \Omega$, all $0 \leqslant |x| \leqslant \delta_0$;

(*vi*) there exist constants $C_- < 0 < C_+$ and $m \in \mathbb{N}$, $m \geqslant 2$ such that

$$[\hat{\lambda}_{m+1} - \xi(z)]C_+ + f(z, C_+) \leqslant 0 \leqslant [\hat{\lambda}_{m+1} - \xi(z)]C_- + f(z, C_-) \text{ for a.a. } z \in \Omega;$$

(*vii*) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z, x) + \hat{\xi}_\rho x$ is nondecreasing on $[-\rho, \rho]$.

Remark. Hypotheses $H_1(ii), (iii)$ imply that

$$\lim_{x\to\pm\infty} \frac{f(z, x)}{x} = \pm\infty \text{ uniformly for a.a. } z \in \Omega.$$

Hence for a.a. $z \in \Omega$, the function $f(z, \cdot)$ is superlinear. However, this superlinearity of the perturbation term is not expressed using the AR-condition, which is common in the literature when dealing with superlinear problems. Recall that the AR-condition says that there exist $q > 2$ and $M > 0$ such that

$$0 < qF(z, x) \leqslant f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geqslant M \tag{5a}$$

$$0 < \operatorname*{essinf}_{\Omega} F(\cdot, \pm M) \tag{5b}$$

(see Mugnai [6]). Integrating (5a) and using (5b), we obtain the weaker condition

$$\begin{aligned} &C_0|x|^q \leqslant F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } |x| \geqslant M, \\ &\quad \Rightarrow C_0|x|^q \leqslant f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geqslant M \text{ (see (5a))}. \end{aligned}$$

So we see that the AR-condition implies that $f(z, \cdot)$ has at least $(q-1)$-polynomial growth. In this paper, instead of the AR-condition, we employ the less restrictive condition $H_1(iii)$, which allows the consideration of superlinear nonlinearities with "slower" growth near $\pm\infty$, which fail to satisfy the AR-condition. The following example illustrates this fact. For the sake of simplicity, we drop the z-dependence of f and assume that $\xi \in L^\infty(\Omega)$. Suppose that for some $m \in \mathbb{N}$, we have $C \geqslant |\hat{\lambda}_{m+2}| + \|\xi\|_\infty$, $C > 0$. Then the function

$$f(x) = \begin{cases} x - (C+1)|x|^{q-2}x, & \text{if } |x| \leqslant 1 \ (2 < q) \\ x \ln |x| - Cx, & \text{if } 1 < x \end{cases}$$

satisfies hypotheses H_1 but fails to satisfy the AR-condition.

For all $\lambda > 0$, let $\varphi_\lambda : H^1(\Omega) \to \mathbb{R}$ denote the energy functional associated to problem (P_λ), which is defined by

$$\varphi_\lambda(u) = \frac{1}{2}\gamma(u) - \frac{\lambda}{2}\|u\|_2^2 - \int_\Omega F(z, u)dz \text{ for all } u \in H^1(\Omega).$$

We have $\varphi_\lambda \in C^2(H^1(\Omega))$.

3. Constant Sign Solutions

In this section we prove the existence of four nontrivial smooth constant sign solutions when $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$.

Proposition 1. *If hypotheses H_0, H_1 hold and $\hat{\lambda}_m < \lambda < \hat{\lambda}_{m+1}$ (see $H_1(vi)$), then problem (P_λ) has at least four nontrivial solutions of constant sign*

$$\begin{aligned} &u_0, \hat{u} \in \operatorname{int} C_+, \ u_0 \neq \hat{u}, \\ &v_0, \hat{v} \in -\operatorname{int} C_+, \ v_0 \neq \hat{v}. \end{aligned}$$

Proof. Let $\mu > 0$ be as in (2) and consider the Carathéodory function $g_\lambda^+(z, x)$ defined by

$$g_\lambda^+(z,x) = \begin{cases} (\lambda+\mu)x^+ + f(z,x^+), & \text{if } x \leqslant C_+ \\ (\lambda+\mu)C_+ + f(z,C_+), & \text{if } C_+ < x. \end{cases} \tag{5}$$

We set $G_\lambda^+(z,x) = \int_0^x g_\lambda^+(z,s)ds$ and consider the C^1-functional $\Psi_\lambda^+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\Psi_\lambda^+(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega G_\lambda^+(z,u)dz \text{ for all } u \in H^1(\Omega).$$

From (2) and (5), we see that $\Psi_\lambda^+(\cdot)$ is coercive. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that $\Psi_\lambda^+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in H^1(\Omega)$ such that

$$\Psi_\lambda^+(u_0) = \inf\left\{\Psi_\lambda^+(u) :\ u \in H^1(\Omega)\right\}. \tag{6}$$

Let $t > 0$ be small so that $t\hat{u}_1(z) \leqslant \min\{C_+, \delta_0\}$ for all $z \in \overline{\Omega}$ (recall that $\hat{u}_1 \in \operatorname{int} C_+$). Using (5) and hypothesis $H_1(v)$ we have

$$\begin{aligned} &\Psi_\lambda^+(t\hat{u}_1) \leqslant \frac{t^2}{2}[\hat{\lambda}_1 - \lambda] < 0 \text{ (since } \lambda > \hat{\lambda}_1, \|\hat{u}_1\|_2 = 1), \\ &\quad \Rightarrow \Psi_\lambda^+(u_0) < 0 = \Psi_\lambda^+(0) \text{ (see (6))}, \\ &\quad \Rightarrow u_0 \neq 0. \end{aligned}$$

From (6) we have

$$\begin{aligned} &(\Psi_\lambda^+)'(u_0) = 0, \\ &\quad \Rightarrow \langle A(u_0), h\rangle + \int_\Omega [\xi(z)+\mu]u_0 h dz + \int_{\partial\Omega} \beta(z)u_0 h d\sigma = \int_\Omega g_\lambda^+(z,u_0)h dz \\ &\quad \text{for all } h \in H^1(\Omega). \end{aligned} \tag{7}$$

In (7) first we choose $h = -u_0^- \in H^1(\Omega)$. Then

$$\begin{aligned} &\gamma(u_0^-) + \mu\|u_0^-\|_2^2 = 0 \text{ see (5)}, \\ &\quad \Rightarrow \hat{C}\|u_0^-\|^2 \leqslant 0 \text{ (see (2))}, \\ &\quad \Rightarrow u_0 \geqslant 0,\ u_0 \neq 0. \end{aligned}$$

Next, in (7) we choose $h = (u_0 - C_+)^+ \in H^1(\Omega)$. We have

$$\begin{aligned}
&\langle A(u_0), (u_0 - C_+)^+\rangle + \int_\Omega [\xi(z) + \mu] u_0 (u_0 - C_+)^+ dz + \int_{\partial\Omega} \beta(z) u_0 (u_0 - C_+)^+ d\sigma \\
&= \int_\Omega [(\lambda + \mu) C_+ + f(z, C_+)] (u_0 - C_+)^+ dz \text{ (see (5))} \\
&\leqslant \int_\Omega \left[(\hat{\lambda}_{m+1} + \mu) C_+ + f(z, C_+)\right] (u_0 - C_+)^+ dz \text{ (since } \lambda < \hat{\lambda}_{m+1}) \\
&\leqslant \int_\Omega [\xi(z) + \mu] C_+ (u_0 - C_+)^+ dz \text{ (see hypotheses } H_1(vi)), \\
&\Rightarrow u_0 \leqslant C_+.
\end{aligned}$$

So, we have proved that

$$u_0 \in [0, C_+], \ u_0 \neq 0. \tag{8}$$

From (8), (5) and (7) it follows that u_0 is a positive solution of problem (P_λ) and we have

$$\begin{cases} -\Delta u_0(z) + \xi(z) u_0(z) = \lambda u_0(z) + f(z, u_0(z)) \text{ for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 \text{ on } \partial\Omega \end{cases} \tag{9}$$

(see Papageorgiou & Rădulescu [9]).

We consider the following functions

$$\hat{\vartheta}_\lambda(z) = \begin{cases} 0, & \text{if } 0 \leqslant u_0(z) \leqslant 1 \\ \lambda - \xi(z) + \frac{f(z, u_0(z))}{u_0(z)}, & \text{if } 1 < u_0(z) \end{cases}$$

and

$$\hat{\gamma}_\lambda(z) = \begin{cases} (\lambda - \xi(z)) u_0(z) + f(z, u_0(z)), & \text{if } 0 \leqslant u_0(z) \leqslant 1 \\ 0, & \text{if } 1 < u_0(z). \end{cases}$$

On account of hypotheses H_0, we have

$$\hat{\vartheta}_\lambda \in L^s(\Omega) \ (s > N) \text{ and } |\hat{\vartheta}_\lambda(z)| \leqslant |\lambda - \xi(z)| + C_1 [1 + u_0(z)^{r-1}]$$
$$\text{for a.a. } z \in \Omega, \text{ some } C_1 > 0.$$

If $N \geqslant 3$ (the case $N = 2$ is clear since then $2^* = +\infty$), then

$$(r - 2)\frac{N}{2} < \left[\frac{2N}{N-2} - 2\right] \frac{N}{2} = \frac{2N}{N-2} = 2^*.$$

Since $u_0 \in H^1(\Omega)$, by the Sobolev embedding theorem we have

$$\begin{aligned} & u_0^{(r-2)N/2} \in L^1(\Omega) \\ & \Rightarrow \hat{\vartheta}_\lambda \in L^{\frac{N}{2}}(\Omega). \end{aligned}$$

From (9) we have

$$\begin{cases} -\Delta u_0(z) = \hat{\vartheta}_\lambda(z) u_0(z) + \hat{\gamma}_\lambda(z) \text{ for a.a. } z \in \Omega, \\ \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 \text{ on } \partial\Omega. \end{cases}$$

By Lemma 5.1 of Wang [21], we obtain that

$$u_0 \in L^\infty(\Omega).$$

Then the Calderon-Zygmund estimates (see Lemma 5.2 of Wang [21]) imply that $u_0 \in W^{2,s}(\Omega)$. By the Sobolev embedding theorem we have $W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{s} > 0$. So, $u_0 \in C^{1,\alpha}(\overline{\Omega})$.

Let $\rho = \|u\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(vii)$. From (9) we have

$$\begin{aligned} &\Delta u_0(z) \leqslant (\|\xi^+\|_\infty + \hat{\xi}_\rho)u_0(z) \text{ for a.a. } z \in \Omega \\ &\quad \text{(see hypotheses } H_0), \\ &\quad \Rightarrow u_0 \in \operatorname{int} C_+ \text{ (by the maximum principle).} \end{aligned}$$

Evidently, choosing $\hat{\xi}_\rho > 0$ even bigger if necessary, we deduce that for a.a. $z \in \Omega$, the function

$$x \mapsto [\lambda + \hat{\xi}_\rho]x + f(z,x)$$

is nondecreasing on $[-\rho, \rho]$ ($\rho = \|u_0\|_\infty$). We have

$$\begin{aligned} &-\Delta u_0(z) + [\xi(z) + \hat{\xi}_\rho]u_0(z) \\ &\quad = [\lambda + \hat{\xi}_\rho]u_0(z) + f(z, u_0(z)) \\ &\quad \leqslant [\lambda + \hat{\xi}_\rho]C_+ + f(z, C_+) \text{ (see (8))} \\ &\quad \leqslant [\xi(z) + \hat{\xi}_\rho]C_+ \text{ for a.a. } z \in \Omega \text{ (see hypothesis } H_1(vi)), \\ &\quad \Rightarrow \Delta(C_+ - u_0)(z) \leqslant \left[\|\xi^+\|_\infty + \hat{\xi}_\rho\right](C_+ - u_0(z)) \text{ for a.a. } z \in \Omega, \\ &\quad \Rightarrow C_+ - u_0 \in \operatorname{int} C_+, \\ &\quad \Rightarrow u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[0, C_+]. \end{aligned} \tag{10}$$

Let $\varphi_\lambda^+ : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_\lambda^+(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u^-\|_2^2 - \frac{\lambda}{2}\|u^+\|_2^2 - \int_\Omega F(z, u^+)dz \text{ for all } u \in H^1(\Omega).$$

From (5) it is clear that

$$\begin{aligned} &\varphi_\lambda^+|_{[0,C_+]} = \Psi_\lambda^+|_{[0,C_+]} \\ &\quad \Rightarrow u_0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \varphi_\lambda^+ \text{ (see(10))}, \\ &\quad \Rightarrow u_0 \text{ is a local } H^1(\Omega) - \text{ minimizer of } \varphi_\lambda^+ \\ &\quad \text{(see Papageorgiou \&R}\breve{\text{a}}\text{dulescu [9]).} \end{aligned}$$

It is easy to see that

$$\begin{aligned} &K_{\varphi_\lambda^+} \subseteq C_+ \text{ (regularity theory)}, \\ &\quad \Rightarrow K_{\varphi_\lambda^+} \subseteq \operatorname{int} C_+ \cup \{0\} \text{ (maximum principle).} \end{aligned}$$

So, we may assume that $K_{\varphi_\lambda^+}$ is finite. Otherwise we already have an infinity of positive smooth solutions and so we are done. Then on account of Theorem 5.7.6 of Papageorgiou, Rădulescu & Repovš [14, p. 449], we can find $\rho_0 \in (0,1)$ small such that

$$\varphi_\lambda^+(u_0) < \inf\left\{\varphi_\lambda^+(u) : \|u-u_0\| = \rho_0\right\} = m_\lambda^+. \tag{11}$$

Hypothesis $H_1(ii)$ implies that

$$\varphi_\lambda^+(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty. \tag{12}$$

Claim. The functional φ_λ^+ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$|\varphi_\lambda^+(u_n)| \leqslant C_2 \text{ for some } C_2 > 0, \text{ all } n \in \mathbb{N}, \tag{13}$$

$$(1+\|u_n\|)(\varphi_\lambda^+)'(u_n) \to 0 \text{ in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{14}$$

From (14) we have

$$\left|\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h dz + \int_{\partial\Omega} \beta(z) u_n h d\sigma - \int_\Omega \mu u_n^- h dz - \int_\Omega [\lambda u_n^* + f(z,u_n^+)]h dz\right| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|}$$
$$\text{for all } h \in H^1(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{15}$$

In (15) we choose $h = -u_n^- \in H^1(\Omega)$. Then

$$\begin{aligned} &\gamma(u_n^-) + \mu\|u_n^-\|_2^2 \leqslant \varepsilon_n \text{ for all } n \in \mathbb{N}, \\ &\Rightarrow \hat{C}\|u_n^-\|^2 \leqslant \varepsilon_n \text{ for all } n \in \mathbb{N} \text{ (see (2))}, \\ &\Rightarrow u_n^- \to 0 \text{ in } H^1(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{16}$$

Next, we choose $h = u_n^+ \in H^1(\Omega)$ in (15). We obtain

$$-\gamma(u_n^+) + \int_\Omega [\lambda(u_n^+)^2 + f(z,u_n^+)u_n^+]dz \leqslant \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{17}$$

On the other hand from (13) and (16), we have

$$\gamma(u_n^+) - \int_\Omega [\lambda(u_n^+)^2 + 2F(z,u_n^+)]dz \leqslant C_3 \text{ for some } C_3 > 0, \text{ all } n \in \mathbb{N}. \tag{18}$$

We add (17) and (18) and obtain

$$\int_\Omega [f(z,u_n^+)u_n^+ - 2F(z,u_n^+)]dz \leqslant C_4 \text{ for some } C_4 > 0, \text{ all } n \in \mathbb{N}. \tag{19}$$

Hypotheses $H_1(i),(iii)$ imply that we can find $\hat{\beta}_1 \in (0,\hat{\beta}_0)$ and $C_5 > 0$ such that

$$\hat{\beta}_1|x|^\tau - C_5 \leqslant f(z,x)x - 2F(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{20}$$

We use (20) in (19) and obtain that

$$\{u_n^+\}_{n\geqslant 1} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{21}$$

First assume that $N \geqslant 3$. From hypothesis $H_1(iii)$ we see that without any loss of generality, we may assume that $\tau < r < 2^*$. So, we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{2^*}. \tag{22}$$

From the interpolation inequality (see Proposition 2.3.17 of Papageorgiou & Winkert [15, p. 116]), we have

$$\begin{aligned} &\|u_n^+\|_r \leqslant \|u_n^+\|_\tau^{1-t} \|u_n^+\|_{2^*}^t \\ &\quad \Rightarrow \|u_n^+\|_r^r \leqslant C_6 \|u_n^+\|^{tr} \text{ for some } C_6 > 0, \text{ all } n \in \mathbb{N} \\ &\qquad \text{(see (21) and use the Sobolev embedding theorem).} \end{aligned} \tag{23}$$

From hypothesis $H_1(i)$ we have

$$f(z,x)x \leqslant C_7[1 + x^r] \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0, \text{ some } C_7 > 0. \tag{24}$$

In (15) we choose $h = u_n^+ \in H^1(\Omega)$. Then

$$\begin{aligned} \gamma(u_n^+) + \mu \|u_n^+\|_2^2 &\leqslant [\lambda + \mu] \|u_n^+\|_2^2 + \int_\Omega f(z, u_n^+) u_n^+ dz + \varepsilon_n \\ &\leqslant [|\lambda| + \mu] \, \|u_n^+\|_2^2 + C_8[1 + \|u_n^+\|^{tr}] \\ &\qquad \text{for some } C_8 > 0 \text{ (see(24) and (23))} \\ &\leqslant C_9 \left[1 + \|u_n^+\|^{tr}\right] \\ &\qquad \text{for some } C_9 > 0 \text{ (recall that } 2 \leqslant \tau \text{ and see (21))}, \\ \Rightarrow \hat{C} \|u_n^+\|^2 &\leqslant C_9 \left[1 + \|u_n^+\|^{tr}\right] \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{25}$$

Using (22) and the fact that $\tau > (r-2)\frac{N}{2}$ (see hypothesis $H_1(iii)$ and recall that $N \geqslant 3$), we see that $tr < 2$. So, from (25) it follows that

$$\begin{aligned} &\{u_n^+\}_{n \geqslant 1} \subseteq H^1(\Omega) \text{ is bounded,} \\ &\quad \Rightarrow \{u_n\}_{n \geqslant 1} \subseteq H^1(\Omega) \text{ is bounded (see (16)).} \end{aligned} \tag{26}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ as } n \to \infty. \tag{27}$$

In (15) we choose $h = u_n - u \in H^1(\Omega)$, pass tot the limit as $n \to \infty$ and use (23), the Sobolev embedding theorem and the compactness of the trace map. We obtain

$$\begin{aligned} &\lim_{n \to \infty} \langle A(u_n), u_n - u \rangle = 0, \\ &\quad \Rightarrow \|Du_n\|_2 \to \|Du\|_2. \end{aligned} \tag{28}$$

From (27), (28) and the Kadec-Klee property of $H^1(\Omega)$, we infer that

$$u_n \to u \text{ in } H^1(\Omega) \text{ as } n \to \infty. \tag{29}$$

This proves that φ_λ^+ satisfies the C-condition when $N \geqslant 3$.

If $N = 2$, then $2^* = +\infty$ and by the Sobolev embedding theorem, we have $H^1(\Omega) \hookrightarrow L^\eta(\Omega)$ compactly for all $1 \leqslant \eta < \infty$. Then for the previous argument to work, we replace $2^*(=+\infty)$ with $\eta > r > \tau$. We choose $t \in (0,1)$ such that

$$\begin{aligned} \frac{1}{r} &= \frac{1-t}{\tau} + \frac{t}{\eta} \\ &\Rightarrow tr = \frac{\eta(r-t)}{\eta-\tau}, \\ &\Rightarrow tr \to r-\tau \text{ as } \eta \to +\infty \text{ and } r-\tau < 2 \text{ (see } H_1(iii)). \end{aligned}$$

So, we choose $\eta > r$ big enough so that $tr < 2$ and reasoning as above, we obtain (26) and then from that and the Kadec-Klee property, we reach again (29). We conclude that φ_λ^+ satisfies the C-condition. This proves the Claim.

Then (11), (12) and the Claim, permit the use of the mountain pass theorem. So, we can find $\hat{u} \in H^1(\Omega)$ such that

$$\hat{u} \in K_{\varphi_\lambda^+} \subseteq \operatorname{int} C_+ \cup \{0\} \text{ and } m_\lambda^+ \leqslant \varphi_\lambda^+(\hat{u}) \text{ (see(11))}. \tag{30}$$

From (11) and (30) it follows that $\hat{u} \neq u_0$. If we show that $\hat{u} \neq 0$, then this will be the second positive solution of (P_λ).

On account of hypotheses $H_1(i), (iv)$, we have

$$|f(z,x)| \leqslant C_{10}[|x| + |x|^{r-1}] \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } C_{10} > 0. \tag{31}$$

We have

$$\begin{aligned} |\varphi_\lambda(u) - \varphi_\lambda^+(u)| &\leqslant \frac{\mu + |\lambda|}{2}\|u_n^-\|_2^2 + \int_\Omega |F(z,-u^-)|dz \\ &\leqslant C_{11}\left[\|u\|^2 + \|u\|^r\right] \text{ for some } C_{11} > 0 \text{ (see (31))}. \end{aligned} \tag{32}$$

Also for $h \in H^1(\Omega)$ we have

$$\begin{aligned} &\left|\langle \varphi_\lambda'(u) - (\varphi_\lambda^+)'(u), h\rangle\right| \leqslant C_{12}\left[\|u\| + \|u\|^{r-1}\right]\|h\| \text{ for some } C_{12} > 0, \\ &\quad \Rightarrow \|\varphi_\lambda'(u) - (\varphi_\lambda^+)'(u)\|_{H^1(\Omega)^*} \leqslant C_{12}\left[\|u\| + \|u\|^{r-1}\right]. \end{aligned} \tag{33}$$

From (32), (33) and the C^1-continuity of critical groups (see Theorem 6.3.4 of Papageorgiou, Rădulescu & Repovš [14, p. 503]), we have

$$C_k(\varphi_\lambda, 0) = C_k(\varphi_\lambda^+, 0) \text{ for all } k \in \mathbb{N}_0. \tag{34}$$

By hypothesis, $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ and $m \geqslant 2$. So, $u = 0$ is a nondegenerate critical point of φ_λ with Morse index $d_m = \dim \overline{H}_m \geqslant 2$ (since $m \geqslant 2$). Then by Proposition 6.2.6 of [14, p. 479], we have

$$\begin{aligned} &C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \\ &\quad \Rightarrow C_k(\varphi_\lambda^+, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (34))}. \end{aligned} \tag{35}$$

On the other hand, from the previous part of the proof we know that $\hat{u} \in K_{\varphi_\lambda^+}$ is of mountain pass type. Therefore Theorem 6.5.8 of Papageorgiou, Rădulescu & Repovš [14, p. 527] implies that

$$C_1(\varphi_\lambda^+, \hat{u}) \neq 0. \tag{36}$$

From (36), (35) and since $d_m \geqslant 2$, we conclude that $\hat{u} \neq 0$ and so $\hat{u} \in \operatorname{int} C_+$ is the second positive solution of (P_λ) distinct from u_0.

For the negative solutions, we consider the Carathéodory function $g_\lambda^-(z, x)$ defined by

$$g_\lambda^-(z,x) = \begin{cases} (\lambda+\mu)C_- + f(z, C_-), & \text{if } x \leqslant C_- \\ (\lambda+\mu)(-x^-) + f(z, -x^-), & \text{if } C_- < x. \end{cases}$$

We set $G_\lambda^-(z,x) = \int_0^x g_\lambda^-(z,s)ds$ and consider the C^1-functionals $\Psi_\lambda^-, \varphi_\lambda^- : H^1(\Omega) \to \mathbb{R}$ defined by

$$\Psi_\lambda^-(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega G_\lambda^-(z,u)dz$$
$$\text{and } \varphi_\lambda^-(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u^+\|_2^2 - \frac{\lambda}{2}\|u^-\|_2^2 - \int_\Omega F(z, -u^-)dz$$

for all $u \in H^1(\Omega)$.

Working with these two functionals as above, we produce two negative solutions $v_0, \hat{v} \in -\operatorname{int} C_+$, $v_0 \neq \hat{v}$. $\square$

4. Nodal Solutions

In this section we show that when λ is close to $\hat{\lambda}_{m+1}$ (near resonance) we can generate two nodal (sign-changing) solutions.

Proposition 2. *If hypotheses H_0, H_1 hold and $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ (see $H_1(vi)$), then we can find $\hat{\delta} > 0$ such that for all $\lambda \in (\hat{\lambda}_{m+1} - \hat{\delta}, \hat{\lambda}_{m+1})$ problem (P_λ) has at least two nodal solutions $y_0, \hat{y} \in C^1(\overline{\Omega})$.*

Proof. From Proposition 2.3 of Rabinowitz, Su & Wang [18], we know that there exists $\delta_1 > 0$ such that for all $\lambda \in (\hat{\lambda}_{m+1} - \delta_1, \hat{\lambda}_{m+1})$ problem (P_λ) has at least two nontrivial solutions $y_0, \hat{y} \in H^1(\Omega)$. As before, using the regularity theory of Wang [21], we obtain that $y_0, \hat{y} \in C^1(\overline{\Omega})$. Note that the result of Rabinowitz, Su & Wang [18] is for Dirichlet problems with $\xi \equiv 0$. However, their proof is based on the abstract bifurcation theorem of Rabinowitz (see Theorem 2.1 in [18]) and so it applies verbatim in our case, too.

We will show that we can have these two solutions $y_0, \hat{y} \in C^1(\overline{\Omega})$ to be nodal. From the proof of Proposition 2.3 of Rabinowitz, Su & Wang [18] and using hypothesis $H_1(iv)$, we see that given $\varepsilon \in \left(0, \frac{\lambda - \hat{\lambda}_1}{2}\right)$ (recall that $\lambda > \hat{\lambda}_1$), we can find $0 < \hat{\delta} \leqslant \delta_1$ such that

$$\lambda \in (\hat{\lambda}_{m+1} - \hat{\delta}, \hat{\lambda}_{m+1}) \Rightarrow |f(z, w(z))| \leqslant \varepsilon w(z) \text{ for a.a. } z \in \Omega, \tag{37}$$

with $w = y_0$ or $w = \hat{y}$. Suppose that $w \in \operatorname{int} C_+$ (the reasoning is similar if $w \in -\operatorname{int} C_+$). We have

$$\begin{aligned}
&\hat{\lambda}_1 \int_\Omega w\hat{u}_1 dz \\
&\quad = \langle A(\hat{u}_1), w\rangle + \int_\Omega \xi(z)\hat{u}_1 w dz + \int_{\partial\Omega} \beta(z)\hat{u}_1 w d\sigma \\
&\quad = \int_\Omega (-\Delta w)\hat{u}_1 dz + \int_{\partial\Omega} \frac{\partial w}{\partial n}\hat{u}_1 d\sigma + \int_\Omega \xi(z)\hat{u}_1 w dz + \int_{\partial\Omega} \beta(z)\hat{u}_1 w d\sigma \\
&\quad \text{(using Green's identity)} \\
&\quad = \int_\Omega [\lambda w - f(z,w)]\hat{u}_1 dz \text{ (since } w \text{ is a solution of } (P_\lambda)) \\
&\quad \geqslant \int_\Omega \left[\lambda w - \frac{\lambda - \hat{\lambda}_1}{2} w\right] \hat{u}_1 dz \text{ (see (37) and recall that } 0 < \varepsilon \leqslant \frac{\lambda - \hat{\lambda}_1}{2}) \\
&= \int_\Omega \frac{\lambda + \hat{\lambda}_1}{2} w\hat{u}_1 dz \\
&> \hat{\lambda}_1 \int_\Omega w\hat{u}_1 dz, \text{ a contradiction.}
\end{aligned}$$

So, $w = y_0$ or $w = \hat{y}$ cannot be constant sign and so $y_0, \hat{y} \in C^1(\overline{\Omega})$ are nodal solutions of (P_λ) for $\lambda \in (\hat{\lambda}_{m+1} - \hat{\delta}, \hat{\lambda}_{m+1})$. $\square$

5. The Seventh Nontrivial Solution

In this section we prove the existence of a seventh nontrivial solution for problem (P_λ) when $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$. However, we are unable to provide sign information for this seventh solution.

Proposition 3. *If hypotheses H_0, $H_1(i),(iv)$ hold and $\lambda < \hat{\lambda}_{m+2}$, then there exists $\rho > 0$ such that*

$$\varphi_\lambda|_{\hat{H}_{m+2} \cap \partial B_\rho} \geqslant \tilde{C}_0 > 0$$
$$\text{with } \hat{H}_{m+2} = \overline{\underset{k \geqslant m+2}{\oplus} E(\hat{\lambda}_k)}, B_\rho = \{u \in H^1(\Omega) : \|u\| < \rho\}.$$

Proof. Hypotheses $H_1(i),(iv)$ imply that given $\varepsilon > 0$, we can find $C_\varepsilon > 0$ such that

$$|F(z,x)| \leqslant \frac{\varepsilon}{2}x^2 + C_\varepsilon |x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{38}$$

Let $u \in \hat{H}_{m+2}$. We have

$$\begin{aligned}\varphi_\lambda(u) &\geqslant \frac{1}{2}\gamma(u) - \frac{\lambda}{2}\|u\|_2^2 - \frac{\varepsilon}{2}\|u\|^2 - \hat{C}_\varepsilon\|u\|^r \\ &\quad \text{for some } \hat{C}_\varepsilon > 0 \text{ (see (38))} \\ &\geqslant \frac{C_{13}-\varepsilon}{2}\|u\|^2 - \hat{C}_\varepsilon\|u\|^r \text{ for some } C_{13} > 0 \text{ (recall that } \lambda < \hat{\lambda}_{m+2}).\end{aligned}$$

Choose $\varepsilon \in (0, C_{13})$. Then we obtain

$$\varphi_\lambda(u) \geqslant C_{14}\|u\|^2 - \hat{C}_\varepsilon\|u\|^r \text{ for some } C_{14} > 0, \text{ all } u \in \hat{H}_{m+2}.$$

Since $2 < r$, we can find $\rho \in (0,1)$ small such that

$$\varphi_\lambda(u) \geqslant \tilde{C}_0 > 0 \text{ for all } u \in \hat{H}_{m+2} \cap \partial B_\rho.$$

The proof is now complete. □

Let $\hat{u}_{m+2} \in E(\hat{\lambda}_{m+2})$ with $\|\hat{u}_{m+2}\| = 1$ and let $V = \overline{H}_{m+1} \oplus \mathbb{R}\hat{u}_{m+2}$, with $\overline{H}_{m+1} = \overset{m+1}{\underset{k=1}{\oplus}} E(\hat{\lambda}_k)$. For $\rho_1 > 0$, we introduce the set

$$C = \{u = \overline{u} + \vartheta\hat{u}_{m+2} : \overline{u} \in \overline{H}_{m+1}, \vartheta \geqslant 0, \|u\| \leqslant \rho_1\}.$$

Evidently we have

$$\begin{aligned}&\partial C = C_0 = \big\{u = \overline{u} + \vartheta\hat{u}_{m+2} : \big(\overline{u} \in \overline{H}_{m+1}, \vartheta \geqslant 0, \|u\| = \rho_1\big) \text{ or} \\ &\quad \big(\overline{u} \in \overline{H}_{m+1}, \|\overline{u}\| \leqslant \rho_1, \vartheta = 0\big)\big\}.\end{aligned}$$

Proposition 4. *If hypotheses H_0, $H_1(i),(ii),(iv),(v)$ hold and $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$, then there exist $\rho_1 > 0$ and $\tilde{\delta} > 0$ such that*

$$\varphi_\lambda|_{C_0} \leqslant \tilde{C}_1 < \tilde{C}_0$$

with $\tilde{C}_0 > 0$ as in Proposition 3.

Proof. From hypotheses $H_1(i),(ii),(v)$ given $\eta > 0$, we can find $\hat{C}_\eta^* > 0$ such that

$$F(z,x) \geqslant \frac{\eta}{2}x^2 - \hat{C}_\eta^*|x|^q \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{39}$$

The space V is finite dimensional and so all norms are equivalent. Let $u \in V$. We have

$$\begin{aligned}\varphi_\lambda(u) &\leqslant \frac{1}{2}\gamma(u) - \frac{\lambda}{2}\|u\|_2^2 - \frac{\eta}{2}\|u\|_2^2 + \frac{\hat{C}_\eta^*}{q}\|u\|_q^q \text{ (see (39))} \\ &\leqslant C_{15}\left[\hat{\lambda}_{m+2} - \lambda - \eta\right]\|u\|^2 + C_{16}\|u\|^q \text{ for some } C_{15}, C_{16} = C_{16}(\eta) > 0.\end{aligned}$$

Since $\eta > 0$ arbitrary, choosing $\eta > 0$ big, we have

$$\varphi_\lambda(u) \leqslant C_{16}\|u\|^q - C_{17}\|u\|^2 \text{ for some } C_{17} > 0.$$

Recall that $q > 2$. Then we can find $\rho_1 \in (0, 1)$ small such that

$$\varphi_\lambda|_{V \cap \partial B_\rho} \leqslant 0 < \tilde{C}_0 \text{ (see Proposition 3)}.$$

If $\overline{u} \in \overline{H}_{m+1}$, $\|\overline{u}\| \leqslant \rho_1$, then

$$\begin{aligned} \varphi_\lambda(\overline{u}) &\leqslant \frac{1}{2}\gamma(\overline{u}) - \frac{\lambda}{2}\|\overline{u}\|_2^2 + C^*\|\overline{u}\|_q^q \\ &\leqslant \frac{1}{2}\left[\hat{\lambda}_{m+1} - \lambda\right]\|\overline{u}\|_2^2 + C^*\|\overline{u}\|_q^q \text{ (see } H_1(v)) \\ &\leqslant C_{18}\rho_1^2 \text{ (since } q > 2,\ \lambda < \hat{\lambda}_{m+1} \text{ and } \rho_1 \in (0, 1)). \end{aligned}$$

Choosing $\rho_1 \in (0, 1)$ even smaller if necessary, we have

$$\varphi_\lambda|_{\overline{H}_{m+1} \cap \partial B_{\rho_1}} \leqslant \tilde{C}_1 < \tilde{C}_0$$

for all $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ and with $\tilde{C}_0 > 0$ (as in Proposition 3).

Therefore we conclude that

$$\varphi_\lambda|_{C_0} \leqslant \tilde{C}_1 < \tilde{C}_0 \text{ for } \lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1}).$$

The proof is now complete. □

Now we are ready to produce the seventh nontrivial smooth solution of problem (P_λ).

Proposition 5. *If hypotheses H_0, H_1 hold and $\lambda \in (\hat{\lambda}_{m+1} - \hat{\delta}, \hat{\lambda}_{m+1})$ (see Proposition 2), then problem (P_λ) has a seventh nontrivial solution $\tilde{y} \in C^1(\overline{\Omega})$.*

Proof. Let $D = \overline{H}_{m+1} \cap \partial B_{\rho_1}$. From Proposition 6.6.5 of Papageorgiou, Rădulescu & Repovš [14, p. 532], we know that

$$\{C, C_0\} \text{ and } D \text{ homologically link in dimension } d_{m+1} + 1$$

with $d_{m+1} = \dim \overline{H}_{m+1}$. Then Propositions 3 and 4 and Corollary 6.6.8 of [14], imply that there exists $\tilde{y} \in K_{\varphi_\lambda} \subseteq C^1(\overline{\Omega})$ (see Wang [21]) such that

$$C_{d_{m+1}+1}(\varphi_\lambda, \tilde{y}) \neq 0. \tag{40}$$

From the proof of Proposition 1, we know that $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ are local minimizers of φ_λ^+ and of φ_λ^- respectively. Note that

$$\varphi_\lambda|_{C_+} = \varphi_\lambda^+|_{C_+} \text{ and } \varphi_\lambda|_{-C_+} = \varphi_\lambda^+|_{-C_+}. \tag{41}$$

So, it follows that $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ are also local minimizers of φ_λ (see [9]). Therefore we have

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{42}$$

Also, again from the proof of Proposition 1, we know that the solutions $\hat{u} \in \operatorname{int} C_+$ and $\hat{v} \in -\operatorname{int} C_+$ are critical points of mountain pass type of the functionals φ_λ^+ and φ_λ^- respectively. Therefore we have

$$C_1(\varphi_\lambda^+, \hat{u}) \neq 0 \text{ and } C_1(\varphi_\lambda^-, \hat{v}) \neq 0 \text{ (see (36))}. \tag{43}$$

From (41) and since $\hat{u} \in \operatorname{int} C_+$, $\hat{v} \in -\operatorname{int} C_+$, we have

$$C_k\left(\varphi_\lambda^+|_{C^1(\overline{\Omega})}, \hat{u}\right) = C_k\left(\varphi_\lambda|_{C^1(\overline{\Omega})}, \hat{u}\right) \text{ and } C_k\left(\varphi_\lambda^-|_{C^1(\overline{\Omega})}, \hat{v}\right) = C_k\left(\varphi_\lambda|_{C^1(\overline{\Omega})}, \hat{v}\right) \tag{44}$$

for all $k \in \mathbb{N}_0$.

But on account of Theorem 6.6.26 of Papageorgiou, Rădulescu, Repovš [14, p. 545], we have

$$C_k(\varphi_\lambda^+, \hat{u}) = C_k(\varphi_\lambda, \hat{u}) \text{ and } C_k(\varphi_\lambda^-, \hat{v}) = C_k(\varphi_\lambda, \hat{v}) \tag{45}$$

for all $k \in \mathbb{N}_0$.

Since $\varphi_\lambda \in C^2(H^1(\Omega))$, from (42), (43), (45) and Proposition 6.5.9 of Papageorgiou, Rădulescu & Repovš [14, p. 529], we infer that

$$C_k(\varphi_\lambda, \hat{u}) = C_k(\varphi_\lambda, \hat{v}) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{46}$$

Recall that

$$C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ (see (35))}. \tag{47}$$

Moreover, from Corollary 6.2.40 of Papageorgiou, Rădulescu & Repovš [14, p. 449], we have

$$C_k(\varphi_\lambda, y_0) = C_k(\varphi_\lambda, \hat{y}) = 0 \text{ for } k \notin [d_m, d_{m+1}] \text{ (recall that } d_m \geqslant 2). \tag{48}$$

From (40), (42), (46), (47), (48), we infer that

$$\begin{aligned}
&\tilde{y} \notin \{u_0, v_0, \hat{u}, \hat{v}, 0, y_0, \hat{y}\},\\
&\quad \Rightarrow \tilde{y} \in C^1(\overline{\Omega}) \text{ is the seventh nontrivial solution of } (P_\lambda)\\
&\quad (\lambda \in (\hat{\lambda}_{m+1} - \hat{\delta}, \hat{\lambda}_{m+1})).
\end{aligned}$$

The proof is now complete. □

So, summarizing our findings for problem (P_λ), we can state the following multiplicity theorem.

Theorem 6. *If hypotheses H_0, H_1 hold, then there exists $\hat{\hat{\delta}} > 0$ such that for all $\lambda \in (\hat{\lambda}_{m+1} - \hat{\hat{\delta}}, \hat{\lambda}_{m+1})$ problem (P_λ) has at least seven distinct nontrivial smooth solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+,\ \ v_0, \hat{v} \in -\operatorname{int} C_+,\ \ y_0, \hat{y} \in C^1(\overline{\Omega}) \text{ nodal}$$
$$\tilde{y} \in C^1(\overline{\Omega}).$$

Remark. Is it possible to show that $\tilde{y}$ is nodal (see [3,8])? Also, it seems that we cannot generate more than seven solutions without symmetry hypotheses (see [1]).

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