A Note on Integrability of the Norm of a Perturbed Semigroup

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Abstract. Recently, Michael Gil' has provided a condition for integrability of norms of semigroups generated by operators of the form $A + B$ where B is bounded, in terms of the norm of the commutator of A and B . We show that his argument, when slightly refined, leads to a bit more general result than the one presented in his paper. We also argue, by examining an example, that the size of the commutator may not be a good indicator of whether the perturbed semigroup has integrable norm or not.

Keywords. Commutator, asymptotic behavior, semigroups of operators.

1. Introduction

Recently, in [6], Michael Gil' has proved the following result. (See also his related article [7] devoted to the case where $\{e^{tA}, t \geq 0\}$ is holomorphic.) Let $\mathbb{B}$ be a Banach space, and let $\mathcal{L}(\mathbb{B})$ be the algebra of bounded linear operators in $\mathbb{B}$. Suppose that A is the generator of a strongly continuous semigroup $\{e^{tA}, t \geq 0\}$ in $\mathbb{B}$, and let $B \in \mathcal{L}(\mathbb{B})$ be such that

- (a) $\mathcal{J} := \int_0^\infty \|e^{tA}\| \|e^{tB}\| dt < \infty$,
- (b) B maps $\mathcal{D}(A)$ into itself and there is a $K \in \mathcal{L}(\mathbb{B})$ such that

$$Kx = BAx - ABx, \quad x \in \mathcal{D}(A),$$

- (c) $\|K\| \mathcal{J}^2 < 1$.

Then

$$\int_0^\infty \|e^{t(A+B)}\| dt < \infty. \quad (1)$$

(For the history of this kind of theorems, dating back to R.S. Phillips, see the survey article [10].) It is perhaps worth noting here that existence of the semigroup $\{e^{t(A+B)}, t \geq 0\}$ is guaranteed by the Phillips Perturbation Theorem ([1, 2, 4, 8] or [9]); also, Lebesgue measurability of $t \mapsto \|e^{tG}\|$ for any strongly continuous semigroup $\{e^{tG}, t \geq 0\}$ is not an issue (see [9] p. 306).

In this note we show that condition (c) may be relaxed: it suffices to assume, instead of (c), that

$$(c') \quad \|K\| \mathcal{I} < 1,$$

where

$$\mathcal{I} := \int_0^\infty \|e^{tB}\| \int_0^t \|e^{(t-s)A}\| \|e^{sA}\| \, ds \, dt.$$

This implication is established in Sect. 2. In Sect. 3, we discuss an illustrative example showing that in certain situations this criterion is significantly stronger than that of M. Gil'.

In Sect. 4, we argue that, unfortunately, ‘the size of the commutator’ seems to be a rather poor criterion for checking whether (1) holds. More specifically, we show that for any natural n there are semigroups $\{e^{tA}, t \geq 0\}$ and $\{e^{tB}, t \geq 0\}$ such that (i) conditions (a) and (b) of the theorem of Gil' are satisfied, (ii) the commutator of A and B , though bounded, has norm larger than n , and (iii) inequality (1) nevertheless holds.

2. Main Result

We start by noting that, by Tonelli's Theorem and the fact that $t \mapsto \|e^{tB}\|$ is submultiplicative,

$$\mathcal{I} \leq \int_0^\infty \|e^{sA}\| \|e^{sB}\| \int_s^\infty \|e^{(t-s)B}\| \|e^{(t-s)A}\| \, dt \, ds = \mathcal{J}^2$$

with equality if $t \mapsto \|e^{tB}\|$ is multiplicative. This shows that assumption (c') is weaker than (c).

That (c') can be substantially weaker is shown by the following example: if A is chosen so that $\|e^{tA}\| = 1, t \geq 0$ and B is chosen so that, for a $\kappa > 1$ and $\omega > 0$, $\|e^{tB}\| = \kappa e^{-\omega t}, t \geq 0$, then $\mathcal{I} = \frac{\kappa}{\omega^2}$ and $\mathcal{J}^2 = \frac{\kappa^2}{\omega^2} > \mathcal{I}$ (see also our Sect. 3).

Our argument that (c') may successfully replace (c) is a straightforward refinement of that of M. Gil'. He proved (see equation (2.5) and the top of page 26 in [6]) that

$$\|e^{t(A+B)}\| \leq \|e^{tA}e^{tB}\| + \int_0^t \|e^{(t-s)(A+B)}\| w(s) \, ds$$

where

$$0 \leq w(t) \leq \|K\| \|e^{tB}\| \int_0^t \|e^{(t-s)A}\| \|e^{sA}\| \, ds, \quad t \geq 0.$$

By assumption (c'), w is integrable over $\mathbb{R}^+$ with $\|w\|_{L^1} := \int_0^\infty w(t) dt = \|K\|\mathcal{I} < 1$. Introducing, for simplicity of notation, $\psi(t) = \|e^{tA}e^{tB}\|$ and $\phi(t) = \|e^{t(A+B)}\|$ we have thus

$$0 \leq \phi(t) \leq \psi(t) + \int_0^t \phi(t-s)w(s) ds, \quad t \geq 0.$$

Since, for any $T > 0$ (comp. p. 26 in [6])

$$\int_0^T \int_0^t \phi(t-s)w(s) ds dt = \int_0^T w(s) \int_s^T \phi(t-s) dt ds \leq \|w\| \int_0^T w(t) dt$$

we obtain

$$\int_0^T \phi(t) dt \leq \int_0^T \psi(t) dt + \|w\|_{L^1} \int_0^T \phi(t) dt$$

and then

$$\int_0^T \phi(t) dt \leq \frac{\|\psi\|_{L^1}}{1 - \|w\|_{L^1}},$$

where $\|\psi\|_{L^1} = \int_0^\infty \psi(t) dt = \mathcal{J}$. Since T is arbitrary, $\int_0^\infty \|e^{t(A+B)}\| dt = \int_0^\infty \phi(t) dt \leq \frac{\mathcal{J}}{1 - \|K\|\mathcal{I}}$. $\square$

We also note that M. Gil's estimates for the L^1 and supremum norms of $e^{tA}e^{tB} - e^{t(A+B)}$ (i.e., formulae (1.4) and (1.5) in [6]) may be similarly improved.

3. An Example

In this section we construct a specific example of semigroups for which $\mathcal{I} < \mathcal{J}^2$, i.e. our criterion is significantly stronger than that of M. Gil'. Our starting point is a semigroup $\{e^{tA}, t \geq 0\}$ of isometries; its generator A is then perturbed by a family of operators B_ω , where $\omega > 0$ is a parameter. We will show that (c') allows concluding (1) for a larger range of ω than (c).

3.1. Semigroup Generated by A

Let $\mathbb{B}$ be the space $C[0, 1]$ of real-valued continuous functions f on the unit interval, and let for $t \geq 0$, $S(t)$ be the operator in $\mathbb{B}$ given by

$$S(t)f(x) = \frac{1 + xe^{-t}}{1 + x} f(xe^{-t}), \quad f \in \mathbb{B}, x \in [0, 1], t \geq 0.$$

It can be checked that $\{S(t), t \geq 0\}$ is a strongly continuous semigroup of non-negative operators in $\mathbb{B}$. As we shall see later (see part (c) of the proof of Proposition 1, later on), the domain $\mathcal{D}(A)$ of the generator of this semigroup is composed of $f \in \mathbb{B}$ such that the derivative of $x \mapsto (1+x)f(x)$ exists for all $x \in (0, 1]$ and $\lim_{x \rightarrow 0+} x[(1+x)f(x)]' = 0$; moreover, for $f \in \mathcal{D}(A)$,

$$Af(0) = 0 \text{ and } Af(x) = \frac{-x}{1+x} [(1+x)f(x)]', \quad x \in (0, 1].$$

Obviously, $\|S(t)f\| \leq \|f\|$, $f \in \mathbb{B}$, $t \geq 0$; since $S(t)f(0) = f(0)$, it is thus clear that

$$\|e^{tA}\| = 1, \quad t \geq 0. \quad (2)$$

3.2. Operators B_ω , $\omega > 0$ and the Related Semigroups

Next, given a continuously differentiable, non-negative function $c \in \mathbb{B}$, such that $c(1) = 0$ and $\int_0^1 c = 1$, we consider $C \in \mathcal{L}(\mathbb{B})$ given by

$$Cf(x) = \frac{1}{1+x} \int_0^1 c(y)(1+y)f(y) \, dy, \quad f \in \mathbb{B}, x \in [0, 1].$$

Then, for

$$\kappa := \int_0^1 c(y)(1+y) \, dy \in (1, 2),$$

we have $\|Cf\| \leq \kappa\|f\|$, $f \in \mathbb{B}$. Since for $f \equiv 1$, $\|Cf\| = Cf(0) = \kappa$, we see that $\|C\| = \kappa$. A straightforward argument (see e.g. [2] p. 6) based on the easy-to-establish formula $C^n = C$, $n \geq 1$ (use $\int_0^1 c = 1$) shows that

$$e^{tC} = (e^t - 1)C + I,$$

where I is the identity operator in $\mathbb{B}$. Hence, arguing as above, we check that $\|e^{tC}\| = (e^t - 1)\kappa + 1$. Thus, introducing

$$B_\omega := C - (\omega + 1)I$$

where $\omega > 0$ is a parameter, we obtain

$$\|e^{tB_\omega}\| = e^{-\omega t}[(1 - e^{-t})\kappa + e^{-t}], \quad t \geq 0. \quad (3)$$

3.3. The Commutator

It is easy to check that C maps $\mathbb{B}$ into $\mathcal{D}(A)$ and that $ACf = 0$, $f \in \mathbb{B}$. Also, assumption $c(1) = 0$ (used here for the first time) and integration by parts yield

$$\begin{aligned} CAf(x) &= \frac{-1}{1+x} \int_0^1 yc(y)[(1+y)f(y)]' \, dy \\ &= \frac{1}{1+x} \int_0^1 [yc(y)]'(1+y)f(y) \, dy, \quad f \in \mathcal{D}(A), x \in [0, 1]. \end{aligned}$$

Therefore the map $\mathcal{D}(A) \ni f \mapsto CAf - ACf$ is a restriction of the bounded linear operator given by

$$Kf(x) = \frac{1}{1+x} \int_0^1 [yc(y)]'(1+y)f(y) \, dy, \quad f \in \mathbb{B}, x \in [0, 1].$$

Since $B_\omega Af - AB_\omega f = CAf - ACf$ for all $f \in \mathcal{D}(A)$ and $\omega > 0$ (i.e., a shift of C does not change the commutator) it follows that assumptions (a) and (b)

of Sect. 2 are satisfied for our A and $B = B_\omega$ for all $\omega > 0$. Moreover, by (2) and (3),

$$\mathcal{J}_\omega := \int_0^\infty \|e^{tA}\| \|e^{tB_\omega}\| dt = \frac{1}{1+\omega} \left(1 + \frac{\kappa}{\omega}\right),$$

$$\mathcal{I}_\omega := \int_0^\infty \|e^{tB_\omega}\| \int_0^t \|e^{(t-s)A}\| \|e^{sA}\| ds dt = \frac{1}{(1+\omega)^2} \left(1 + \frac{2\kappa}{\omega} + \frac{\kappa}{\omega^2}\right),$$

and so $\mathcal{J}_\omega^2$ and $\mathcal{I}_\omega$ differ by $\frac{\kappa(\kappa-1)}{\omega^2(1+\omega)^2} > 0$. It is thus clear that (c') is satisfied for a larger range of ω than is (c).

4. Is ‘The Size of the Commutator’ a Good Criterion for (1)?

The criterion proved in Sect. 2 guarantees that

$$\int_0^\infty \|e^{t(A+B_\omega)}\| dt < \infty \quad (4)$$

for the operators A and B_ω of our example, if

$$\frac{\|K\|}{(1+\omega)^2} \left(1 + \frac{2\kappa}{\omega} + \frac{\kappa}{\omega^2}\right) < 1 \quad (5)$$

i.e., for all sufficiently large ω . On the other hand, a look at the definitions of κ and K reveals that the function c may be modified, without affecting κ , so that $\|K\|$ is made arbitrarily large (see the example at the end of this section for details). In such a case the criterion implies (4) only for really large ω .

It is our goal in this section to establish the following statement.

Proposition 1. *For the operators A and B_ω of the previous section,*

$$\int_0^\infty \|e^{t(A+B_\omega)}\| dt < \infty$$

for all $\omega > 0$ regardless of how large $\|K\|$ is.

In view of our previous analysis, this proposition suggests that ‘the size of the commutator’ is probably not the best possible criterion for condition (1). We also note that this proposition holds even if we do not assume that c is differentiable and $c(1) = 0$; these two assumptions were used to check that the commutator is a bounded operator.

Proof (of Proposition 1). (a) We start by recalling (see e.g. [1, 2, 4]) that if $\{e^{tG}, t \geq 0\}$ is a strongly continuous semigroup in a Banach space $\mathbb{B}$ and $J \in \mathcal{L}(\mathbb{B})$ is an isomorphism, then $\{J^{-1}e^{tG}J, t \geq 0\}$ is also a strongly continuous semigroup, often termed similar or isomorphic to $\{e^{tG}, t \geq 0\}$. The generator, say $G^\sharp$, of the latter semigroup is characterized as follows: $f \in \mathcal{D}(G^\sharp)$ iff $Jf \in \mathcal{D}(G)$, and then $G^\sharp f = J^{-1}GJf$. Since $\|e^{tG^\sharp}\| \leq \|J^{-1}\| \|e^{tG}\| \|J\|$ and

the roles of the semigroups $\{e^{tG}, t \geq 0\}$ and $\{e^{tG^\sharp}, t \geq 0\}$ are symmetric, it is clear that $\int_0^\infty \|e^{tG}\| dt < \infty$ iff $\int_0^\infty \|e^{tG^\sharp}\| dt < \infty$

(b) Thus our task reduces to finding a J such that $\int_0^\infty \|J^{-1}e^{t(A+B_\omega)}J\| dt < \infty$. We will show that

$$\|J^{-1}e^{t(A+B_\omega)}J\| = e^{-\omega t}, \quad t \geq 0,$$

for the J defined by

$$Jf(x) = \frac{f(x)}{1+x}, \quad f \in \mathbb{B}, x \in [0, 1], \quad (6)$$

or, which is the same, that

$$\|J^{-1}e^{t(A+C-I)}J\| = 1, \quad t \geq 0. \quad (7)$$

(c) Let $T(t) := J^{-1}e^{tA}J = J^{-1}S(t)J$. A calculation shows that $T(t)f(x) = f(xe^{-t})$, $f \in \mathbb{B}, x \in [0, 1], t \geq 0$. The generator $A^\sharp$ of the semigroup $\{T(t), t \geq 0\}$ is known (see [2] p. 7) to have domain $\mathcal{D}(A^\sharp)$ composed of $f \in \mathbb{B}$ such that $f'(x)$ exists for all $x \in (0, 1]$ and $\lim_{x \rightarrow 0+} xf'(x) = 0$; for such f , $A^\sharp f(0) = 0$ and $A^\sharp f(x) = -xf'(x)$, $x \in (0, 1]$. (This, by the way, when combined with point (a) above, shows the claim on the generator of $\{S(t), t \geq 0\}$ made in Sect. 3.) It is clear from the formula for $T(t) = e^{tA^\sharp}$ that

$$\|e^{tA^\sharp}\| = 1. \quad (8)$$

(d) Next, since the operator $C^\sharp := J^{-1}CJ$ is given by

$$C^\sharp f(x) = \int_0^1 c(y)f(y) dy, \quad f \in \mathbb{B}, x \in [0, 1],$$

it is non-negative and has norm 1. Therefore, also the operators $J^{-1}e^{t(C-I)}J = e^{-t}e^{tC^\sharp}$, $t \geq 0$ are non-negative and have norms equal to 1. This together with (8) implies that the so-called stability condition is satisfied and thus the Trotter Product Formula (see e.g. [1, 2, 4, 5] or [8]; compare [3]) applies:

$$J^{-1}e^{t(A+C-I)}J = e^{t(A^\sharp+C^\sharp-I)} = \lim_{n \rightarrow \infty} \left[e^{\frac{t}{n}A^\sharp} e^{-\frac{t}{n}} e^{\frac{t}{n}C^\sharp} \right]^n$$

(strongly). This in turn forces (7), completing the proof. $\square$

The operator $A^\sharp + C^\sharp - I$ is in fact a Feller generator (see e.g. [1, 5]). In the stochastic process described by this generator, a particle starting at an $x \in [0, 1]$ travels along the trajectory $t \mapsto xe^{-t}$ for an exponential time with parameter 1 and then jumps to a randomly chosen $y \in [0, 1]$, the probability that it jumps to a Borel set $\mathcal{B} \subset [0, 1]$ equaling $\int_{\mathcal{B}} c$, to continue his travel along $t \mapsto ye^{-t}$, and so on.

We complete the paper with the promised example showing that c may be chosen so that $\|K\|$ in (5) is arbitrarily large, without affecting κ . First, let

$K^\sharp = J^{-1}KJ$ where J was defined in (6). A calculation shows that $K^\sharp f(x) = \int_0^1 (yc(y))' f(y) dy$, $x \in [0, 1]$. It follows that $\|K^\sharp\| = \int_0^1 |(yc(y))'| dy$ and thus

$$\|K\| \geq \frac{1}{2} \int_0^1 |(yc(y))'| dy$$

because $\|J\| = 1$ and $\|J^{-1}\| = 2$.

Second, given $n \in \mathbb{N}$, let c , in addition to satisfying the assumptions given at the beginning of Sect. 3.2, be symmetric about $x = \frac{1}{2}$ (note that for such c , $\kappa = \int_0^1 (1+y)c(y) dy = \frac{3}{2}$), non-decreasing in $[0, \frac{1}{2}]$, and such that $c(\frac{1}{2}) = 2n + 1$. Then,

$$\begin{aligned} \int_0^1 |(yc(y))'| dy &= \int_0^1 |c(y) + yc'(y)| dy \geq \int_0^1 |yc'(y)| dy - \int_0^1 c(y) dy \\ &= \int_0^{\frac{1}{2}} yc'(y) dy - \int_{\frac{1}{2}}^1 yc'(y) dy - 1 = c(\tfrac{1}{2}) - 1 = 2n, \end{aligned}$$

(previous-to-last equality by integration by parts) implying $\|K\| \geq n$, whereas $\kappa = \frac{3}{2}$.

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