Surjectivity, Closed Range, and Fredholmness of the Composition and Multiplication Operators Between Possibly Distinct Orlicz Spaces

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Abstract. We give criteria for the closely related concepts of surjectivity, closed range, and Fredholmness of the composition and multiplication operators between possibly different Orlicz spaces over non-atomic measure spaces.

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1. Introduction

We continue our investigation of the composition and multiplication operators between possibly different Orlicz spaces over non-atomic measure spaces. It was initiated in the papers [1, 2] (see also [16]). For results on the composition and multiplication operators between the same Orlicz spaces, see, e.g. [3, 4, 6, 8].

In this paper we consider the closely related concepts of surjectivity, closed range, and Fredholmness of these operators. Here the criterion for the continuity of these operators, based on Ishii's inclusion theorem (see [5]), again proves its usefulness. Some results on the Fredholm property of the multiplication operator between Lebesgue spaces were published in [15] and between Orlicz spaces in [6].

The criteria for surjectivity of the composition and multiplication operators given below extend and/or partly correct the criteria found in the papers [1, 4].

2. Preliminaries

We explain some of the terms used in this paper.

We will call an **Orlicz function** a function $\Phi : \mathbb{R} \rightarrow [0, \infty)$ that is convex, even, vanishes at 0 and only at 0.

The function $\Phi^* : \mathbb{R} \rightarrow [0, \infty]$ defined by

$$\Phi^*(x) := \sup\{|y|x| - \Phi(y) : y \geq 0\} \quad (x \in \mathbb{R})$$

is called the **function complementary to Φ in the sense of Young**.

An Orlicz function Φ is said to satisfy the Δ_2 -**condition at ∞** if for every $a > 1$ there are constants $K > 0$ and $x_0 > 0$ such that

$$\Phi(ax) \leq K\Phi(x) \quad (x \geq x_0).$$

If $x_0 = 0$, then the Orlicz function Φ is said to satisfy the Δ_2 -**condition globally**.

Let Φ, Ψ be Orlicz functions. Then Φ is called **stronger** than Ψ at ∞ if

$$\Psi(x) \leq \Phi(ax) \quad (x \geq x_0)$$

for some $a \geq 0$ and some $x_0 > 0$.

Let $(\Omega, \mathcal{M}, \mu)$ be a σ -finite non-atomic complete measure space and let $L^0(\Omega, \mathcal{M}, \mu)$ be the linear space of equivalence classes of $\mathcal{M}$ -measurable real-valued functions on Ω . We often write Ω instead of $(\Omega, \mathcal{M}, \mu)$, for brevity. Throughout the paper all measures are assumed to be positive.

If Φ is an Orlicz function, then the space

$$L^\Phi(\Omega) := \left\{ f \in L^0(\Omega, \mathcal{M}, \mu) : \exists \lambda > 0 \ I_\Phi(\lambda f) < \infty \right\}$$

equipped with the norm

$$\|f\|_{L^\Phi(\Omega)} := \inf \left\{ \lambda > 0 : I_\Phi\left(\frac{f}{\lambda}\right) \leq 1 \right\},$$

where $I_\Phi(f) := \int_\Omega \Phi(f(\omega)) d\mu(\omega)$, is a Banach space called **Orlicz space**. The functional I_Φ is called a modular (see [7, 9]).

If $A \in \mathcal{M}$, by $I_\Phi(f, A)$ we mean the value of the modular I_Φ at f in the space $L^\Phi(A, \mathcal{M} \cap A, \mu|_A)$.

For $f_n, f \in L^\Phi(\Omega)$ ($n \in \mathbb{N}$), we have

$$\|f_n - f\|_{L^\Phi(\Omega)} \rightarrow 0 \quad \text{iff} \quad I_\Phi(\lambda(f_n - f)) \rightarrow 0 \quad \text{for every} \quad \lambda > 0$$

(cf. [10, Theorem 1.3]).

If Φ satisfies the Δ_2 -condition globally, then F is a bounded linear functional on $L^\Phi(\Omega)$ if and only if it has the form $F(g) = \int_\Omega g(\omega) \tilde{g}(\omega) d\mu(\omega)$, for some $\tilde{g} \in L^{\Phi^*}(\Omega)$ and every $g \in L^\Phi(\Omega)$ (cf. [10, Theorem 9.3]).

Let $(\Omega, \mathcal{M}, \mu)$ and $(\Xi, \mathcal{N}, \nu)$ be general measure spaces and let $T : \Xi \rightarrow \Omega$ be a measurable transformation in the sense that the set $T^{-1}(A)$ is $\mathcal{N}$ -measurable for any $\mathcal{M}$ -measurable set A . The transformation T is said to be **nonsingular** if $\nu(T^{-1}(F)) = 0$ for any $F \in \mathcal{M}$ with $\mu(F) = 0$. This property ensures that the measure $\nu \circ T^{-1}$ defined by $\nu \circ T^{-1}(A) := \nu(T^{-1}(A))$ for any $A \in \mathcal{M}$ is **absolutely continuous** with respect to the measure μ . Under this condition, the Radon–Nikodym theorem guarantees the existence of a non-negative locally integrable function $h : \Omega \rightarrow \mathbb{R}_+$, called the **Radon–Nikodym derivative** of $\nu \circ T^{-1}$ with respect to μ , such that

$$\nu \circ T^{-1}(A) = \int_A h(\omega) d\mu(\omega) \quad (A \in \mathcal{M}).$$

Upon extending it, we may and shall assume that the Radon–Nikodym derivative h is 0 on $\Omega \setminus T(\Xi)$.

In case $T : \Xi \rightarrow \Omega$ is measurable, nonsingular, and injective, we can define the function $T^{-1} : T(\Xi) \rightarrow \Xi$. If, additionally, we assume that the function T^{-1} measurable and nonsingular, then $h(\omega) > 0$ for a.e. $\omega \in T(\Xi)$. Indeed, suppose that $h = 0$ on a $\mathcal{M}$ -measurable set $A \subset T(\Xi)$ with $\mu(A) > 0$. Then $\nu \circ T^{-1}(A) = \int_A h(\omega) d\mu(\omega) = 0$. Hence, for the $\mathcal{N}$ -measurable set $B := T^{-1}(A)$, we have $\nu(B) = 0$ and also $\mu((T^{-1})^{-1}(B)) = \mu(T(B)) = \mu(A) > 0$, which means that T^{-1} would not be nonsingular.

Instead of $T : \Xi \rightarrow \Omega$ being injective (as we will be doing all along), we might assume that $T : \Xi \rightarrow \Omega$ is essentially injective in the sense that $\nu(\{\xi \in \Xi : \text{there is } \tilde{\xi} \in \Xi \text{ such that } \xi \neq \tilde{\xi} \text{ and } T(\xi) = T(\tilde{\xi})\}) = 0$. Then a function $T^{-1} : T(\Xi) \rightarrow \Xi$ may be defined, for any $\xi \in \Xi$, by $T(\xi) \mapsto \tilde{\xi}$ with $\tilde{\xi} \in \Xi$ such that $T(\xi) = T(\tilde{\xi})$ chosen arbitrarily. Moreover, if any such function T^{-1} is measurable and nonsingular, then all of them are.

Let $(\Omega, \mathcal{M}, \mu)$ and $(\Xi, \mathcal{N}, \nu)$ be measure spaces and let $T : \Xi \rightarrow \Omega$ be a nonsingular measurable transformation. The linear transformation C_T defined by

$$(C_T f)(\xi) := f(T(\xi)) \quad (\xi \in \Xi, f \in L^0(\Omega, \mathcal{M}, \mu))$$

is called a **composition operator** induced by the transformation T .

The nonsingularity of the transformation $T : \Xi \rightarrow \Omega$ ensures that the composition operator C_T is well defined, i.e. if $f = g$ μ -a.e. on Ω then $C_T f = C_T g$ ν -a.e. on Ξ (see [14, p. 18]).

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Given a measurable function $u \in L^0_+(\Omega, \mathcal{M}, \mu)$ (with non-negative values), the linear transformation M_u defined by

$$(M_u f)(\omega) := u(\omega) f(\omega) \quad (f \in L^0(\Omega, \mathcal{M}, \mu)),$$

is called a **multiplication operator** induced by the function u .

A function $\varphi : \Omega \times \mathbb{R} \rightarrow [0, \infty]$ such that $\varphi(\omega, \cdot)$ is an Orlicz function for μ -a.e. $\omega \in \Omega$ and $\varphi(\cdot, x)$ is a $\mathcal{M}$ -measurable function for every $x \in \mathbb{R}$ is called a **Musiela–Orlicz function**.

The space $L^\varphi(\Omega, \mathcal{M}, \mu)$ of all equivalence classes of $\mathcal{M}$ -measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that

$$I_\varphi(\lambda f) := \int_\Omega \varphi(\omega, \lambda f(\omega)) d\mu(\omega) < \infty$$

for some $\lambda > 0$, equipped with the Luxemburg norm $\|f\|_\varphi$ defined analogously as for the Orlicz space, is called a **Musiela–Orlicz space** generated by the Musiela–Orlicz function φ . The Musiela–Orlicz space is a Banach space (cf. [11, 12]).

We are only interested in a special class of Musiela–Orlicz spaces. Firstly, the **Orlicz weighted space** $L_h^\Psi(\Omega)$ is the Musiela–Orlicz space $L^\psi(\Omega)$ generated by the Musiela–Orlicz function $\psi(\omega, x) := \Psi(x)h(\omega)$ for $\omega \in \Omega$ and $x \in \mathbb{R}$, where $h \in L_+^0(\Omega)$ is the Radon–Nikodym derivative related to the transformation T inducing the composition operator C_T .

Secondly, the **Orlicz weighted space** $L^{\Psi_u}(\Omega)$ is the Musiela–Orlicz space $L^\psi(\Omega)$ generated by the Musiela–Orlicz function $\psi(\omega, x) := \Psi_u(\omega, x) := \Psi(u(\omega)x)$ for $\omega \in \Omega$ and $x \in \mathbb{R}$, where $u \in L_+^0(\Omega)$ is the function inducing the multiplication operator M_u .

The consideration of the composition and multiplication operators between Orlicz spaces rather naturally leads to Orlicz weighted spaces.

Recall that a continuous linear operator between Banach spaces is said to be **Fredholm** if the codimension of its range and the dimension of its kernel are finite and the range is closed. It is a consequence of the Open Mapping Theorem that, actually, finite codimension of the range of a continuous linear operator between Banach spaces already implies that its range is closed (cf. [13, p. 267]).

We will take advantage of the following two theorems proved elsewhere. For the following theorem on the composition operator, see Theorem 2.1 and part of the proof of Theorem 2.2 in [1] (when the Radon–Nikodym derivative h is assumed to vanish outside the image of T , as we do here, the sufficient conditions in [1, Theorem 2.1] are also necessary).

Theorem 1. *Let $T : \Xi \rightarrow \Omega$ be a nonsingular measurable transformation. The composition operator C_T acts continuously from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$ if and only if one of the following equivalent conditions is satisfied.*

- (1) $L^\Phi(\Omega) \subset L_h^\Psi(\Omega)$.
- (2) There are $K > 0$ and $g \in L_+^1(\Omega)$ such that $\Psi(x)h(\omega) \leq \Phi(Kx) + g(\omega)$ for every $x \geq 0$ and a.e. $\omega \in \Omega$.
- (3) The function $\sup_{x \geq 0} \{\Psi(x)h(\omega) - \Phi(Kx)\}$ is integrable over Ω for some $K > 0$.

For the following corresponding theorem on the multiplication operator, see the proof of Theorem 3.1 in [1].

Theorem 2. *Let $u : \Omega \rightarrow \mathbb{R}_+$ be a measurable function. The multiplication operator M_u acts continuously from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$ if and only if one of the following equivalent conditions is satisfied.*

- (1) $L^\Phi(\Omega) \subset L^{\Psi_u}(\Omega)$.
- (2) There are $K > 0$ and $g \in L^1_+(\Omega)$ such that $\Psi(u(\omega)x) \leq \Phi(Kx) + g(\omega)$ for every $x \geq 0$ and a.e. $\omega \in \Omega$.
- (3) The function $\sup_{x \geq 0} \{\Psi(u(\omega)x) - \Phi(Kx)\}$ is integrable over Ω for some $K > 0$.

3. Surjectivity

3.1. Surjectivity of C_T

We would like to give necessary and sufficient conditions for a composition operator to be surjective. The following theorem extends Theorem 4.1. from [4], and Theorems 2.4. and 2.5. from [1], as well as corrects their points (2), (ii), and (2), respectively. More precisely, the surjectivity of the operator C_T is in fact not equivalent to $L^\Phi(\Omega) = L^\Psi_h(\Omega)$, as it was claimed in [1, Fact 2.1] and used in the proof of Theorems 2.4 (ii) and 2.5 (2) in [1].

Theorem 3. *Let $T : \Xi \rightarrow \Omega$ be a nonsingular measurable transformation which is injective and such that the mapping $T^{-1} : T(\xi) \mapsto \xi$ is measurable and nonsingular. Then a continuous composition operator from $C_T : L^\Phi(\Omega)$ to $L^\Psi(\Xi)$ is surjective if and only if one of the following equivalent conditions is satisfied.*

- (1) $L^\Psi(\Xi) \subset L^{\Phi_{\frac{1}{h \circ T}}}(\Xi)$.
- (2) There is $K > 0$ and $g \in L^1_+(\Xi)$ such that

$$\Phi(x) \frac{1}{h \circ T(\xi)} \leq \Psi(Kx) + g(\xi)$$

for all $x \geq 0$ and a.e. $\xi \in \Xi$.

- (3) There is $K > 0$ for which the function

$$\sup_{x \geq 0} \left\{ \Phi(x) \frac{1}{h \circ T(\xi)} - \Psi(Kx) \right\}$$

is integrable over Ξ .

Proof. A continuous composition operator C_T from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$ is surjective if and only if for every $g \in L^\Psi(\Xi)$ there is $f \in L^\Phi(\Omega)$ such that $C_T f = g$, i.e. $f(T(\xi)) = g(\xi)$ for a.e. $\xi \in \Xi$. Since T is injective and T^{-1} is nonsingular, we have $f(\omega) = g(T^{-1}(\omega))$ for a.e. $\omega \in T(\Xi)$, i.e. $f = C_{T^{-1}} g$. Therefore C_T is surjective if and only if $C_{T^{-1}}$ acts continuously from $L^\Psi(\Xi)$ to $L^\Phi(\Omega)$, supposing $C_{T^{-1}}$ is well defined. But the mapping $T^{-1} : T(\Xi) \rightarrow \Xi$ is measurable

and nonsingular, hence $C_{T^{-1}}$ is indeed well defined. By the Radon–Nikodym theorem, there is a locally integrable function $\tilde{h} : \Xi \rightarrow R_+$ such that for any measurable set $A \subset \Xi$,

$$\mu \circ T(A) = \int_A \tilde{h}(\xi) d\nu(\xi).$$

In fact, $\tilde{h} = \frac{1}{h \circ T}$ a.e. on Ξ because

$$\begin{aligned} \int_A \frac{1}{h(T(\xi))} d\nu(\xi) &= \int_{T(A)} \frac{1}{h(\omega)} d\nu \circ T^{-1}(\omega) = \int_{T(A)} \frac{1}{h(\omega)} h(\omega) d\mu(\omega) \\ &= \mu(T(A)) = \int_A \tilde{h}(\xi) d\nu(\xi), \end{aligned}$$

for any measurable $A \subset \Xi$. Now by Theorem 1 (substituting $T^{-1} : T(\Xi) \rightarrow \Xi$ for $T : \Xi \rightarrow \Omega$ and $\frac{1}{h \circ T}$ for h) we infer that C_T is surjective if and only if $L^\Psi(\Xi) \subset L^{\Phi, \frac{1}{h \circ T}}(\Xi)$. The same theorem yields the other two equivalent conditions for the surjectivity of C_T .

Remark 1. It is possible to prove that if a continuous nonzero composition operator $C_T : L^\Phi(\Omega) \rightarrow L^\Psi(\Xi)$ is surjective, then the functions Φ and Ψ are equivalent at ∞ ; more precisely, there are constants $K, L, a, x_0 > 0$ such that $\Psi(Kx) \leq a\Phi(x) \leq \Psi(Lx)$ for all $x \geq x_0$.

In the special case when $\Psi = \Phi$, a much simpler condition for the surjectivity of a continuous composition operator can be derived.

Corollary 1. *Let $T : \Xi \rightarrow \Omega$ be a nonsingular measurable transformation which is injective and such that the mapping $T^{-1} : T(\xi) \mapsto \xi$ is measurable and nonsingular. Then a continuous composition operator $C_T : L^\Phi(\Omega) \rightarrow L^\Phi(\Xi)$ is surjective if there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in T(\Xi)$. Moreover, assuming that Φ satisfies the Δ_2 -condition at ∞ , if a continuous composition operator $C_T : L^\Phi(\Omega) \rightarrow L^\Phi(\Xi)$ is surjective, then this condition is also necessary.*

Proof. Sufficiency. Let $h(\omega) \geq \varepsilon$ for a.e. $\omega \in T(\Xi)$ and some $\varepsilon > 0$. We may assume that $\varepsilon \leq 1$. By the convexity of Φ and the fact that $\Phi(0) = 0$, we have

$$\Phi(x) \frac{1}{h \circ T(\xi)} \leq \Phi(x) \frac{1}{\varepsilon} \leq \Phi\left(\frac{1}{\varepsilon}x\right)$$

for all $x \geq 0$ and a.e. $\xi \in \Xi$. Hence, by the preceding theorem (with $K = \frac{1}{\varepsilon}$ and $g = 0$), the operator C_T is surjective.

Necessity. Let $C_T : L^\Phi(\Omega) \rightarrow L^\Phi(\Xi)$ be surjective and let $K > 0$ be arbitrary. Since Φ satisfies the Δ_2 -condition at ∞ , there is $a > 1$ and $x_0 \geq 0$ such that $\Phi(Kx) \leq a\Phi(x)$ for $x \geq x_0$. Suppose that the set $\{\xi \in \Xi : h \circ T(\xi) < \varepsilon\}$ has positive measure, for every $\varepsilon > 0$. Let $\varepsilon_0 := \frac{1}{2a}$ and set $A := \{\xi \in \Xi : h \circ T(\xi) < \varepsilon_0\}$. Notice that

$$\Phi(x) \frac{1}{h \circ T(\xi)} - \Phi(Kx) > \Phi(x) \frac{1}{\varepsilon_0} - \Phi(Kx) \geq 2a\Phi(x) - a\Phi(x) = a\Phi(x)$$

for $x \geq x_0$ and $\xi \in A$. Therefore, for every $K > 0$,

$$\int_{\Xi} \sup_{x \geq 0} \left\{ \Phi(x) \frac{1}{h \circ T(\xi)} - \Phi(Kx) \right\} d\nu(\xi) \geq \int_A \sup_{x \geq x_0} a\Phi(x) d\nu(\xi) = \infty,$$

which, by Theorem 3, implies that C_T is not surjective. This contradiction shows that there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in T(\Xi)$. $\square$

Proceeding in exactly the same manner one can prove that if there is $M > 0$ such that $h(\omega) \leq M$ for a.e. $\omega \in T(\Xi)$ then C_T is continuous from $L^\Phi(\Omega)$ to $L^\Phi(\Xi)$, and that if Φ satisfies the Δ_2 -condition at ∞ , then the same condition is also necessary for the continuity of $C_T : L^\Phi(\Omega) \rightarrow L^\Phi(\Xi)$. Hence the last corollary may be written in a more specific way.

Corollary 2. *Let $T : \Xi \rightarrow \Omega$ be a nonsingular measurable transformation which is injective and such that the mapping $T^{-1} : T(\xi) \mapsto \xi$ is measurable and nonsingular. Then the composition operator $C_T : L^\Phi(\Omega) \rightarrow L^\Phi(\Xi)$ is continuous and surjective if there are $\varepsilon, M > 0$ such that $\varepsilon \leq h(\omega) \leq M$ for a.e. $\omega \in T(\Xi)$. Moreover, assuming that Φ satisfies the Δ_2 -condition at ∞ , if the composition operator C_T is continuous from $L^\Phi(\Omega)$ to $L^\Phi(\Xi)$ and surjective, then this condition is also necessary.*

3.2. Surjectivity of M_u

We proceed to the surjectivity of the multiplication operator. The following theorem corrects point (ii) of Theorem 3.2. from [1]. The surjectivity of M_u is a different property from the equality $L^\Phi(\Omega) = L^{\Psi_u}(\Omega)$, as was claimed there.

Theorem 4. *Let $u : \Omega \rightarrow \mathbb{R}_+$ be a measurable function such that $u(\omega) > 0$ for a.e. $\omega \in \Omega$. Then a continuous multiplication operator from $M_u : L^\Phi(\Omega)$ to $L^\Psi(\Omega)$ is surjective if and only if one of the following equivalent conditions is satisfied.*

- (1) $L^\Psi(\Omega) \subset L^{\Phi \frac{1}{u}}(\Omega)$.
- (2) There is $K > 0$ and $g \in L^1_+(\Omega)$ such that

$$\Phi\left(\frac{x}{u(\omega)}\right) \leq \Psi(Kx) + g(\omega)$$

for all $x \geq 0$ and a.e. $\omega \in \Omega$.

- (3) There is $K > 0$ for which the function

$$\sup_{x \geq 0} \left\{ \Phi\left(\frac{x}{u(\omega)}\right) - \Psi(Kx) \right\}$$

is integrable over Ω .

Proof. A continuous multiplication operator M_u from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$ is surjective if and only if for every $g \in L^\Psi(\Omega)$ there is $f \in L^\Phi(\Omega)$ such that $M_u f = g$. But, since $u(\omega) > 0$ for a.e. $\omega \in \Omega$, we may write, equivalently,

$f = M_{\frac{1}{u}}g$, which means that a continuous multiplication operator M_u from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$ is surjective if and only if $M_{\frac{1}{u}}$ acts continuously from $L^\Psi(\Omega)$ to $L^\Phi(\Omega)$.

Now by Theorem 2 (substituting $\frac{1}{u}$ for u) we infer that M_u is surjective if and only if $L^\Psi(\Omega) \subset L^{\Phi \frac{1}{u}}(\Omega)$, and the same theorem also yields the other two equivalent conditions for the surjectivity of M_u , which finishes the proof.

Remark 2. Analogously to Remark 1, the surjective continuous multiplication operator $M_u : L^\Phi(\Omega) \rightarrow L^\Psi(\Omega)$ implies the equivalence of the functions Φ and Ψ at ∞ .

A simpler condition for the surjectivity of a continuous multiplication operator can be obtained in the case when $\Psi = \Phi$.

Corollary 3. *Let $u : \Omega \rightarrow \mathbb{R}_+$ be a measurable function such that $u(\omega) > 0$ for a.e. $\omega \in \Omega$. Then a continuous multiplication operator $M_u : L^\Phi(\Omega) \rightarrow L^\Phi(\Omega)$ is surjective if there is $\varepsilon > 0$ such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$. Moreover, assuming that Φ satisfies the Δ_2 -condition at ∞ , if a continuous multiplication operator $M_u : L^\Phi(\Omega) \rightarrow L^\Phi(\Omega)$ is surjective, then there is $\varepsilon > 0$ such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$.*

We omit the proof of this corollary as it is analogous to the proof of Corollary 1.

4. Closed Range

We have just found necessary and sufficient conditions for the surjectivity of a continuous composition operator from $L^\Phi(\Omega)$ to $L^\Phi(\Xi)$ and a continuous multiplication operator from $L^\Phi(\Omega)$ to $L^\Phi(\Omega)$. It turns out that basically the same conditions are necessary and sufficient for the range of these operators to be closed when the generating functions are possibly different. As is to be expected, they look somewhat stronger than the conditions we obtained for the surjectivity of $C_T : L^\Phi(\Omega) \rightarrow L^\Psi(\Xi)$ and $M_u : L^\Phi(\Omega) \rightarrow L^\Psi(\Omega)$.

4.1. C_T with Closed Range

The sufficient condition for closedness of the range of a continuous composition operator C_T we give below requires that C_T acts between Orlicz spaces over the same measure space.

Theorem 5. *Suppose that $T : \Omega \rightarrow \Omega$ is an injective, nonsingular measurable transformation, the composition operator C_T is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$, and $L^\Phi(T(\Omega)) \cap L^\Psi(T(\Omega))$ is a closed subset of $L^\Psi(T(\Omega))$. If there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in T(\Omega)$, then the range of C_T is closed.*

Proof. Suppose that C_T is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$ and there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in T(\Omega)$ but the range of C_T is *not* closed. Then there is a sequence $(f_n)_{n=1}^\infty \subset L^\Phi(\Omega)$ and a function $g \in L^\Psi(\Omega)$ such that

$$(C_T f_n)_{n=1}^\infty \subset \mathcal{R}(C_T), \quad C_T f_n \rightarrow g \text{ in } L^\Psi(\Omega) \text{ as } n \rightarrow \infty \quad \text{but} \quad g \notin \mathcal{R}(C_T).$$

The fact that $g \notin \mathcal{R}(C_T)$ implies that there is no $f \in L^\Phi(\Omega)$ for which $C_T f = g$, or, equivalently, $f(T(\omega)) = g(\omega)$ ($\omega \in \Omega$). Hence there is no $f \in L^\Phi(\Omega)$ such that $f(\omega) = g(T^{-1}(\omega))$ ($\omega \in T(\Omega)$) since, by assumption, T is injective. In other words, $g \notin \mathcal{R}(C_T)$ is equivalent to $g \circ T^{-1} \notin L^\Phi(T(\Omega))$.

Since C_T is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$, Theorem 1 implies that there are $K > 0$ and $\tilde{g} \in L_+^1(T(\Omega))$ such that

$$\Psi(x)h(\omega) \leq \Phi(Kx) + \tilde{g}(\omega)$$

for all $x \geq 0$ and a.e. $\omega \in T(\Omega)$. Now the condition $h(\omega) \geq \varepsilon > 0$ for a.e. $\omega \in T(\Omega)$ yields that

$$\varepsilon \Psi(x) \leq \Phi(Kx) + \tilde{g}(\omega)$$

for all $x \geq 0$ and a.e. $\omega \in T(\Omega)$. Thus, if $f \in L^\Phi(T(\Omega))$, then for $\lambda > 0$ such that $I_\Phi(\lambda f, T(\Omega)) < \infty$ we have

$$\varepsilon I_\Psi\left(\frac{\lambda}{K}f, T(\Omega)\right) \leq I_\Phi(\lambda f, T(\Omega)) + \int_{T(\Omega)} \tilde{g}(\omega) d\mu(\omega) < \infty,$$

i.e. $f \in L^\Psi(T(\Omega))$. Hence $L^\Phi(T(\Omega)) \subset L^\Psi(T(\Omega))$. In particular, $(f_n)_{n=1}^\infty \subset L^\Phi(\Omega) \subset L^\Phi(T(\Omega)) \subset L^\Psi(T(\Omega))$. Further, since we assume that $L^\Phi(T(\Omega)) \cap L^\Psi(T(\Omega))$ is a closed subset of $L^\Psi(T(\Omega))$, we conclude that $L^\Phi(T(\Omega))$ is a closed subspace of $L^\Psi(T(\Omega))$. Consequently, there is no sequence in $L^\Phi(T(\Omega))$ that converges to $g \circ T^{-1} \notin L^\Phi(T(\Omega))$ in $L^\Psi(T(\Omega))$. However, applying the fact that $C_T f_n \rightarrow g$ in $L^\Psi(\Omega)$, which is equivalent to $I_\Psi(\lambda(f_n \circ T - g), \Omega) \rightarrow 0$ for every $\lambda > 0$, and the fact that $h(\omega) \geq \varepsilon > 0$ for a.e. $\omega \in T(\Omega)$, we obtain

$$\begin{aligned} \varepsilon I_\Psi\left(\lambda(f_n - g \circ T^{-1}), T(\Omega)\right) &= \varepsilon \int_{T(\Omega)} \Psi\left(\lambda(f_n(\omega) - g \circ T^{-1}(\omega))\right) d\mu(\omega) \\ &\leq \int_{T(\Omega)} \Psi\left(\lambda(f_n(\omega) - g \circ T^{-1}(\omega))\right) h(\omega) d\mu(\omega) \\ &= \int_{T(\Omega)} \Psi\left(\lambda(f_n(\omega) - g \circ T^{-1}(\omega))\right) d\mu \circ T^{-1}(\omega) \\ &= \int_{\Omega} \Psi\left(\lambda(f_n \circ T(\omega) - g(\omega))\right) d\mu(\omega) \\ &= I_\Psi\left(\lambda(f_n \circ T - g), \Omega\right) \rightarrow 0, \end{aligned}$$

for every $\lambda > 0$. Therefore $f_n \rightarrow g \circ T^{-1}$ in $L^\Psi(T(\Omega))$. This contradiction finishes the proof. $\square$

The following theorem gives a necessary condition for the closedness of the range of a continuous composition operator C_T from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$. (Note that here measure spaces may be different.)

Theorem 6. *Suppose that $T : \Xi \rightarrow \Omega$ is a nonsingular measurable transformation such that $h(\omega) > 0$ for a.e. $\omega \in T(\Xi)$, and the composition operator C_T is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$. If the range of C_T is closed and Ψ satisfies the Δ_2 -condition at ∞ , then there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in T(\Xi)$.*

Proof. Let C_T be continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$ and suppose that for every $\varepsilon > 0$ we have $\mu(\{\omega \in T(\Xi) : h(\omega) < \varepsilon\}) > 0$.

We will show that there is a sequence $(g_n)_{n=1}^\infty \subset \mathcal{R}(C_T) \subset L^\Psi(\Xi)$ and a function $g \in L^\Psi(\Xi)$ such that $g_n \rightarrow g$ in $L^\Psi(\Xi)$ and $g \notin \mathcal{R}(C_T)$. This will imply that the range of C_T is not closed, proving the theorem.

For each $k \in \mathbb{N}$ define the set

$$A_k := \left\{ \omega \in T(\Xi) : \frac{1}{(k+1)^2} \leq h(\omega) < \frac{1}{k^2} \right\}.$$

By our assumptions, infinitely many sets A_k have nonzero measure and, upon taking a subsequence, one may and shall assume that the sequence $(\mu(A_k))_k$ is decreasing to 0.

For each $n \in \mathbb{N}$, define the function

$$f_n := \sum_{k=1}^n \Psi^{-1} \left(\frac{\chi_{A_k}}{\mu(A_k)} \right) \in L^\Phi(\Omega).$$

Set $g_n := C_T f_n \in L^\Psi(\Xi)$, for each $n \in \mathbb{N}$, and

$$g := \sum_{k=1}^{\infty} \Psi^{-1} \left(\frac{\chi_{A_k} \circ T}{\mu(A_k)} \right).$$

Note that, since T is injective, the function

$$f = \sum_{k=1}^{\infty} \Psi^{-1} \left(\frac{\chi_{A_k}}{\mu(A_k)} \right)$$

is the only function measurable on Ω such that $g = C_T f$. Thus the theorem will be proved once we have shown that $g \in L^\Psi(\Xi)$ but $f \notin L^\Phi(\Omega)$.

Let $\lambda \geq 1$. Since Ψ satisfies the Δ_2 -condition at ∞ , there are $b, x_0 > 0$ such that $\Psi(\lambda x) \leq b\Psi(x)$ for all $x \geq x_0$. Then for every $n \geq n_0$, where n_0 is the smallest positive integer such that $\Psi^{-1} \left(\frac{1}{\mu(A_{n_0})} \right) \geq x_0$, we have

$$\begin{aligned}
I_\Psi(\lambda(g - g_n)) &= \sum_{k=n+1}^{\infty} \int_{\Xi} \Psi \left(\lambda \Psi^{-1} \left(\frac{\chi_{A_k} \circ T(\xi)}{\mu(A_k)} \right) \right) d\nu(\xi) \\
&= \sum_{k=n+1}^{\infty} \int_{T(\Xi)} \Psi \left(\lambda \Psi^{-1} \left(\frac{\chi_{A_k}(\omega)}{\mu(A_k)} \right) \right) h(\omega) d\mu(\omega) \\
&\leq \sum_{k=n+1}^{\infty} \int_{T(\Xi)} b \frac{\chi_{A_k}(\omega)}{\mu(A_k)} h(\omega) d\mu(\omega) \\
&\leq b \sum_{k=n+1}^{\infty} \frac{1}{k^2}.
\end{aligned}$$

Hence $I_\Psi(\lambda(g - g_n)) \rightarrow 0$. Since the function $x \mapsto \Psi(x)$ is nondecreasing for $x > 0$, the above convergence holds for all $\lambda > 0$. By [10, Theorem 1.3] referred to in the Preliminaries, this implies that $g \in L^\Psi(\Xi)$ and $g_n \rightarrow g$ in $L^\Psi(\Xi)$. But the continuity of C_T from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$ easily implies that there are $a, x_1 \geq 0$ such that $\Psi(x) \leq \Phi(ax)$ for $x \geq x_1$ (i.e. Ψ is stronger than Φ at ∞). Let $\lambda \in (0, 1)$. Since Ψ satisfies the Δ_2 -condition at ∞ , there are $b, x_2 > 0$ such that $\Psi(x) \leq \frac{1}{b}\Psi(\lambda x)$ for all $x \geq x_2$. Therefore, for $N \in \mathbb{N}$ so large that $\lambda \Psi^{-1}\left(\frac{1}{\mu(A_N)}\right) \geq x_1$ and $\Psi^{-1}\left(\frac{1}{\mu(A_N)}\right) \geq x_2$, we have

$$\begin{aligned}
I_\Phi(\lambda a f) &= \sum_{k=1}^{\infty} \int_{\Omega} \Phi \left(a \lambda \Psi^{-1} \left(\frac{\chi_{A_k}(\omega)}{\mu(A_k)} \right) \right) d\mu(\omega) \\
&\geq \sum_{k=N}^{\infty} \int_{\Omega} \Psi \left(\lambda \Psi^{-1} \left(\frac{\chi_{A_k}(\omega)}{\mu(A_k)} \right) \right) d\mu(\omega) \\
&\geq \sum_{k=N}^{\infty} \int_{\Omega} b \frac{\chi_{A_k}(\omega)}{\mu(A_k)} d\mu(\omega) = b \sum_{k=N}^{\infty} 1 = \infty,
\end{aligned}$$

and the proof is complete. $\square$

Combining Theorem 6 and Corollary 1, we obtain the following corollary.

Corollary 4. *Suppose that $T : \Xi \rightarrow \Omega$ is an injective, nonsingular measurable transformation, T^{-1} is measurable and nonsingular, Φ satisfies the Δ_2 -condition at ∞ , and the composition operator C_T is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$. Then C_T is surjective if and only if C_T has closed range.*

4.2. M_u with Closed Range

Now we provide a necessary and sufficient condition for a continuous multiplication operator M_u from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$ to have closed range.

Theorem 7. *Suppose that $u : \Omega \rightarrow \mathbb{R}$ is a measurable function, the multiplication operator M_u is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$, and $L^\Phi(\Omega) \cap L^\Psi(\Omega)$ is a closed subset of $L^\Psi(\Omega)$. If there is $\varepsilon > 0$ such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$, then the range of M_u is closed.*

Proof. Assume that M_u is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$, and let $\varepsilon > 0$ be such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$ but suppose that the range of M_u is *not* closed. Then there is a sequence $(f_n)_{n=1}^\infty \subset L^\Phi(\Omega)$ and a function $g \in L^\Psi(\Omega)$ such that

$$(M_u f_n)_{n=1}^\infty \subset \mathcal{R}(M_u), \quad M_u f_n \rightarrow g \text{ in } L^\Psi(\Omega) \text{ as } n \rightarrow \infty, \quad \text{but } g \notin \mathcal{R}(M_u).$$

The fact that $g \notin \mathcal{R}(M_u)$ means that there is no $f \in L^\Phi(\Omega)$ for which $M_u f = g$, i.e. $f = \frac{g}{u}$. Hence $g \notin \mathcal{R}(M_u)$ is equivalent to $\frac{g}{u} \notin L^\Phi(\Omega)$.

Since M_u is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$, Theorem 2 implies that there are $K > 0$ and $\tilde{g} \in L^1_+(\Omega)$ such that

$$\Psi(u(\omega)x) \leq \Phi(Kx) + \tilde{g}(\omega)$$

for all $x \geq 0$ and a.e. $\omega \in \Omega$. The assumption that $u(\omega) \geq \varepsilon > 0$ for a.e. $\omega \in \Omega$ yields

$$\Psi(\varepsilon x) \leq \Phi(Kx) + \tilde{g}(\omega)$$

for all $x \geq 0$ and a.e. $\omega \in \Omega$. Thus, if $f \in L^\Phi(\Omega)$, then for $\lambda > 0$ such that $I_\Phi(\lambda f) < \infty$ we have

$$I_\Psi\left(\frac{\lambda\varepsilon}{K}f\right) \leq I_\Phi(\lambda f) + \int_\Omega \tilde{g}(\omega) d\mu(\omega) < \infty,$$

hence $f \in L^\Psi(\Omega)$. Therefore $L^\Phi(\Omega) \subset L^\Psi(\Omega)$. In particular, $(f_n)_{n=0}^\infty \subset L^\Psi(\Omega)$. Since $L^\Phi(\Omega) \cap L^\Psi(\Omega)$ is a closed subset of $L^\Psi(\Omega)$, we conclude that $L^\Phi(\Omega)$ is a closed subspace of $L^\Psi(\Omega)$. Consequently, no sequence in $L^\Phi(\Omega)$ converges to $\frac{g}{u} \notin L^\Phi(\Omega)$. However, applying the fact that $M_u f_n \rightarrow g$ in $L^\Psi(\Omega)$, which is equivalent to $I_\Psi(\lambda(u f_n - g)) \rightarrow 0$ for every $\lambda > 0$, and the fact that $u(\omega) \geq \varepsilon > 0$ for a.e. $\omega \in \Omega$, we obtain

$$I_\Psi\left(\lambda\varepsilon\left(f_n - \frac{g}{u}\right)\right) \leq I_\Psi\left(\lambda u\left(f_n - \frac{g}{u}\right)\right) \leq I_\Psi(\lambda(u f_n - g)) \rightarrow 0$$

for every $\lambda > 0$. Therefore $f_n \rightarrow \frac{g}{u}$ in $L^\Psi(\Omega)$, and this contradiction finishes the proof. $\square$

Theorem 8. Suppose that $u(\omega) > 0$ for a.e. $\omega \in \Omega$ and the multiplication operator M_u is continuous from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$. Assume, additionally, Ψ satisfies the Δ_2 -condition at ∞ . If the range of M_u is closed, then there is $\varepsilon > 0$ such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$.

Proof. Suppose that $\mu(\{\omega \in \Omega : u(\omega) < \varepsilon\}) > 0$ for every $\varepsilon > 0$. We claim that there is a sequence $(g_n)_{n=1}^\infty \subset \mathcal{R}(M_u) \subset L^\Psi(\Omega)$ and a function $g \in L^\Psi(\Omega)$ such that $g_n \rightarrow g$ in $L^\Psi(\Omega)$ and $g \notin \mathcal{R}(M_u)$. This will imply that the range of M_u is *not* closed, proving the necessity of the condition $u(\omega) \geq \varepsilon > 0$ for a.e. $\omega \in \Omega$.

For each $k \in \mathbb{N}$ define the set

$$A_k := \left\{ \omega \in \Omega : \frac{1}{(k+1)^2} \leq u(\omega) < \frac{1}{k^2} \right\}.$$

As in the proof of Theorem 6, upon taking a subsequence, we may and shall assume that the sequence $(\mu(A_k)_k)$ is decreasing to 0.

For each $n \in \mathbb{N}$, define the function

$$f_n := \sum_{k=1}^n \Psi^{-1}\left(\frac{1}{\mu(A_k)}\right) \chi_{A_k} \in L^\Phi(\Omega).$$

Let $g_n := M_u f_n \in L^\Psi(\Omega)$ for each $n \in \mathbb{N}$ and

$$g := \sum_{k=1}^{\infty} u \Psi^{-1}\left(\frac{1}{\mu(A_k)}\right) \chi_{A_k}.$$

Since $u > 0$, the function

$$f = \sum_{k=1}^{\infty} \Psi^{-1}\left(\frac{1}{\mu(A_k)}\right) \chi_{A_k}$$

is the only function measurable on Ω such that $g = M_u f$. We will show that $g \in L^\Psi(\Omega)$ but $f \notin L^\Phi(\Omega)$.

For every $\lambda > 0$ and all $n \geq n_0$, where n_0 is the smallest positive integer satisfying $\frac{\lambda}{n_0^2} \leq 1$, we have

$$\begin{aligned} I_\Psi(\lambda(g - g_n)) &= \sum_{k=n+1}^{\infty} \int_{\Omega} \Psi\left(\lambda u(\omega) \Psi^{-1}\left(\frac{1}{\mu(A_k)}\right)\right) \chi_{A_k}(\omega) d\mu(\omega) \\ &\leq \sum_{k=n+1}^{\infty} \Psi\left(\frac{\lambda}{k^2} \Psi^{-1}\left(\frac{1}{\mu(A_k)}\right)\right) \mu(A_k) \\ &\leq \lambda \sum_{k=n+1}^{\infty} \frac{1}{k^2}. \end{aligned}$$

Hence $I_\Psi(\lambda(g - g_n)) \rightarrow 0$ for every $\lambda > 0$. By [10, Theorem 1.3] cited in the Preliminaries, this implies that $g \in L^\Psi(\Omega)$ and $g_n \rightarrow g$ in $L^\Psi(\Omega)$. But the continuity of M_u from $L^\Phi(\Omega)$ to $L^\Psi(\Omega)$ implies there is $a > 0$ and $x_0 \geq 0$ such that $\Psi(x) \leq \Phi(ax)$ for $x \geq x_0$. Let $\lambda \in (0, 1)$. Since Ψ satisfies the Δ_2 -condition at ∞ , there are $b, x_1 > 0$ such that $\Psi(x) \leq \frac{1}{b} \Psi(\lambda x)$ for all $x \geq x_1$. Therefore, for $N \in \mathbb{N}$ so large that $\lambda \Psi^{-1}\left(\frac{1}{\mu(A_N)}\right) \geq x_0$ and $\Psi^{-1}\left(\frac{1}{\mu(A_N)}\right) \geq x_1$, we have

$$\begin{aligned}
I_\Phi(\lambda a f) &= \sum_{k=1}^{\infty} \int_{\Omega} \Phi \left(a \lambda \Psi^{-1} \left(\frac{1}{\mu(A_k)} \right) \right) \chi_{A_k}(\omega) d\mu(\omega) \\
&\geq \sum_{k=N}^{\infty} \Psi \left(\lambda \Psi^{-1} \left(\frac{1}{\mu(A_k)} \right) \right) \mu(A_k) \\
&\geq b \sum_{k=N}^{\infty} 1 = \infty,
\end{aligned}$$

which completes the proof of the theorem. $\square$

From Theorem 8 and Corollary 3 we derive the following corollary.

Corollary 5. *Suppose that $u(\omega) > 0$ for a.e. $\omega \in \Omega$, Φ satisfies the Δ_2 -condition at ∞ , and the multiplication operator M_u is continuous from $L^\Phi(\Omega)$ to $L^\Phi(\Omega)$. Then M_u is surjective if and only if M_u has closed range.*

5. Fredholm Property

Finally, we present, partially based on the preceding results, necessary and sufficient conditions for a continuous composition and a multiplication operator to be Fredholm. The conditions will be reasonably simple if in the respective theorems we limit ourselves to the case $\Psi = \Phi$.

5.1. Fredholmness of C_T

In the following lemma we show that a finite codimension of the range of C_T implies the surjectivity of this operator.

Lemma 1. *Let Ψ satisfy the Δ_2 -condition globally, and let the composition operator C_T act continuously from $L^\Phi(\Omega)$ to $L^\Psi(\Xi)$. Suppose, further, that for any measurable set $B \subset \Xi$ of positive measure, there is a measurable set $A \subset T(\Xi)$ such that $T^{-1}(A) \subset B$ and $\nu(T^{-1}(A)) > 0$. If the codimension of the range of C_T is finite, then C_T is surjective.*

Proof. From $\text{codim } \mathcal{R}(C_T) < \infty$ it follows that $\mathcal{R}(C_T)$ is closed. Suppose that C_T is not surjective. Then there is a function $g \in L^\Psi(\Xi)$ which does not belong to $\mathcal{R}(C_T)$. Since Ψ satisfies the Δ_2 -condition globally and $\mathcal{R}(C_T)$ is a closed subspace of $L^\Psi(\Xi)$, there is $\tilde{g} \in L^{\Psi^*}(\Xi)$ such that

$$\langle g, \tilde{g} \rangle = \int_{\Xi} g(\xi) \tilde{g}(\xi) d\nu(\xi) = 1 \quad (1)$$

and

$$\langle C_T f, \tilde{g} \rangle = \int_{\Xi} f(T(\xi)) \tilde{g}(\xi) d\nu(\xi) = 0 \quad \text{for every } f \in L^\Phi(\Omega). \quad (2)$$

In virtue of (1), there is $\varepsilon > 0$ such that the set $B := \{\xi \in \Xi : g(\xi) \tilde{g}(\xi) > \varepsilon\}$ has positive measure. Note that $\tilde{g}$ is nonzero on B . Since Ξ is non-atomic, a

sequence $(B_n)_{n=1}^\infty$ of pairwise disjoint subsets of B can be found such that $0 < \nu(B_n) < \infty$ for $n \in \mathbb{N}$. By assumption, for each $n \in \mathbb{N}$, there is a measurable subset $A_n \subset T(\Xi)$ such that $T^{-1}(A_n) \subset B_n$ and $T^{-1}(A_n)$ has positive measure. Define the functions $\tilde{g}_n := \tilde{g}\chi_{T^{-1}(A_n)}$, for $n \in \mathbb{N}$; they are all nonzero, linearly independent, and belong to $L^{\Psi^*}(\Xi)$. Now, by (2), for every $f \in L^\Phi(\Omega)$ and each $n \in \mathbb{N}$, we have

$$\begin{aligned}\langle f, C_T^* \tilde{g}_n \rangle &= \langle C_T f, \tilde{g}_n \rangle = \langle C_T f, \tilde{g} \chi_{T^{-1}(A_n)} \rangle = \langle \chi_{T^{-1}(A_n)} C_T f, \tilde{g} \rangle \\ &= \langle (\chi_{A_n} \circ T)(f \circ T), \tilde{g} \rangle = \langle C_T(f \chi_{A_n}), \tilde{g} \rangle = 0.\end{aligned}$$

Therefore, each $\tilde{g}_n$ belongs to $\mathcal{N}(C_T^*)$, the null space of the adjoint operator of C_T . But

$$\mathcal{N}(C_T^*) = \mathcal{R}(C_T)^\perp = \{\tilde{g} \in L^{\Psi^*}(\Xi) : \langle g, \tilde{g} \rangle = 0 \text{ for all } g \in \mathcal{R}(C_T)\}.$$

Hence $\mathcal{R}(C_T)^\perp$ contains an infinite number of linearly independent functions. Consequently, the quotient space $L^\Psi(\Xi)/\mathcal{R}(C_T)$ also contains an infinite number of linearly independent functions. Since $\text{codim } \mathcal{R}(C_T) = \dim L^\Psi(\Xi)/\mathcal{R}(C_T)$, this contradicts the assumption that $\text{codim } \mathcal{R}(C_T) < \infty$. Therefore, C_T is surjective. $\square$

We will use the following lemma in the proof of the next theorem. The notation ' $T(\Xi) = \Omega$ essentially' means that $\mu(\Omega \setminus T(\Xi)) = 0$.

Lemma 2. *Let $T : \Xi \rightarrow \Omega$ be a nonsingular measurable transformation.*

- (1) *If there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$, then $\text{Ker}(C_T) = \{0\}$.*
- (2) *If $\dim \text{Ker}(C_T) < \infty$, then $T(\Xi) = \Omega$ essentially.*

Proof. (1) First note that if $h \geq \varepsilon > 0$ on Ω , then $T(\Xi) = \Omega$ essentially. Indeed, if not, we can take a measurable $B \subset \Omega \setminus T(\Xi)$ such that $\mu(B) > 0$. Then

$$0 = \nu(\emptyset) = \nu \circ T^{-1}(B) = \int_B h(\omega) d\mu(\omega) \geq \int_B \varepsilon d\mu(\omega) = \varepsilon \mu(B) > 0.$$

Now if $C_T f = 0$, then

$$\begin{aligned}0 = I_\Psi(C_T f) &= \int_\Xi \Psi(f \circ T(\xi)) d\nu(\xi) = \int_{T(\Xi)} \Psi(f(\omega)) h(\omega) d\mu(\omega) \\ &\geq \varepsilon \int_\Omega \Psi(f(\omega)) d\mu(\omega).\end{aligned}$$

Hence $\mu(\text{supp } f) = 0$, which implies that $f = 0$ (as usual, up to sets of measure zero). Thus $\text{Ker}(C_T) = \{0\}$.

- (2) If it is not true that $T(\Xi) = \Omega$ essentially, then there is an infinite sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint subsets of $\Omega \setminus T(\Xi)$ with nonzero measure. Since each function $f_n := \chi_{A_n}$ is nonzero and $C_T f_n = \chi_{T^{-1}(A_n)} = 0$, we conclude that $\dim \text{Ker}(C_T) = \infty$. $\square$

Theorem 9. *Suppose that $T : \Xi \rightarrow \Omega$ is a nonsingular measurable transformation which is injective and such that the mapping $T^{-1} : T(\xi) \mapsto \xi$ is nonsingular. Suppose, further, that for any measurable set $B \subset \Xi$ of positive measure, there is a measurable set $A \subset T(\Xi)$ such that $T^{-1}(A) \subset B$ and $\nu(T^{-1}(A)) > 0$. If Φ satisfies the Δ_2 -condition globally and C_T is a continuous operator from $L^\Phi(\Omega)$ to $L^\Phi(\Xi)$, then the following conditions are equivalent.*

- (1) C_T is a bijection.
- (2) C_T is Fredholm.
- (3) There is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$.

Proof. The implication (a) $\Rightarrow$ (b) is obvious.

Let C_T be Fredholm. Since $\dim \text{Ker}(C_T) < \infty$, by Lemma 2, we get that $T(\Xi) = \Omega$ essentially. Since $\text{codim } \mathcal{R}(C_T) < \infty$, Lemma 1 implies that C_T is surjective. Applying Corollary 1 or Theorem 6 (with $\Psi = \Phi$), we infer that there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in T(\Xi) = \Omega$. Hence (b) $\Rightarrow$ (c) holds.

Finally we prove (c) $\Rightarrow$ (a). If there is $\varepsilon > 0$ such that $h(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$, then, by Corollary 1, the continuous composition operator C_T is surjective. This immediately implies that $\mathcal{R}(C_T)$ is closed and $\text{codim } \mathcal{R}(C_T) = 0$. Moreover, Lemma 2 yields that $\dim \text{Ker}(C_T) = 0$. Therefore C_T is a bijection. $\square$

5.2. Fredholmness of M_u

Lemma 3. *Suppose that Φ satisfies the Δ_2 -condition globally and the multiplication operator M_u acts continuously from $L^\Phi(\Omega)$ to $L^\Phi(\Omega)$. If the codimension of M_u is finite, then the operator M_u is surjective.*

The proof of this lemma is so similar to the proof of Lemma 1 that we omit it.

To complete the picture we prove necessary and sufficient conditions for Fredholmness of the multiplication operator. This theorem was proved in [6] by essentially different methods.

Theorem 10. *Suppose that $u : \Omega \rightarrow \mathbb{R}_+$ is a measurable function such that $u(\omega) > 0$ for a.e. $\omega \in \Omega$. If Φ satisfies the Δ_2 -condition globally and M_u is a continuous operator from $L^\Phi(\Omega)$ to $L^\Phi(\Omega)$, then the following conditions are equivalent.*

- (1) M_u is a bijection.
- (2) M_u is Fredholm.
- (3) The codimension of M_u is finite.
- (4) There is $\varepsilon > 0$ such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious.

If $\text{codim } \mathcal{R}(M_u) < \infty$, then $\mathcal{R}(M_u)$ is closed. Lemma 3 implies that M_u is surjective. Therefore, by Corollary 3 or Theorem 8 (with $\Psi = \Phi$), we get that there is $\varepsilon > 0$ such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$. Hence (c) $\Rightarrow$ (d) holds.

We only need to show $(d) \Rightarrow (a)$. If there is $\varepsilon > 0$ such that $u(\omega) \geq \varepsilon$ for a.e. $\omega \in \Omega$, by Corollary 3, the continuous multiplication operator M_u is surjective. This immediately implies that $\mathcal{R}(M_u)$ is closed and $\text{codim } \mathcal{R}(M_u) < \infty$. It is obvious that $\dim \text{Ker}(M_u) = 0$. Therefore M_u is a bijection. $\square$

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