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Results in Mathematics

Weyl–Schrödinger Representations of Heisenberg Groups in Infinite Dimensions

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Abstract. We investigate the group $\mathcal{H}_\mathbb{C}$ of complexified Heisenberg matrices with entries from an infinite-dimensional complex Hilbert space H. Irreducible representations of the Weyl–Schrödinger type on the space L^2_χ of quadratically integrable $\mathbb{C}$-valued functions are described. Integrability is understood with respect to the projective limit $\chi = \varprojlim \chi_i$ of probability Haar measures χ_i defined on groups of unitary $i \times i$-matrices $U(i)$. The measure χ is invariant under the infinite-dimensional group $U(\infty) = \bigcup U(i)$ and satisfies the abstract Kolmogorov consistency conditions. The space L^2_χ is generated by Schur polynomials on Paley–Wiener maps. The Fourier-image of L^2_χ coincides with the Hardy space H^2_β of Hilbert–Schmidt analytic functions on H generated by the correspondingly weighted Fock space $\varGamma_\beta(H)$. An application to heat equation over $\mathcal{H}_\mathbb{C}$ is considered.

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1. Introduction

An aim of this work is to investigate irreducible Weyl–Schrödinger representations of the complexified Heisenberg group $\mathcal{H}_\mathbb{C}$ (see [17, n.9]), consisting of matrix elements $X(a, b, t)$ with any $a, b \in H$ and $t \in \mathbb{C}$ such that

$$X(a,b,t)=\begin{bmatrix}1 & a & t\\ 0 & \mathbb{1} & b\\ 0 & 0 & 1\end{bmatrix},$$
$$X(a,b,t)\cdot X(a',b',t')=\begin{bmatrix}1 & a+a' & t+t'+\langle a\mid b'\rangle\\ 0 & \mathbb{1} & b+b'\\ 0 & 0 & 1\end{bmatrix} \tag{1}$$

where H is an infinite-dimensional complex Hilbert space and $\mathbb{1}$ is its identity map.

The group $\mathcal{H}_{\mathbb{C}}$ has the unit $X(0,0,0)$ and inverse elements of the form $X(a,b,t)^{-1}=X\left(-a,-b,-t+\langle a\mid b\rangle\right)$.

In what follows, we consider the infinite-dimensional unitary group $U(\infty)=\bigcup U(i)$, containing all subgroups $U(i)$ of unitary $i\times i$-matrices, which acts irreducibly on a complex Hilbert space $\{H,\langle\cdot\mid\cdot\rangle\}$ with an orthonormal basis $(\mathfrak{e}_i)_{i\in\mathbb{N}}$.

To find the desired representation, we use the space L^2_χ of $\mathbb{C}$-valued functions that are quadratically integrable with respect to the probability measure χ. Wherein, according to our assumption χ has a structure of the projective limit $\chi=\varprojlim\chi_i$ of probability Haar's measures χ_i on $U(i)$, satisfying the Kolmogorov consistency conditions in an abstract Bochner's formulation (see [23,27]).

In [21,24] it was shown that the projective limit $\chi=\varprojlim\chi_i$ is well defined over the projective limit $\mathfrak{U}=\varprojlim U(i)$ with respect to the Livšic transforms $\pi_i^{i+1}\colon U(i+1)\to U(i)$ such that $\chi_i=\pi_i^{i+1}(\chi_{i+1})$. In this paper, we prove that for such χ each function from L^2_χ admit a superposition (linearization in the sense of [5]) on Paley–Wiener maps associated with $U(\infty)$. As a result, it is shown that Schur polynomials form an orthonormal basis in L^2_χ and the Fourier-image of L^2_χ consists of Hilbert-Schmidt analytic functions on H.

Note also that projective limits of probability measures over various infinite-dimensional manifolds with similar properties were investigated in [25,34,35].

If instead of the unitary group $U(\infty)$ we take the infinite-dimensional linear space with a Gaussian measure γ, a similar construction of the appropriate space L^2_γ can be found in the well-known works [1,2]. In this case, the Fourier-image of L^2_γ coincides with the Segal–Bargmann space of entire analytic functions over which the Schrödinger type irreducible representations of Heisenberg groups are well defined. In the present paper, we change γ by the unitarily-invariant projective limit $\chi=\varprojlim\chi_i$ and, as a result, we obtain another irreducible representation, called to be the Weyl–Schrödinger type.

Infinite-dimensional Heisenberg groups over $\mathbb{R}$ was considered in [19] by using the reproducing kernel Hilbert spaces. The Schrödinger representation of such groups using Gaussian measures over a real Hilbert space was described in [3]. Since the group $\mathcal{H}_{\mathbb{C}}$ in the case of matrix entries $a,b,t\in\mathbb{R}$ coincides with the classical Heisenberg group over $\mathbb{R}$ (see, e.g. [11]), the results of the present paper can be considered as a complexification of previous studies. The Weyl–Schrödinger representation obtained here is not equivalent to that was described earlier.

Further, let us briefly describe the main results. Consider the following mapping $\phi\colon H \ni h \longmapsto \phi_h \in L^2_\chi$ defined by Paley–Wiener maps

$$\phi_h(\mathfrak{u}) := \sum \phi_i(\mathfrak{u})\, \mathfrak{e}_i^*(h) \quad \text{with} \quad \phi_i(\mathfrak{u}) := \langle u_i(\mathfrak{e}_i) \mid \mathfrak{e}_i \rangle\,, \quad u_i = \pi_i(\mathfrak{u}), \tag{2}$$

where $\mathfrak{e}_i^*(\cdot) := \langle \cdot \mid \mathfrak{e}_i \rangle$ and the projections $\pi_i\colon \mathfrak{U} \ni \mathfrak{u} \to u_i \in U(i)$ are uniquely defined by π_i^{i+1}. Every function ϕ_h of variable $\mathfrak{u} \in \mathfrak{U}$ satisfies the equality (Corollary 3)

$$\int \exp\left\{ \operatorname{Re} \phi_h \right\} d\chi = \exp\left\{ \frac{1}{4} \|h\|^2 \right\}, \quad h \in H.$$

The space L^2_χ can be generated by two orthonormal bases, consisting of Schur polynomials and power polynomials of variables $\phi_\imath = \left(\phi_{\imath_1}, \ldots, \phi_{\imath_\eta}\right)$, respectively,

$$s_\imath^\lambda(\mathfrak{u}) := \frac{\det\left[\phi_{\imath_i}^{\lambda_j+\eta-j}(\mathfrak{u})\right]_{1\le i,j\le\eta}}{\prod_{1\le i<j\le\eta}[\phi_{\imath_i}(\mathfrak{u}) - \phi_{\imath_j}(\mathfrak{u})]} \quad \text{and} \quad \phi_\imath^\lambda := \phi_{\imath_1}^{\lambda_1} \ldots \phi_{\imath_\eta}^{\lambda_\eta}. \tag{3}$$

These bases are indexed by tabloids $\imath^\lambda$ with strictly ordered $\imath = (\imath_1, \ldots, \imath_\eta) \in \mathbb{N}^\eta$ where $\lambda = (\lambda_1, \ldots, \lambda_\eta) \in \mathbb{N}^\eta$ is a partition of $n \in \mathbb{N}$ and $\eta = \eta(\lambda)$ stands for the length of λ. Then we write briefly $\imath^\lambda \vdash n$. The orthogonal expansion $L^2_\chi = \bigoplus L^{2,n}_\chi$ holds (Theorem 1) where $L^{2,n}_\chi$ are formed by n-homogeneous polynomials $\phi_\imath^\lambda$, normed as follows

$$\|\phi_\imath^\lambda\|^2_\chi = \int |\phi_\imath^\lambda|^2 d\chi = \beta_\lambda \lambda!, \qquad \beta_\lambda := \frac{(\eta-1)!}{(\eta-1+n)!}, \quad \lambda! := \lambda_1! \ldots \lambda_\eta!.$$

It is also shown that the surjective linear isometry $\Psi\colon H^2_\beta \ni \psi_f^* \longmapsto f \in L^2_\chi$ holds (Lemma 5), where $H^2_\beta = \sum P^n_\beta(H)$ means the Hardy space of entire analytic functions $\psi_f^*(h)$ of variable $h \in H$ and $P^n_\beta(H)$ is generated by the n-homogeneous Hilbert–Schmidt polynomials $\mathfrak{e}_\imath^{*\lambda} := \mathfrak{e}_{\imath_1}^{*\lambda_1} \ldots \mathfrak{e}_{\imath_\eta}^{*\lambda_\eta}$, normed as $\|\mathfrak{e}_\imath^{*\lambda}\|_{H^2_\beta} = \left(\beta_\lambda \lambda!\right)^{1/2}$.

If the basis of symmetric tensor elements $\mathfrak{e}_\imath^{\odot\lambda} := \mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \odot \ldots \odot \mathfrak{e}_{\imath_\eta}^{\otimes\lambda_\eta}$ (associated with $\mathfrak{e}_\imath^{*\lambda}$) in the correspondingly weighted Fock space $\Gamma_\beta(H)$ is normed as $\|\mathfrak{e}_\imath^{\odot\lambda}\|_{\Gamma_\beta} = \|\mathfrak{e}_\imath^{*\lambda}\|_{H^2_\beta}$ then each function $f \in L^2_\chi$ admits the superposition

$$f = \Psi \circ \psi_f^*, \qquad \psi_f^*(h) = \sum_{n\ge0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \mathfrak{e}_\imath^{*\lambda}(h) \left\langle \mathfrak{e}_\imath^{\odot\lambda} \mid \psi_f \right\rangle_{\Gamma_\beta}, \quad h \in H,$$

where the Taylor expansion on the right-hand side of any analytic function $\psi_f^* \in H^2_\beta$ on H is uniquely determined by the corresponding element $\psi_f \in \Gamma_\beta(H)$.

Our further goal is to analyze the inverse isomorphism $\varPsi^{-1}$ which can be described by the Fourier transform under the measure χ in following way

$$\hat{f}(h) = \int \exp(\bar{\phi}_h) f \, d\chi \quad \text{where} \quad F = \varPsi^{-1} \colon L^2_\chi \ni f \longmapsto \hat{f} := \psi^*_f \in H^2_\beta.$$

The Fourier transform F acts isometrically on the Hardy space of analytic functions H^2_β (Theorem 2). So, F acts as an analytic extension of the mapping ϕ.

Applying the superposition with $\varPsi$, we describe two different representations of the additive group $(H, +)$ over L^2_χ defined by shift and multiplicative groups (Lemma 7). Using this we show (in Theorem 3) that an irreducible representation of the Heisenberg group $\mathcal{H}_\mathbb{C}$ can be realized on L^2_χ in the Weyl–Schrödinger form

$$X(a, b, z) \longmapsto \exp(z) W^\dagger(a, b), \quad W^\dagger(a, b) := \exp\left\{\frac{1}{2}\langle a \mid b\rangle\right\} T^\dagger_b M^\dagger_{a^*}$$

for all $a, b \in H$ and $z \in \mathbb{C}$, where $T^\dagger_b$ and $M^\dagger_{a^*}$ are defined by shift and multiplicative groups, respectively. It is also proved that the Weyl system $W^\dagger(a, b)$ has the densely-defined generator $\mathfrak{p}^\dagger_{a,b} := \partial^\dagger_b + \bar{\phi}_a$ which satisfies the commutation relation

$$W^\dagger(a, b) W^\dagger(a', b') = \exp\left\{-\left[\mathfrak{p}^\dagger_{a,b}, \mathfrak{p}^\dagger_{a',b'}\right]\right\} W^\dagger(a', b') W^\dagger(a, b)$$

where the groups $M^\dagger_{a^*}$ and $T^\dagger_b$ are generated by $\bar{\phi}_a$ and $\partial^\dagger_b$, respectively.

Applying the Weyl–Schrödinger representation to the associated with $\mathcal{H}_\mathbb{C}$ heat equation, we prove (Theorem 4) that the following Cauchy problem with $\partial^\dagger_i := \partial^\dagger_{\mathfrak{e}_i}$,

$$\frac{dw(r)}{dr} = -\sum \left(\partial^\dagger_i + \bar{\phi}_i\right)^2 w(r), \quad w(0) = f, \quad r > 0,$$

has the unique solution $w(r) = \mathfrak{G}^\dagger_r f$ for any function f from a finite sum $\bigoplus L^{2,n}_\chi$, where the 1-parameter Gaussian semigroup $\mathfrak{G}^\dagger_r$ has the form

$$\mathfrak{G}^\dagger_r f = \frac{1}{\sqrt{4\pi r}} \int_{c_0} \exp\left\{ - \frac{\|\tau\|^2_{w_0}}{4r}\right\} W^\dagger_\tau f \, d\mathfrak{w}(\tau),$$

$$W^\dagger_\tau f := \lim_{n\to\infty} \exp\left\{ - \frac{\|p^\sim_n(\tau)\|^2_{w_0}}{2}\right\} \prod_{i=1}^n T^\dagger_{\mathbb{i}\tau_i \mathfrak{e}_i} M^\dagger_{-\mathbb{i}\tau_i \mathfrak{e}^*_i}.$$

Here $\tau = (\tau_i)$ belongs to the abstract Wiener space $\{w_0, \|\cdot\|_{w_0}\}$ defined by the injections $l_2 \hookrightarrow w_0 \hookrightarrow c_0$ of real Banach spaces and endowed with the Wiener measure $\mathfrak{w}$ in according to the known Gross' theorem [10], whereas the sequence of projectors $(p^\sim_n)$ onto $\mathbb{R}^n$ is convergent to the identity map on w_0.

Finally, note that this work is a continuation of previous publications [16, 17]. The novelty results from the observation that the system of Schur polynomials with variables on Paley–Wiener maps form an orthonormal basis in L^2_χ. This allowed us to investigate irreducible Weyl–Schrödinger representations and Weyl systems of the Heisenberg group $\mathcal{H}_\mathbb{C}$ on the whole space L^2_χ.

2. Invariant Probability Measure

Consider the unitary group $U(\infty) = \bigcup U(m)$ with $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{1} = U(0)$, irreducibly acting on a separable Hilbert space H, where subgroups $U(m)$ are identified with ranges of injections $U(m) \ni u_m \longmapsto \begin{bmatrix} u_m & 0 \\ 0 & \mathbb{1} \end{bmatrix} \in U(\infty)$. Following to [21,24], we use the Livšic transforms $\pi_m^{m+1} \colon U(m+1) \to U(m)$ of the form

$$\pi_m^{m+1} \colon u_{m+1} := \begin{bmatrix} z_m & a \\ b & t \end{bmatrix} \longmapsto u_m := \begin{cases} z_m - [a(1+t)^{-1}b] : & t \neq -1 \\ z_m \quad : & t = -1 \end{cases} \tag{4}$$

with $z_m \in U(m)$ defined by excluding $x_1 = y_1 \in \mathbb{C}$ from $\begin{bmatrix} y_m \\ y_1 \end{bmatrix} = \begin{bmatrix} z_m & a \\ -b & -t \end{bmatrix} \begin{bmatrix} x_m \\ x_1 \end{bmatrix}$ for $x_m, y_m \in \mathbb{C}^m$ and $a, b \in \mathbb{C}$ [24, Lem. 3.1]. It is surjective (not continuous) Borel mapping [24, Lem. 3.11].

The projective limit $\mathfrak{U} := \varprojlim U(m)$ under π_m^{m+1} has surjective Borel (not group homomorphisms) projections

$$\pi_m \colon \mathfrak{U} \ni \mathfrak{u} \longmapsto u_m \in U(m) \quad \text{such that} \quad \pi_m = \pi_m^{m+1} \circ \pi_{m+1}.$$

Their elements $\mathfrak{u} \in \mathfrak{U}$ are called the *virtual unitary matrices*. The right action

$$\mathfrak{U} \ni \mathfrak{u} \longmapsto \mathfrak{u}.g \in \mathfrak{U} \quad \text{with} \quad g = (v, w) \in U(\infty) \times U(\infty)$$

is defined to be $\pi_m(\mathfrak{u}.g) = w^{-1}\pi_m(\mathfrak{u})v$, where m is large enough that $v, w \in U(m)$. On $\mathfrak{U}$ the involution $\mathfrak{u} \mapsto \mathfrak{u}^\star = (u_k^\star)$ is well defined, where $u_k^\star = u_k^{-1}$ is adjoint to $u_k \in U(k)$. Thus, $[\pi_m(\mathfrak{u}.g)]^\star = \pi_m(\mathfrak{u}^\star.g^\star)$ for all $g^\star = (w^\star, v^\star) \in U(\infty) \times U(\infty)$.

There exists the dense embedding $U(\infty) \looparrowright \mathfrak{U}$ (see [24, n.4]) which assigns the stabilized sequence $\mathfrak{u} = (u_k)$ to each $u_m \in U(m)$ such that

$$\begin{aligned} &U(m) \ni u_m \longmapsto (u_k) \in \mathfrak{U}, \\ &u_k = \begin{cases} \pi_k^m(u_m) = (\pi_k^{k+1} \circ \ldots \circ \pi_{m-1}^m)(u_m) : k < m, \\ u_m \quad : k \geq m. \end{cases} \end{aligned} \tag{5}$$

We always assume that the group $U(m)$ is endowed with the probability Haar measure χ_m. Using the Kolmogorov consistency theorem (see, e.g. [24, Lem.4.8], [27, Thm 2.2], [30, Cor.4.2]), we determine the probability measure on $\mathfrak{U}$ to be the projective limit

$$\chi := \varprojlim \chi_m \quad \text{under} \quad \chi_m = \pi_m^{m+1}(\chi_{m+1})$$

where $\pi_m^{m+1}(\chi_{m+1})$ means an image-measure and $\chi_0 = 1$. As is known [30, Thm 2.5], the measure χ is Radon. We now describe the necessary properties of χ.

Consider the Hilbert space L^2_χ of functions $f\colon \mathfrak{U} \to \mathbb{C}$ with the following norm and inner product

$$\|f\|_\chi = \langle f \mid f\rangle_\chi^{1/2}, \quad \langle f_1 \mid f_2\rangle_\chi := \int f_1 \bar{f}_2 \, d\chi.$$

Let L^∞_χ be the space of χ-essentially bounded functions $f\colon \mathfrak{U} \to \mathbb{C}$ with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{\mathfrak{u}\in\mathfrak{U}}|f(\mathfrak{u})|$. The embedding $L^\infty_\chi \hookrightarrow L^2_\chi$ holds and $\|f\|_\chi \le \|f\|_\infty$.

Lemma 1. *For any $f \in L^\infty_\chi$ there exists the limit*

$$\int f \, d\chi = \lim \int f \, d(\chi_m \circ \pi_m) = \lim \int (f \circ \pi_m^{-1}) \, d\chi_m. \tag{6}$$

Moreover, the measure χ is invariant under the right action, which means that

$$\int f(\mathfrak{u}.g) \, d\chi(\mathfrak{u}) = \int f(\mathfrak{u}) \, d\chi(\mathfrak{u}), \quad g \in U(\infty) \times U(\infty), \tag{7}$$

$$\int f \, d\chi = \int d\chi(\mathfrak{u}) \int f(\mathfrak{u}.g) \, d(\chi_m \otimes \chi_m)(g). \tag{8}$$

Proof. The sequence $\{(\chi_m \circ \pi_m)(\mathcal{K})\}$ is decreasing for any compact set $\mathcal{K}$ in $\mathfrak{U}$, since $\pi_m = \pi_m^{m+1} \circ \pi_{m+1}$ yields $\pi_{m+1}(\mathcal{K}) \subseteq (\pi_m^{m+1})^{-1}[\pi_m(\mathcal{K})]$. It follows

$$\begin{aligned}(\chi_m \circ \pi_m)(\mathcal{K}) &= \pi_m^{m+1}(\chi_{m+1})\,[\pi_m(\mathcal{K})] \\ &= \chi_{m+1}\left[(\pi_m^{m+1})^{-1}[\pi_m(\mathcal{K})]\right] \ge (\chi_{m+1} \circ \pi_{m+1})(\mathcal{K}).\end{aligned} \tag{9}$$

This ensures that the necessary and sufficient conditions of the Prokhorov theorem [4, Thm IX.52] and its modification from [30, Thm 4.2] are satisfied.

Indeed, let $\check{U}(m) \subset U(m)$ be the set of matrices with no eigenvalue $\{-1\}$ for $m \ge 1$. As is known [24, n.3], $\check{U}(m)$ is open in $U(m)$ and $\chi_m(U(m)\backslash\check{U}(m)) = 0$. In virtue of [24, Lem. 3.11] the restrictions $\pi_m^{m+1}\colon \check{U}(m+1) \to \check{U}(m)$ are continuous and surjective. The projective limit $\varprojlim \check{U}(m)$ under these restrictions has continuous surjective projections $\pi_m\colon \varprojlim \check{U}(m) \to \check{U}(m)$. Restrict χ_m to $\check{U}(m)$. By [30, Thm 6], a probability measure $\check{\chi}$ satisfying conditions $\pi_m(\check{\chi}) = \chi_m|_{\check{U}(m)}$ is well defined iff for every $\varepsilon > 0$ there exists a compact set $\mathcal{K} \subset \varprojlim \check{U}(m)$ such that

$$(\chi_m \circ \pi_m)(\mathcal{K}) \ge 1 - \varepsilon \quad \text{for all} \quad m \in \mathbb{N}.$$

Then by the Prokhorov theorem $\check{\chi}$ is uniquely determined as

$$\check{\chi}(\mathcal{K}) = \inf(\chi_m \circ \pi_m)(\mathcal{K}) \quad \text{for all} \quad \mathcal{K} \subset \varprojlim \check{U}(m). \tag{10}$$

Let $\varepsilon > 0$ and $K_1 \subset \check{U}(1)$ be a compact set such that $\chi_1(K_1) > 1 - \varepsilon$. Let a compact sets $K_m \subset \check{U}(m)$ be defined inductively such that

$$\pi_m^{m+1}(K_{m+1}) \subset K_m \quad \text{and} \quad \chi_{m+1}(K_{m+1}) > 1 - \varepsilon \quad \text{for all} \quad m \ge 1.$$

Assume that $K_1, \ldots, K_m$ are constructed. Since $\chi_m = \pi_m^{m+1}(\chi_{m+1})$, we get

$$\chi_m(K_m) = \chi_{m+1}[(\pi_m^{m+1})^{-1}(K_m)] > 1 - \varepsilon.$$

By regularity of $\chi_{m+1}|_{\check{U}(m)}$, there exists a compact set

$$K_{m+1} \subset (\pi_m^{m+1})^{-1}(K_m) \quad \text{such that} \quad \chi_{m+1}(K_{m+1}) > 1 - \varepsilon.$$

The induction is complete. Then $\mathcal{K} = \varprojlim K_m$ with $K_0 = \mathbb{1}$ is compact. By virtue of (10), we have $\check{\chi}(\mathcal{K}) \geq 1 - \varepsilon$. Hence, the projective limit $\check{\chi} = \varprojlim \chi_m|_{\check{U}(m)}$ is well defined on $\varprojlim \check{U}(j)$ by the Prokhorov criterion.

The measure $\check{\chi}$ can be extended to $\varprojlim U(m) \backslash \varprojlim \check{U}(m)$ as zero, since each χ_m is zero on $U(m) \backslash \check{U}(m)$. The uniqueness of the projective limits yields $\check{\chi} = \chi$. So, $\chi = \varprojlim \chi_m$ is also well defined and by (9) and (10) we get

$$\chi(\mathcal{K}) = \inf(\chi_m \circ \pi_m)(\mathcal{K}) = \lim(\chi_m \circ \pi_m)(\mathcal{K}) \quad \text{for all compact} \quad \mathcal{K} \subset \mathfrak{U}.$$

By the known Portmanteau theorem [14, Thm 13.16] it follows that the limit (6) exists. Whereas, the property (7) is a consequence of the equalities

$$\chi(\mathcal{K}.g) = \lim \chi_m(K_m.g) = \lim \chi_m(K_m) = \chi(\mathcal{K})$$

for all $g = (v, w) \in U(\infty) \times U(\infty)$ where m is large enough that $v, w \in U(m)$.

Finally, the function $(\mathfrak{u}, g) \mapsto f(\mathfrak{u}.g)$ with any $f \in L^\infty_\chi$ is integrable over $\mathfrak{U} \times U(m) \times U(m)$, hence

$$\int d\chi(\mathfrak{u}) \int f(\mathfrak{u}.g)\, d(\chi_m \otimes \chi_m)(g) = \int d(\chi_m \otimes \chi_m)(g) \int f(\mathfrak{u}.g)\, d\chi(\mathfrak{u})$$

by the Fubini theorem. It yields (8) since the internal integral on the right-hand side is independent of g by (7) and $\int d(\chi_m \otimes \chi_m)(g) = 1$. The proof is complete. □

We now note the concentration property of Haar measures sequence (χ_m) satisfying the Kolmogorov conditions $\chi_m = \pi_m^{m+1}(\chi_{m+1})$ if each group $U(m)$ is endowed with the normalized Hilbert–Schmidt metric

$$d_{HS}(u, v) = \sqrt{m^{-1} \operatorname{tr} |u - v|_{HS}} \quad \text{where} \quad |u - v|_{HS} = \sqrt{(u - v)^\star (u - v)}.$$

As is well known (see [9,31]), $(U(m), d_{HB}, \chi_m)$ is a Lévy family. Namely, the following sequence of isoperimetric constants dependent on $\varepsilon > 0$

$$\alpha(U(m), \varepsilon) = 1 - \inf\left\{\chi_m[(\Omega_m)_\varepsilon] \colon \Omega_m \text{ be Borel set in } U(m), \chi_m(\Omega_m) > 1/2\right\}$$

with $(\Omega_m)_\varepsilon = \{u_m \in U(m) \colon d_{HS}(u_m, \Omega_m) < \varepsilon\}$ is such that

$$\alpha(U(m), \varepsilon) \to 0 \quad \text{as} \quad m \to \infty.$$

Taking into account the Lemma 1, we can formulate the following conclusion.

Corollary 1. *For any Borel set $\Omega_\varepsilon = \varprojlim (\Omega_m)_\varepsilon$ with $\chi_m(\Omega_m) > 1/2$ in the projective limit $\mathfrak{U} = \varprojlim U(m)$ the equality*

$$\chi(\Omega_\varepsilon) = \lim_{m \to \infty} \chi_m\left[(\Omega_m)_\varepsilon\right] = 1$$

holds. Consequently, all Borel sets $\mathfrak{U}\backslash\Omega_\varepsilon$ *with* $\chi_m(\Omega_m) > 1/2$ *and any* $\varepsilon > 0$ *are* χ*-measure zero, i.e., the measure* $\chi = \underleftarrow{\lim}\,\chi_m$ *is concentrated outside these sets.*

3. Polynomials on Paley–Wiener Maps

Let $\mathscr{I}_\eta := \{\imath = (\imath_1, \ldots, \imath_\eta) \in \mathbb{N}^\eta : \imath_1 < \imath_2 < \ldots < \imath_\eta\}$ be an integer alphabet of length η and $\mathscr{I} = \bigcup \mathscr{I}_\eta$. Let $\lambda = (\lambda_1, \ldots, \lambda_\eta) \in \mathbb{N}^\eta$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\eta$ be a partition of an n-letter word $\imath^\lambda = \{\square_{ij} : 1 \leq i \leq \eta, j = 1, \ldots, \lambda_i\}$ with $\imath \in \mathscr{I}_\eta$. A Young λ-tableau with a partition λ is a result of filling the word $\imath^\lambda$ onto the matrix $[\imath^\lambda] = \begin{matrix} \square_{11} & \ldots & \ldots & \square_{1\lambda_1} \\ \vdots & \vdots & \ddots & \\ \square_{\eta 1} & \ldots & \square_{\eta\lambda_\eta} & \end{matrix}$ with n nonzero entries in some way without repetitions. So, each λ-tableau $[\imath^\lambda]$ can be identified with a bijection $[\imath^\lambda] \to \imath^\lambda$. The conjugate partition $\lambda^\intercal$ corresponds to the transpose matrix $[\imath^\lambda]^\intercal$.

A Young tableau $[\imath^\lambda]$ is called *standard* (*semistandard*) if its entries are strictly (weakly) ordered along each row and strictly ordered down each column. Let $\mathbb{Y}$ denote all Young tabloids $[\imath^\lambda]$ and $\mathbb{Y}_n$ be its subset such that $\imath^\lambda \vdash n$. Assume that $\mathbb{Y}_0 = \{\emptyset \in \mathbb{Y} : |\emptyset| = 0\}$ and $\eta(\emptyset) = 0$.

As before, $\{H, \langle\cdot \mid \cdot\rangle\}$ is a separable complex Hilbert space with an orthonormal basis $\{\mathfrak{e}_i : i \in \mathbb{N}\}$ and $\|\cdot\| = \langle\cdot \mid \cdot\rangle^{1/2}$. For its adjoint space H^* the conjugate-linear isometry $*: H^* \to H^{**} = H$ is defined via $a^*(h) = \langle h \mid a\rangle$ for all $a, h \in H$. The Fourier expansion $h = \sum \mathfrak{e}_i^*(h)\mathfrak{e}_i$ with $\mathfrak{e}_i^*(h) := \langle h \mid \mathfrak{e}_i\rangle$ holds. The tensor power $H^{\otimes n}$, spanned by elements $\psi_n = h_1 \otimes \ldots \otimes h_n$ with $h_i \in H$ $(i = 1, \ldots, n)$, is endowed with the norm $\|\psi_n\| = \langle\psi_n \mid \psi_n\rangle^{1/2}$ where $\langle\psi_n \mid \psi_n'\rangle := \langle h_1 \mid h_1'\rangle \ldots \langle h_n \mid h_n'\rangle$.

Let S_n be the group of n-elements permutations $\sigma(\psi_n) := h_{\sigma(1)} \otimes \ldots \otimes h_{\sigma(n)}$. An orthogonal basis in $H^{\otimes n}$ is formed by elements $\sigma(\mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \otimes \ldots \otimes \mathfrak{e}_{\imath_\eta}^{\otimes\lambda_\eta})$ with $\imath^\lambda \vdash n$ and $\eta = \eta(\lambda)$, additionally indexed by all $\sigma \in S_n$. The symmetric tensor power $H^{\odot n} \subset H^{\otimes n}$ is defined to be a range of the orthogonal projector $\mathcal{S}_n : H^{\otimes n} \ni \psi_n \longmapsto h_1 \odot \ldots \odot h_n := (n!)^{-1}\sum_{\sigma\in S_n}\sigma(\psi_n)$. We assume that $H^{\otimes n}$ is completed and that $H^{\otimes 0} = \mathbb{C}$. Let $\psi_n := h^{\otimes n}$ for $h = h_i$. The embedding $\{h^{\otimes n} : h \in H\} \subset H^{\odot n}$ is total by the polarization formula [7, n.1.5]

$$h_1 \odot \ldots \odot h_n = \frac{1}{2^n n!}\sum_{\theta_1,\ldots,\theta_n=\pm 1}\theta_1 \ldots \theta_n h^{\otimes n}, \quad h = \sum_{i=1}^{n}\theta_i h_i. \tag{11}$$

Let $H_\eta \subset H$ be spanned by $\{\mathfrak{e}_{\imath_1}, \ldots, \mathfrak{e}_{\imath_\eta}\}$. We can uniquely assign to any semistandard tableau $[\imath^\lambda]$ with $\imath^\lambda \vdash n$ the element in $H_\eta^{\otimes n}$ for which there exists the permutation $\sigma' \in S_n$ such that $\sigma'(\mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \otimes \ldots \otimes \mathfrak{e}_{\imath_\eta}^{\otimes\lambda_\eta}) = \mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \odot \ldots \odot \mathfrak{e}_{\imath_\eta}^{\otimes\lambda_\eta}$

$\in H_\eta^{\odot n}$. Taking all $\imath \in \mathscr{I}$, we conclude that the system indexed by semistandard λ-tabloids

$$\mathfrak{e}^{\mathbb{Y}_n} = \left\{ \mathfrak{e}_\imath^{\odot\lambda} := \mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \odot \ldots \odot \mathfrak{e}_{\imath_\eta}^{\otimes\lambda_\eta} : \imath^\lambda \vdash n,\ \lambda \in \mathbb{Y}_n,\ \imath \in \mathscr{I} \right\}, \quad \mathfrak{e}_\imath^{\odot\emptyset} = 1$$

$$\text{where} \quad \langle \mathfrak{e}_\imath^{\odot\lambda} \mid \mathfrak{e}_{\imath'}^{\odot\lambda'} \rangle = \begin{cases} \lambda!/n! : \lambda = \lambda & \text{and} \quad \imath = \imath' \\ \ \ 0 \quad : \lambda \neq \lambda' & \text{or} \quad \imath \neq \imath' \end{cases}$$

forms an orthogonal basis in the symmetric tensor power $H_\eta^{\odot n}$.

The system $\left\{ \mathfrak{e}_\imath^{\otimes\lambda} := \mathcal{S}_n \big(\mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \otimes \ldots \otimes \mathfrak{e}_{\imath_\eta}^{\otimes\lambda_\eta} \big) : \imath^\lambda \vdash n,\ \lambda \in \mathbb{Y}_n,\ \imath \in \mathscr{I} \right\}$, additionally indexed by all $\sigma \in S_n$, forms an orthonormal basis in the whole tensor power $H^{\otimes n}$.

As usually, the *symmetric Fock space* is defined to be the Hilbertian orthogonal sum $\varGamma(H) = \bigoplus_{n\geq 0} H^{\odot n}$ with the orthogonal basis $\mathfrak{e}^{\mathbb{Y}} := \bigcup \left\{ \mathfrak{e}^{\mathbb{Y}_n} : n \in \mathbb{N}_0 \right\}$ of elements $\psi = \bigoplus \psi_n$ with $\psi_n \in H^{\odot n}$ endowed with the inner product and norm

$$\langle \psi \mid \psi' \rangle_\varGamma = \sum n! \langle \psi_n \mid \psi_n' \rangle, \quad \|\psi\|_\varGamma = \langle \psi \mid \psi \rangle_\varGamma^{1/2}.$$

Note that by tensor multinomial theorem the Fourier expansion under $\mathfrak{e}^{\mathbb{Y}_n}$

$$h^{\otimes n} = \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \mathfrak{e}_\imath^{\odot\lambda}\, \mathfrak{e}_\imath^{*\lambda}(h), \quad \|h^{\otimes n}\|^2 = \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} |\mathfrak{e}_\imath^{*\lambda}(h)|^2, \quad \mathfrak{e}_\imath^{*\lambda} := \mathfrak{e}_{\imath_1}^{*\lambda_1} \ldots \mathfrak{e}_{\imath_\eta}^{*\lambda_\eta}, \tag{12}$$

holds in $H^{\odot n}$ for all $h \in H$. Consequently, the linearly independent, so-called, coherent states $\left\{ \exp(h) : h \in H \right\}$ in $\varGamma(H)$ have the expansion under the basis $\mathfrak{e}^{\mathbb{Y}}$

$$\exp(h) := \bigoplus_{n\geq 0} \frac{h^{\otimes n}}{n!} = \bigoplus_{n\geq 0} \frac{1}{n!} \left(\sum_{i\geq 0} \mathfrak{e}_i\, \mathfrak{e}_i^*(h) \right)^{\otimes n} = \bigoplus_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \mathfrak{e}_\imath^{\odot\lambda}\, \mathfrak{e}_\imath^{*\lambda}(h) \tag{13}$$

with $h^{\otimes 0} = 1$, that is convergent, since $\|\mathfrak{e}_\imath^{\odot\lambda}\|_\varGamma^2 = n! \|\mathfrak{e}_\imath^{\odot\lambda}\|^2$ and

$$\begin{aligned} \|\exp(h)\|_\varGamma^2 &= \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \left(\frac{n!}{\lambda!} \right)^2 \|\mathfrak{e}_\imath^{\odot\lambda}\|^2 |\mathfrak{e}_\imath^{*\lambda}(h)|^2 = \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} |\mathfrak{e}_\imath^{*\lambda}(h)|^2 \\ &= \sum \frac{1}{n!} \left(\sum |\mathfrak{e}_i^*(h)|^2 \right)^n = \sum \frac{1}{n!} \|h\|^{2n} = \exp \|h\|^2. \end{aligned} \tag{14}$$

Definition 1. For any $h \in H$ and $\mathfrak{u} \in \mathfrak{U}$ the Paley–Wiener maps are defined to be

$$\phi_h(\mathfrak{u}) := \sum \phi_i(\mathfrak{u})\, \mathfrak{e}_i^*(h) \quad \text{with} \quad \phi_i(\mathfrak{u}) := \langle u_i(\mathfrak{e}_i) \mid \mathfrak{e}_i \rangle, \quad u_i = \pi_i(\mathfrak{u})$$

where projections $\pi_i : \mathfrak{U} \ni \mathfrak{u} \to u_i \in U(i)$ are uniquely defined by π_i^{i+1}.

These maps satisfy the orthogonal conditions $\phi_{\mathfrak{e}_i} = \phi_i$ and have the natural extension $\phi_{h^*} = \bar{\phi}_h$ onto the adjoint space H^*.

Note that, as in the case of linear spaces (see e.g. [12, n.4.4], [29]), the Paley–Wiener maps uniquely determine the embedding $\phi\colon H \ni h \longmapsto \phi_h \in L^2_\chi$.

For every $h \in H$ the l_2-valued function $\phi_h(\mathfrak{u})$ of variable $\mathfrak{u} \in \mathfrak{U}$ is well-defined, since $(\mathfrak{e}_i^*(h)) \in l_2$ and $|\langle u_i(\mathfrak{e}_i) \mid \mathfrak{e}_i\rangle| \le 1$. We show that $\phi_h \in L^2_\chi$. Assign for any partition $\lambda = (\lambda_1, \ldots, \lambda_\eta) \in \mathbb{N}^\eta$ of the weight $|\lambda| = \lambda_1 + \ldots + \lambda_\eta$ the constant

$$\beta_\lambda := \frac{(\eta-1)!}{(\eta-1+|\lambda|)!} \le 1, \quad \eta = \eta(\lambda). \tag{15}$$

Lemma 2. *To every semistandard tableau $[\imath^\lambda]$ one can uniquely assign the function*

$$\phi_\imath^\lambda(\mathfrak{u}) := \phi_{\imath_1}^{\lambda_1}(\mathfrak{u}) \ldots \phi_{\imath_\eta}^{\lambda_\eta}(\mathfrak{u}), \quad \phi_\imath^\emptyset \equiv 1 \tag{16}$$

of variable $u \in \mathfrak{U}$ belonging to L^∞_χ. The system of χ-essentially bounded functions

$$\phi^{\mathbb{Y}} := \bigcup \left\{\phi^{\mathbb{Y}_n} : n \in \mathbb{N}_0\right\} \quad \textit{with} \quad \phi^{\mathbb{Y}_n} := \bigcup \left\{\phi_\imath^\lambda \colon \imath^\lambda \vdash n, \imath \in \mathscr{I}_\eta\right\}$$

is orthogonal in the space L^2_χ and is normed as follows

$$\|\phi_\imath^\lambda\|^2_\chi = \int |\phi_\imath^\lambda|^2 d\chi = \lambda! \beta_\lambda, \quad \imath^\lambda \vdash n, \quad \lambda! := \lambda_1! \ldots \lambda_\eta!.$$

Proof. According to (4), we have $(\pi_m \circ \pi_{m+l}^{-1}) u_{m+l}(\mathfrak{e}_m) = u_m(\mathfrak{e}_m)$ for $t = -1$ and $(\pi_m \circ \pi_{m+l}^{-1}) u_{m+l}(\mathfrak{e}_m) = u_m(\mathfrak{e}_m) - [a(1+t)^{-1}b]\mathfrak{e}_m$ for $t \ne -1$ for any integer $l \ge 1$. This means that $(\phi_k \circ \pi_m^{-1})(u_m) = \langle u_m(\mathfrak{e}_m) \mid \mathfrak{e}_k\rangle \not\equiv 0$ for all $k \le m$ and that

$$\begin{aligned}
&(\phi_m \circ \pi_{m+l}^{-1})(u_{m+l}) = \langle u_m(\mathfrak{e}_m) \mid \mathfrak{e}_m\rangle \quad \text{for} \quad t = -1, \\
&(\phi_m \circ \pi_{m+l}^{-1})(u_{m+l}) = \langle u_m(\mathfrak{e}_m) \mid \mathfrak{e}_m\rangle - a(1+t)^{-1} b \,\langle \mathfrak{e}_m \mid \mathfrak{e}_m\rangle \quad \text{for} \quad t \ne -1.
\end{aligned} \tag{17}$$

Let $U(\eta)$ with $\eta = \eta(\lambda)$ be the unitary group acting over the linear complex span $\left\{\mathfrak{e}_{\imath_1}, \ldots, \mathfrak{e}_{\imath_\eta}\right\}$ in H. Let χ_η be the probability Haar measure on $U(\eta)$ and $\pi_\eta \colon \mathfrak{U} \to U(\eta)$ be the corresponding projector. Using (6) and (17), we obtain

$$\begin{aligned}
\int |\phi_\imath^\lambda(\mathfrak{u})|^2 d\chi(\mathfrak{u}) &= \lim \int |(\phi_\imath^\lambda \circ \pi_m^{-1})(u_m)|^2 d\chi_m(u_m) \\
&= \lim \int |(\phi_{\imath_1}^{\lambda_1} \circ \pi_m^{-1})(u_m) \ldots (\phi_{\imath_\eta}^{\lambda_\eta} \circ \pi_m^{-1})(u_m)|^2 d\chi_m(u_m) \\
&= \int |(\phi_{\imath_1}^{\lambda_1} \circ \pi_\eta^{-1})(u_\eta) \ldots (\phi_{\imath_\eta}^{\lambda_\eta} \circ \pi_\eta^{-1})(u_\eta)|^2 d\chi_\eta(u_\eta).
\end{aligned} \tag{18}$$

By (18) and the known integral formula for unitary groups $U(\eta)$ [28, 1.4.9], we get

$$\int |\phi_\imath^\lambda|^2 d\chi = \int \prod_{k=1}^{\eta(\lambda)} |\langle u_\eta(\mathfrak{e}_\eta) \mid \mathfrak{e}_{\imath_k}\rangle|^2 \, d\chi_\eta(u_\eta) = \frac{(\eta(\lambda)-1)!\lambda!}{(\eta(\lambda)-1+|\lambda|)!}.$$

On the other hand, the invariant property (8) provides the formula

$$\int f\, d\chi = \frac{1}{2\pi} \int d\chi(\mathfrak{u}) \int_{-\pi}^{\pi} f\left[\exp(\mathbb{i}\vartheta)\mathfrak{u}\right] d\vartheta, \qquad f \in L_\chi^\infty. \tag{19}$$

From (19) it follows the orthogonality relations $\phi_\jmath^{\lambda'} \perp \phi_\imath^\lambda$ with $|\lambda'| \neq |\lambda|$, since

$$\int \phi_\jmath^{\lambda'} \bar{\phi}_\imath^\lambda \, d\chi = \frac{1}{2\pi} \int \phi_\jmath^{\lambda'} \bar{\phi}_\imath^\lambda \, d\chi \int_{-\pi}^{\pi} \exp\left[\mathbb{i}\big(|\lambda'| - |\lambda|\big)\vartheta\right] d\vartheta = 0$$

for any $\lambda', \lambda \in \mathbb{Y}\backslash\{\emptyset\}$. Let $|\lambda'| = |\lambda|$ and $\eta(\lambda') > \eta(\lambda)$ for definiteness. Then there exists an index k with a nonzero integer λ'_k in $\lambda' = \big(\lambda'_1, \ldots, \lambda'_k, \ldots, \lambda'_{\eta(\lambda')}\big) \in \mathbb{Y}\backslash\{\emptyset\}$ such that $\eta(\lambda) < k \leq \eta(\lambda')$. In this case $\phi_\jmath^{\lambda'} \perp \phi_\imath^\lambda$ because (19) yields

$$\int \phi_\jmath^{\lambda'} \bar{\phi}_\imath^\lambda \, d\chi = \frac{1}{2\pi} \int \phi_\jmath^{\lambda'} \bar{\phi}_\imath^\lambda \, d\chi \int_{-\pi}^{\pi} \exp\left(\mathbb{i}\lambda'_k\vartheta\right) d\vartheta = 0.$$

Consider the case $|\lambda'| = |\lambda|$ and $\eta(\lambda') = \eta(\lambda)$. If $\phi_\jmath^{\lambda'} \neq \phi_\imath^\lambda$ then $\lambda' \neq \lambda$. There exists an index $0 < k \leq \eta(\lambda)$ such that $\lambda'_k \neq \lambda_k$. As above, $\phi_\jmath^{\lambda'} \perp \phi_\imath^\lambda$, because

$$\int \phi_\jmath^{\lambda'} \bar{\phi}_\imath^\lambda \, d\chi = \frac{1}{2\pi} \int \phi_\jmath^{\lambda'} \bar{\phi}_\imath^\lambda \, d\chi \int_{-\pi}^{\pi} \exp\left[\mathbb{i}(\lambda'_k - \lambda_k)\vartheta\right] d\vartheta = 0.$$

This proves that the system $\phi^{\mathbb{Y}}$ is orthogonal. □

4. Orthonormal Basis of Schur Polynomials

Let $\imath^\lambda \vdash n$, $\eta = \eta(\lambda)$ and $t_\imath = (t_{\imath_1}, \ldots, t_{\imath_\eta})$ be a complex variable. Let $t_\imath^\lambda := \prod t_{\imath_j}^{\lambda_j}$. The n-homogenous Schur polynomial is defined (see, e.g. [18]) to be

$$s_\imath^\lambda(t_\imath) := D_\lambda(t_\imath)/\Delta(t_\imath) \quad \text{where} \quad D_\lambda(t_\imath) = \det\left[t_{\imath_i}^{\lambda_j+\eta-j}\right] \text{ with } \lambda_j = 0 \text{ for } j > \eta,$$

$\Delta(t_\imath) = \prod_{1\leq i<j\leq\eta}(t_{\imath_i} - t_{\imath_j})$ is Vandermonde's determinant. It can be written as $s_\imath^\lambda(t_\imath) = \sum_{[\imath^\lambda]} t_\imath^\lambda$ with summation over all semistandard Young tabloids [8, I.2.2].

We construct an orthonormal basis in L_χ^2 consisting of Schur polynomials on Paley–Wiener maps. Assign (uniquely) to $\imath \in \mathscr{I}_\eta$ the vector $\phi_\imath := \big(\phi_{\imath_1}, \ldots, \phi_{\imath_\eta}\big)$. Let $s_\imath^\lambda(\mathfrak{u}) = (s_\imath^\lambda \circ \phi_\imath)(\mathfrak{u})$ be n-homogeneous functions of variable $\mathfrak{u} \in \mathfrak{U}$ with $\lambda \in \mathbb{N}^\eta$, defined by the formulas (3). Denote

$$s_n^{\mathbb{Y}} := \bigcup \left\{s_\imath^\lambda \colon \imath^\lambda \vdash n\right\}, \quad s^{\mathbb{Y}} := \bigcup \left\{s_n^{\mathbb{Y}} \colon n \in \mathbb{N}_0\right\} \quad \text{with} \quad s_0 = s_\imath^\emptyset \equiv 1.$$

Theorem 1. *The system of Schur polynomials $s^{\mathbb{Y}}$ forms an orthonormal basis in L^2_χ and $s^{\mathbb{Y}}_n$ is the same basis in $L^{2,n}_\chi$. The following orthogonal decomposition holds,*

$$L^2_\chi = \mathbb{C} \oplus L^{2,1}_\chi \oplus L^{2,2}_\chi \oplus \dots. \tag{20}$$

For any $h \in H$ the equality (2) uniquely defines the conjugate-linear embedding

$$\phi \colon H \ni h \longmapsto \phi_h \in L^2_\chi \quad \textit{such that} \quad \|\phi_h\|_\chi = \|h\|. \tag{21}$$

Proof. Let $U(\eta)$ be the unitary group over the linear complex span $\{\mathfrak{e}_{\imath_1}, \dots, \mathfrak{e}_{\imath_\eta}\}$ with $\eta = \eta(\lambda)$. Taking into account (17) similarly as (18), we obtain

$$\int s^\lambda_\imath \bar{s}^\mu_\imath \, d\chi = \int s^\lambda_\imath(z_\eta)\, \bar{s}^\mu_\imath(z_\eta)\, d\chi_\eta(z_\eta) = \delta_{\lambda\mu}$$

for all $[\imath^\lambda]$, $[\imath^\mu]$ with $\imath = (\imath_1, \dots, \imath_\eta)$ and $\lambda, \mu \in \mathbb{N}^\eta$. In fact, the corresponding Schur polynomials $\{s^\lambda_\imath \colon \lambda \in \mathbb{N}^\eta\}$ are characters of the group $U(\eta)$. Hence, by the Weyl integration formula, the right-hand side integral is equal to Kronecker's delta $\delta_{\lambda\mu}$ [26, Thm 8.3.2 & Thm 11.9.1].

The family of finite alphabets $\imath \in \mathscr{I}$ is directed and for any $\imath, \imath'$ there exists $\imath''$ such that $\imath \cup \imath' \subset \imath''$. This means that the whole system $s^{\mathbb{Y}}_n$ is orthonormal in L^2_χ.

The property $s^\mu_\jmath \perp s^\lambda_\imath$ with $|\mu| \neq |\lambda|$ for any $\imath, \jmath \in \mathscr{I}$ follows from (19), since

$$\int s^\mu_\jmath \bar{s}^\lambda_\imath \, d\chi = \frac{1}{2\pi} \int s^\mu_\jmath \bar{s}^\lambda_\imath \, d\chi \int_{-\pi}^{\pi} \exp\big(\mathbb{i}(|\mu| - |\lambda|)\vartheta\big)\, d\vartheta = 0$$

for all $\lambda \in \mathbb{Y}$ and $\mu \in \mathbb{Y}\backslash\{\emptyset\}$. This yields $L^{2,|\mu|}_\chi \perp L^{2,|\lambda|}_\chi$ in the space L^2_χ. Taking $\lambda = \emptyset$ with $|\emptyset| = 0$, we get $1 \perp L^{2,|\mu|}_\chi$ for all $\mu \in \mathbb{Y}\backslash\{\emptyset\}$. Hence, (20) is proved.

By Lemma 2 the subsystem $\phi_k = s^1_k$ is orthonormal in L^2_χ, hence by Definition 1 it instantly follows that $\|\phi_h\|^2_\chi = \sum |\mathfrak{e}^*_k(h)|^2 \int |\phi_k|^2 d\chi = \|h\|^2$. It follows the isometric embedding (21).

The set $\check{U}(m)$ of matrices with no eigenvalue $\{-1\}$ has Stone–Čech compactification $\tilde{U}(m)$ such that the mapping $\check{\pi}^{m+1}_m$ has a continuous $U(m)$-valued extension

$$\tilde{\pi}^{m+1}_m \colon \tilde{U}(m+1) \longrightarrow U(m).$$

This fact follows from [33, Thm 19.5] by virtue of that $U(m)$ is compact. Hence, the projective limit $\tilde{\mathfrak{U}} := \varprojlim \tilde{U}(m)$, determined by $\tilde{\pi}^{m+1}_m$, is a compact set in $\mathfrak{U}$ with continuous $U(m)$-valued projections $\tilde{\pi}_m \colon \tilde{\mathfrak{U}} \to U(m)$.

Since $U(\infty)$ on H acts irreducibly, for any $\mathfrak{u}' \neq \mathfrak{u}''$ there is m such that

$$\phi_m(\mathfrak{u}') = \langle \pi_m(\mathfrak{u}')(\mathfrak{e}_m) \mid \mathfrak{e}_m \rangle \neq \langle \pi_m(\mathfrak{u}'')(\mathfrak{e}_m) \mid \mathfrak{e}_m \rangle = \phi_m(\mathfrak{u}''),$$

i.e., $\phi^{\mathbb{Y}}$ separates $\mathfrak{U}$ and so $\tilde{\mathfrak{U}}$. Hence, the system of Schur polynomials $s^{\mathbb{Y}}$ also separates $\tilde{\mathfrak{U}}$. Moreover, each complex-conjugate function $\bar{\phi}_m(\mathfrak{u}) = \langle \mathfrak{e}_m \mid \pi_m(\mathfrak{u}) (\mathfrak{e}_m) \rangle = \langle \pi_m(\mathfrak{u}^\star)(\mathfrak{e}_m) \mid \mathfrak{e}_m \rangle$ belongs to $\phi^{\mathbb{Y}}$. Thus, by the Stone–Weierstrass approximation theorem the complex linear span of polynomials $\phi^{\mathbb{Y}}$, as well as, of $s^{\mathbb{Y}}$, forms a dense subspace in the Banach space of all continuous functions $C(\tilde{\mathfrak{U}})$.

Let $\tilde{\chi}_m$ means the image of χ_m under $\check{U}(m) \looparrowright U(m)$. In Lemma 1 it inductively was shown that for every $\varepsilon > 0$ there exists a compact set $\varprojlim K_m \subset \check{\mathfrak{U}}$ such that

$$\tilde{\chi}_m(K_m) \geq 1 - \varepsilon \quad \text{for all} \quad m$$

where $\tilde{\chi}_m(K_m) = \check{\chi}_m(K_m) = \chi_m(K_m)$, by definition of the measure $\tilde{\chi}_m$ as an image. Hence, by the Prokhorov theorem the projective limit $\tilde{\chi} = \varprojlim \tilde{\chi}_m$, defined by mappings $\tilde{\pi}_m^{m+1}$, possesses the properties

$$\tilde{\chi}(\Omega) = \inf \tilde{\chi}_m(\Omega) = \inf \chi_m(\Omega) = \varprojlim \chi_m(\Omega) = \chi(\Omega)$$

for all Borel Ω in $\check{\mathfrak{U}}$ or otherwise $\tilde{\chi}|_{\check{\mathfrak{U}}} = \chi|_{\check{\mathfrak{U}}}$. Consequently,

$$\tilde{\chi}|_{\check{\mathfrak{U}}} = \chi|_{\check{\mathfrak{U}}} = \chi|_{\check{\mathfrak{U}} \bigsqcup (\mathfrak{U} \setminus \check{\mathfrak{U}})} = \chi|_{\mathfrak{U}} \quad \text{since} \quad \chi(\mathfrak{U} \setminus \check{\mathfrak{U}}) = 0.$$

In particular, $\tilde{\chi} = \varprojlim \tilde{\chi}_m$ is regular on $\tilde{\mathfrak{U}}$ by the Riesz–Markov theorem [20, 1.1].

As a consequence, the space L^2_χ coincides with the completion of $C(\tilde{\mathfrak{U}})$ and for any $f \in L^2_\chi$ there exists a sequence $(f_n) \subset \operatorname{span}(s^{\mathbb{Y}})$ such that $\int |f - f_n|^2 d\chi \to 0$. Hence, the system $s^{\mathbb{Y}}$ forms an orthogonal basis in L^2_χ.

Finally, $s_n^{\mathbb{Y}} \cap L^2_\chi$ is total in $L^{2,n}_\chi$ and $s_n^{\mathbb{Y}} \perp s_m^{\mathbb{Y}}$ if $n \neq m$. This yields (20). $\square$

5. Unitarily-Weighted Symmetric Fock Space

Define on the tensor power $H^{\otimes n}$ the unitarily-weighted norm $\|\cdot\|_{H_\beta^{\otimes n}} = \langle \cdot \mid \cdot \rangle^{1/2}_{H_\beta^{\otimes n}}$ where the inner product $\langle \cdot \mid \cdot \rangle^{1/2}_{H_\beta^{\otimes n}}$ is determined by the relations

$$\langle \mathfrak{e}_\imath^{\otimes\lambda} \mid \mathfrak{e}_{\imath'}^{\otimes\lambda'} \rangle_{H_\beta^{\otimes n}} = \begin{cases} \dfrac{(\eta-1)!}{(\eta-1+n)!} & : \lambda = \lambda' \text{ and } \imath = \imath' \\ 0 & : \lambda \neq \lambda' \ \text{ or } \ \imath \neq \imath'. \end{cases} \tag{22}$$

Here $\mathfrak{e}_\imath^{\otimes\lambda} := \sigma'(\mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \otimes \ldots \otimes \mathfrak{e}_{\imath_\eta}^{\otimes\lambda_\eta})$ with $\eta = \eta(\lambda)$ and $\sigma' \in S_n$ is fixed. Let $H_\beta^{\otimes n}$ be the completion of $\{H^{\otimes n}, \|\cdot\|_{H_\beta^{\otimes n}}\}$. Its closed subspace, defined by the projection

$$\mathcal{S}_n \colon H_\beta^{\otimes n} \ni \mathfrak{e}_\imath^{\otimes\lambda} \longmapsto \mathfrak{e}_\imath^{\odot\lambda} = (n!)^{-1} \sum_{\sigma \in S_n} \sigma(\mathfrak{e}_\imath^{\otimes\lambda})$$

forms an unitarily-weighted symmetric tensor power $H_\beta^{\odot n} \subset H_\beta^{\otimes n}$ with the inner product determined by relations $\langle \mathfrak{e}_\imath^{\odot\lambda} \mid \mathfrak{e}_{\imath'}^{\odot\lambda'} \rangle_{H_\beta^{\otimes n}} = \beta_\lambda \langle \mathfrak{e}_\imath^{\odot\lambda} \mid \mathfrak{e}_{\imath'}^{\odot\lambda'} \rangle$ or more specific

$$\langle \mathfrak{e}_{\imath}^{\odot\lambda} \mid \mathfrak{e}_{\imath'}^{\odot\lambda'} \rangle_{H_\beta^{\otimes n}} = \begin{cases} \dfrac{\lambda!}{n!}\dfrac{(\eta-1)!}{(\eta-1+n)!} & : \lambda = \lambda \ \text{ and } \imath = \imath' \\ 0 & : \lambda \neq \lambda' \ \text{ or } \ \imath \neq \imath'. \end{cases} \tag{23}$$

Definition 2. The *unitarily-weighted symmetric Fock space* is defined to be the Hilbertian orthogonal sum $\mathit{\Gamma}_\beta(H) = \bigoplus_{n\ge 0} H_\beta^{\odot n}$ of elements $\psi = \bigoplus \psi_n$, $\psi_n \in H_\beta^{\odot n}$ with the orthogonal basis $\mathfrak{e}^{\mathbb{Y}} = \bigcup \left\{\mathfrak{e}^{\mathbb{Y}_n} : n \in \mathbb{N}_0\right\}$ and the following inner product and norm

$$\langle \psi \mid \psi' \rangle_\beta = \sum n! \langle \psi_n \mid \psi'_n \rangle_{H_\beta^{\otimes n}}, \quad \|\psi\|_\beta = \langle \psi \mid \psi \rangle_\beta^{1/2}.$$

We immediately notice that $\|h\|_\beta^2 = \sum |\mathfrak{e}_i^*(h)|^2 = \|h\|^2$ for all $h = \sum \mathfrak{e}_i \mathfrak{e}_i^*(h) \in H$.

Lemma 3. *The set of coherent states* $\{\exp(h) \colon h \in H\}$ *is total in* $\mathit{\Gamma}_\beta(H)$ *and the expansion* (13) *is convergent in* $\mathit{\Gamma}_\beta(H)$*. The injections*

$$\mathit{\Gamma}(H) \looparrowright \mathit{\Gamma}_\beta(H) \quad \textit{and} \quad H^{\odot n} \looparrowright H_\beta^{\odot n}$$

are contractive and dense. The $\mathit{\Gamma}_\beta(H)$*-valued function* $H \ni h \longmapsto \exp(h)$ *is entire analytic. The shift group, defined to be*

$$\mathcal{T}_a \exp(h) := \exp(h+a) = \exp(\partial_a)\exp(h) \quad \textit{with} \quad \partial_a \exp(h) = \frac{d\exp(h+za)}{dz}\Big|_{z=0}$$

for $a, h \in H$*, has a unique linear extension* $\mathcal{T}_a \colon \mathit{\Gamma}_\beta(H) \ni \psi \longmapsto \mathcal{T}_a\psi \in \mathit{\Gamma}_\beta(H)$ *such that*

$$\|\mathcal{T}_a\psi\|_\beta^2 \le \exp\left(\|a\|^2\right)\|\psi\|_\beta^2 \quad \textit{and} \quad \mathcal{T}_{a+b} = \mathcal{T}_a\mathcal{T}_b = \mathcal{T}_b\mathcal{T}_a, \quad a, b \in H. \tag{24}$$

Proof. Taking into account that $\beta_\lambda \le 1$, we get the following inequalities

$$\|h^{\otimes n}\|_{H_\beta^{\otimes n}}^2 = \sum_{\imath^\lambda \vdash n} \left(\frac{n!}{\lambda!}\right)^2 \|\mathfrak{e}_\imath^{\odot\lambda}\|_{H_\beta^{\otimes n}}^2 |\mathfrak{e}_\imath^{*\lambda}(h)|^2 = \sum_{\imath^\lambda \vdash n} \beta_\lambda \frac{n!}{\lambda!} |\mathfrak{e}_\imath^{*\lambda}(h)|^2 \le \|h^{\otimes n}\|^2 = \|h\|^{2n},$$

$$\|\exp(h)\|_\beta^2 = \sum_{n\ge 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \beta_\lambda \frac{n!}{\lambda!} |\mathfrak{e}_\imath^{*\lambda}(h)|^2 \overset{(15)}{\le} \exp\|h\|^2 \overset{(14)}{=} \|\exp(h)\|_{\mathit{\Gamma}}^2.$$

Hence, (12), (13) are convergent in $\mathit{\Gamma}_\beta(H)$. This implies that $h \mapsto \exp(h)$ is analytic and inclusions $\mathit{\Gamma}(H) \looparrowright \mathit{\Gamma}_\beta(H)$ and $H^{\odot n} \looparrowright H_\beta^{\odot n}$ are contractive. By the polarization formula (11) their ranges are dense.

Using the binomial formula $(h+za)^{\otimes n} = \bigoplus_{m=0}^n \binom{n}{m} (za)^{\otimes m} \odot h^{\otimes(n-m)}$, we find

$$\partial_a^m \exp(h) = \frac{d^m \exp(h+za)}{dz^m}\Big|_{z=0} = \bigoplus_{n\ge m} \frac{\mathcal{S}_{n/m}[a^{\otimes m} \otimes h^{\otimes(n-m)}]}{(n-m)!}, \quad z \in \mathbb{C}$$

with the orthogonal projector $\mathcal{S}_{n/m}$ defined as $\psi_m \odot \psi_{n-m} = \mathcal{S}_{n/m}\left(\psi_m \otimes \psi_{n-m}\right) \in H_\beta^{\odot n}$ for all $\psi_m \in H_\beta^{\odot m}$ and $\psi_{n-m} \in H_\beta^{\odot(n-m)}$. By orthogonality $\|\mathcal{S}_{n/m}\| \le 1$.

Applying the expansions (12) to $a^{\otimes m}$ and $h^{\otimes(n-m)}$, by (22), we get

$$\|a^{\otimes m}\otimes h^{\otimes(n-m)}\|^2_{H^{\otimes n}_\beta}=\sum_{\substack{\imath^\lambda\vdash m\\ \jmath^\mu\vdash(n-m)}}\Big(\frac{m!}{\lambda!}\frac{(n-m)!}{\mu!}\Big)^2\|\mathfrak{e}^{\odot\lambda}_\imath\otimes\mathfrak{e}^{\odot\mu}_\jmath\|^2_{H^{\otimes n}_\beta}|\mathfrak{e}^{*\lambda}_\imath(a)|^2|\mathfrak{e}^{*\mu}_\jmath(h)|^2$$

with summations over semistandard tableaux $[\imath^\lambda]$, $[\jmath^\mu]$ and $\imath,\jmath\in\mathscr{I}$. Let $(\lambda,\mu)\in\mathbb{N}^{\eta(\lambda,\mu)}$ be the smallest partition of number n with the length $\eta(\lambda,\mu)$ containing the partitions λ for m and μ for $n-m$. Then $\eta(\lambda,\mu)\geq\max\{\eta(\lambda),\eta(\mu)\}$ and so

$$\|\mathfrak{e}^{\odot\lambda}_\imath\otimes\mathfrak{e}^{\odot\mu}_\jmath\|^2_{H^{\otimes n}_\beta}=\frac{(\eta(\lambda,\mu)-1)!}{(\eta(\lambda,\mu)-1+n)!}\leq\min\{\beta_\lambda,\beta_\mu\},$$

since $\frac{(\eta-1)!}{(\eta-1+n)!}$ is decreasing in variable η. Thus, the following inequality

$$\begin{aligned}\|a^{\otimes m}\otimes h^{\otimes(n-m)}\|^2_{H^{\otimes n}_\beta}&\leq\sum_{\substack{\imath^\lambda\vdash m\\ \jmath^\mu\vdash(n-m)}}\Big(\frac{m!}{\lambda!}\frac{(n-m)!}{\mu!}\Big)^2\min\{\beta_\lambda,\beta_\mu\}|\mathfrak{e}^{*\lambda}_\imath(a)|^2|\mathfrak{e}^{*\mu}_\jmath(h)|^2\\&=\|a^{\otimes m}\|^2\|h^{\otimes(n-m)}\|^2_{H^{\otimes(n-m)}_\beta}=\|a\|^{2m}\|h^{\otimes(n-m)}\|^2_{H^{\otimes(n-m)}_\beta}\end{aligned}$$

holds. Using this inequality and that $\|\mathcal{S}_{n/m}\|\leq1$, we find

$$\begin{aligned}\|\partial^m_a\exp(h)\|^2_\beta&=\sum_{n\geq m}\frac{\|\mathcal{S}_{n/m}[a^{\otimes m}\otimes h^{\otimes(n-m)}]\|^2_\beta}{(n-m)!}\leq\sum_{n\geq m}\frac{\|\mathcal{S}_{n/m}\|^2\|a^{\otimes m}\otimes h^{\otimes(n-m)}\|^2_\beta}{(n-m)!}\\&\leq\|a^{\otimes m}\|^2\sum_{n\geq m}\frac{\|\mathcal{S}_{n/m}\|^2\|h^{\otimes(n-m)}\|^2_\beta}{(n-m)!}\leq\|a\|^{2m}\|\exp(h)\|^2_\beta.\end{aligned}$$

Summing with coefficients $1/m!$, we get $\|\mathcal{T}_a\exp(h)\|^2_\beta\leq\exp\big(\|a\|^2\big)\|\exp(h)\|^2_\beta$. This inequality and totality of $\{\exp(x)\colon h\in H\}$ in $\Gamma_\beta(H)$ yield the required inequality (24). It also follows that $\Gamma_\beta(H)$ is invariant under $\mathcal{T}_a$ and that the group property (24) holds, since $\partial_{a+b}=\partial_a+\partial_b$ for all $a,b\in H$ by linearity. □

Lemma 4. *The mapping $\phi\colon H\ni h\longmapsto\phi_h\in L^2_\chi$, extended onto $\mathcal{T}_a\exp(h)$ as*

$$\varPhi\colon\mathcal{T}_a\exp(h)\longmapsto\sum_{n\geq0}\frac{1}{n!}\sum_{\imath^\lambda\vdash n}\frac{n!}{\lambda!}\phi^\lambda_\imath\mathfrak{e}^{*\lambda}_\imath(h+a),\quad a\in H,$$

has the unique isometric conjugate-linear extension

$\varPhi\colon\Gamma_\beta(H)\ni\psi\longmapsto\varPhi\psi\in L^2_\chi$ *with the adjoint mapping* $\varPhi^*\colon L^2_\chi\to\Gamma_\beta(H)$

defined to be $\langle\varPhi\mathfrak{e}^{\odot\lambda}_\imath\mid f\rangle_\chi=\langle\mathfrak{e}^{\odot\lambda}_\imath\mid\varPhi^*f\rangle_\beta$ for all $f\in L^2_\chi$ in such way that

$$\varPhi\colon\mathfrak{e}^{\odot\lambda}_\imath/\|\mathfrak{e}^{\odot\lambda}_\imath\|_\beta\longmapsto\phi^\lambda_\imath/\|\phi^\lambda_\imath\|_\chi\quad\text{for all}\quad\lambda\in\mathbb{Y},\ \imath\in\mathscr{I}_{\eta(\lambda)}.$$

As a result, the conjugate-linear isometries $\Gamma_\beta(H)\overset{\varPhi}{\simeq}L^2_\chi$ and $H^{\odot n}_\beta\overset{\varPhi}{\simeq}L^{2,n}_\chi$ hold.

Proof. By Lemma 3 the $\varGamma_\beta(H)$-valued function $H \ni h \mapsto \mathcal{T}_a \exp(h)$ is well defined for all $a \in H$. Let us use the expansion $\phi_{h+a} = \sum \mathfrak{e}_i^*(h+a)\phi_i$. By Lemma 2 and Theorem 1, $\phi\colon H \ni h \longmapsto \phi_h \in L^2_\chi$ may be extended to $\varPhi$ in following way

$$\begin{aligned}\varPhi\mathcal{T}_a \exp(h) &= \sum_{n\geq 0}\frac{1}{n!}\sum_{\imath^\lambda\vdash n}\frac{n!}{\lambda!}\phi_\imath^\lambda \mathfrak{e}_\imath^{*\lambda}(h+a) = \prod_{i\geq 0}\sum_{n\geq 0}\frac{\phi_i^n}{n!}\mathfrak{e}_i^{*n}(h+a)\\ &= \prod \exp\left(\phi_i\mathfrak{e}_i^*(h+a)\right) = \exp\left(\phi_{h+a}\right) \quad \text{where}\\ \varPhi[(h+a)^{\odot n}] &= \phi_{h+a}^n = \sum_{\imath^\lambda\vdash n}\frac{n!}{\lambda!}\phi_\imath^\lambda\mathfrak{e}_\imath^{*\lambda}(h+a), \quad a\in H\end{aligned}$$

is an orthogonal component of $\varPhi\mathcal{T}_a \exp(h)$ in L^2_χ. It follows that

$$\begin{aligned}\|\exp(\phi_{h+a})\|^2_\chi &= \sum_{n\geq 0}\frac{1}{n!^2}\sum_{\imath^\lambda\vdash n}\|\phi_\imath^\lambda\|^2_\chi\frac{n!^2}{\lambda!^2}|\mathfrak{e}_\imath^{*\lambda}(h+a)|^2\\ &= \sum_{n\geq 0}\frac{1}{n!^2}\sum_{\imath^\lambda\vdash n}\frac{n!^2}{\lambda!}\beta_\lambda|\mathfrak{e}_\imath^{*\lambda}(h+a)|^2 \leq \sum_{n\geq 0}\frac{1}{n!}\sum_{\imath^\lambda\vdash n}\frac{n!}{\lambda!}|\mathfrak{e}_\imath^{*\lambda}(h+a)|^2\\ &= \prod \exp|\mathfrak{e}_i^*(h+a)|^2 = \exp\|h+a\|^2.\end{aligned}$$

Hence, the composition $\mathfrak{U} \ni \mathfrak{u} \longmapsto [\varPhi\exp(h+a)](\mathfrak{u})$ is well defined in L^2_χ.

Now, we consider the ordinary irreducible representation of permutation group S_n on the Specht λ-module $S_\imath^\lambda$ that is corresponded to the standard Young tableau $[\imath^\lambda]$. The following known hook formula (see [8, I.4.3]) holds,

$$\hbar_\lambda := n!\Big(\prod_{i\leq\lambda_j} h(i,j)\Big)^{-1} \quad \text{where} \quad \hbar_\lambda = \dim S_\imath^\lambda, \tag{25}$$

with $h(i,j) = \#\big\{\square_{i'j'} \in [\imath^\lambda]\colon i' \geq i, j' = j\big\} = \#\big\{\square_{i'j'} \in [\imath^\lambda]\colon i' = i, j' \geq j\big\}$ independed of $\imath \in \mathscr{I}$. Assign to $\imath \in \mathscr{I}_\eta$ the vectors

$$\Big(\phi_{\imath_1}(\mathfrak{u})\mathfrak{e}^*_{\imath_1}(h),\ldots,\phi_{\imath_\eta}(\mathfrak{u})\mathfrak{e}^*_{\imath_\eta}(h)\Big) := t_\imath(\mathfrak{u},h).$$

Let $s_\imath^\lambda(\mathfrak{u},h) := s_\imath^\lambda(t_\imath)$ with $t_\imath = t_\imath(\mathfrak{u},h)$ for all $\mathfrak{u}\in\mathfrak{U}$, where polynomial terms are $\phi_\imath^\lambda(\mathfrak{u})\mathfrak{e}_\imath^{*\lambda}(h) = \phi_{\imath_1}^{\lambda_1}(\mathfrak{u})\mathfrak{e}_{\imath_1}^{*\lambda_1}(h)\ldots\phi_{\imath_\eta}^{\lambda_\eta}(\mathfrak{u})\mathfrak{e}_{\imath_\eta}^{*\lambda_\eta}(h)$. Applying the Frobenius formula [18, I.7] and taking into account (2), (3), (25), we obtain

$$\phi_h^n(\mathfrak{u}) = \sum_{\imath^\lambda\vdash n}\hbar_\lambda s_\imath^\lambda(\mathfrak{u},h), \quad h\in H$$

where $s_\imath^\lambda = 0$ if $\lambda_1^\intercal > l_\lambda$ and the summation is over all standard tabloids. Hence, $\big\{\phi_h^n\colon h\in H\big\}$ is total in $L^{2,n}_\chi$ by Theorem 1. In consequence, $\{\exp(\phi_h)\colon h\in H\}$ is total in L^2_χ. This yields surjectivity of $\varPhi$ and of all its restrictions to $H_\beta^{\odot n}$. □

Corollary 2. *The sets* $\big\{\phi_h^n\colon h\in H\big\}$ *in* $L^{2,n}_\chi$ *and* $\{\exp\phi_h\colon h\in H\}$ *in* L^2_χ *are total.*

6. Fourier Analysis on Virtual Unitary Matrices

Consider the isometry $H_\beta^{*\odot n} \overset{\mathcal{P}}{\simeq} P_\beta^n(H)$ (see e.g., [7, 1.6]), where the space $P_\beta^n(H)$ of unitarily-weighted n-homogeneous Hilbert–Schmidt polynomials of variable $h \in H$ is defined to be a restriction to the diagonal in $H \times \ldots \times H$ of the n-linear forms $\mathcal{P} \circ \psi_n$ endowed with the norm $\|\psi_n^*\|_{P_\beta^n} = \|\psi_n\|_{H_\beta^{\otimes n}}$ where

$$\psi_n^*(h) := \langle h^{\otimes n} \mid \psi_n \rangle_{H_\beta^{\otimes n}} \simeq \langle (h, \ldots, h) \mid \mathcal{P} \circ \psi_n \rangle, \quad \psi_n \in H_\beta^{\odot n}.$$

Let $H_\beta^2 = \sum_{n \geq 0} P_\beta^n(H)$ be the direct sum of functions $\psi^*(h) = \sum \psi_n^*(h)$ of variable $h \in H$ with summands $\psi_n^* = \mathcal{P} \circ \psi_n \in P_\beta^n(H)$ where $\psi = \sum \psi_n \in \Gamma_\beta(H)$. Since the set $\{\exp(h) \colon h \in H\}$ is total in $\Gamma_\beta(H)$, elements of H_β^2 can be written as

$$H_\beta^2 = \left\{ \psi^*(h) = \langle \exp(h) \mid \psi \rangle_\beta : \psi = \sum \psi_n \in \Gamma_\beta(H) \right\}.$$

The analyticity of $H \ni h \mapsto \psi^*(h)$ is a result of the composition $\exp(\cdot)$ and $\psi^*(\cdot)$.

Definition 3. Let H_β^2 be defined as a Hardy space of unitarily-weighted Hilbert–Schmidt analytic functions $\psi^*(h)$ of variable $h \in H$ endowed with the inner product

$$\langle \psi^*(\cdot) \mid \varphi^*(\cdot) \rangle_{H_\beta^2} := \langle \varphi \mid \psi \rangle_\beta \quad \text{where} \quad \|\psi^*\|_{H_\beta^2}^2 = \langle \psi^*(\cdot) \mid \psi^*(\cdot) \rangle_{H_\beta^2} = \sum n! \|\psi_n^*\|_{P_\beta^n}^2.$$

The conjugate-linear surjective isometry from H_β^2 onto $\Gamma_\beta(H)$ is realized by the conjugate-linear mapping

$$* \colon \Gamma_\beta(H) \ni \psi \longmapsto \psi^* \in H_\beta^2, \quad \psi = \sum \psi_n.$$

On the other hand, the correspondence $\Phi \colon \mathfrak{e}_\imath^{\odot \lambda} \rightleftarrows \phi_\imath^\lambda$ with $\lambda \in \mathbb{Y}$ and $\imath \in \mathscr{I}_{\eta(\lambda)}$ allows us to determine the conjugate-linear isometry from $\Gamma_\beta(H)$ onto L_χ^2. As a result, the mapping

$$\Psi \colon H_\beta^2 \ni \mathfrak{e}_\imath^{*\lambda} / \|\mathfrak{e}_\imath^{\odot \lambda}\|_\beta \longmapsto \phi_\imath^\lambda / \|\phi_\imath^\lambda\|_\chi \in L_\chi^2$$

defines the surjective isometry

$$\Psi \colon H_\beta^2 \longrightarrow L_\chi^2 \quad \text{and its adjoint} \quad \Psi^* \colon L_\chi^2 \longrightarrow H_\beta^2.$$

Lemma 5. *The systems of Hilbert–Schmidt polynomials of variable $h \in H$,*

$$\mathfrak{e}^{*\mathbb{Y}_n} := \bigcup \left\{ \mathfrak{e}_\imath^{*\lambda} \colon \imath^\lambda \vdash n, \imath \in \mathscr{I} \right\} \quad \text{and} \quad \mathfrak{e}^{*\mathbb{Y}} := \bigcup \left\{ \mathfrak{e}^{*\mathbb{Y}_n} : n \in \mathbb{N}_0 \right\}$$

where $\mathfrak{e}_\imath^{*\emptyset} = 1$, form orthogonal bases in $P_\beta^n(H)$ and H_β^2, respectively, such that

$$\|\mathfrak{e}_\imath^{*\lambda}\|_{P_\beta^n}^2 = \beta_\lambda \|\mathfrak{e}_\imath^{\odot \lambda}\|^2 = \frac{(\eta(\lambda) - 1)!}{(\eta(\lambda) - 1 + n)!} \frac{\lambda!}{n!}, \quad \imath^\lambda \vdash n.$$

Every function $\psi^* \in H^2_\beta$ with $\psi \in \mathit{\Gamma}_\beta(H)$ has the expansion with respect to $\mathfrak{e}^{*\mathbb{Y}}$

$$\psi^*(h) = \langle \exp(h) \mid \psi \rangle_\beta = \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \mathfrak{e}^{*\lambda}_\imath(h) \big\langle \mathfrak{e}^{\odot\lambda}_\imath \mid \psi_n \big\rangle_\beta \tag{26}$$

with summation in the inner sum over all semistandard tabloids $[\imath^\lambda]$ such that $\imath^\lambda \vdash n$. Each function $\psi^* \in H^2_\beta$ is entire Hilbert–Schmidt analytic and can be also written as

$$\begin{gathered}\psi^*(h) = \big\langle \psi^*(\cdot) \mid \exp\langle \cdot \mid h\rangle \big\rangle_{H^2_\beta} = \big\langle \psi^*(\cdot) \mid E(\cdot, h) \big\rangle_{H^2_\beta}, \quad \psi \in \mathit{\Gamma}_\beta(H) \\ \text{where} \quad E(h', h) := |\exp\langle h' \mid h\rangle|^2 / \exp\langle h \mid h\rangle \quad \text{for all} \quad h \in H.\end{gathered} \tag{27}$$

The following linear isometries, defined by linearization via coherent states, hold

$$H^2_\beta \overset{\mathit{\Psi}}{\simeq} L^2_\chi, \quad P^n_\beta(H) \overset{\mathit{\Psi}}{\simeq} L^{2,n}_\chi. \tag{28}$$

Proof. Taking into account (13) and (23), we conclude that every $\psi^* \in H^2_\beta$ such that $\psi = \bigoplus \psi_n \in \mathit{\Gamma}_\beta(H)$ with $\psi_n \in H^{\odot n}_\beta$ has the following expansion

$$\psi^*(h) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \mathfrak{e}^{*\lambda}_\imath(h) \langle \mathfrak{e}^{\odot\lambda}_\imath \mid \psi_n \rangle_\beta \quad \text{where} \quad \psi = \bigoplus_{n\geq 0} \sum_{\imath^\lambda \vdash n} \frac{\langle \mathfrak{e}^{\odot\lambda}_\imath \mid \psi_n \rangle_\beta}{\|\mathfrak{e}^{\odot\lambda}_\imath\|^2_\beta} \mathfrak{e}^{\odot\lambda}_\imath.$$

On the other hand, in relative to the inner product $\langle \cdot \mid \cdot \rangle_\mathit{\Gamma}$, we have

$$\exp\langle h' \mid h\rangle = \bigoplus_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \mathfrak{e}^{*\lambda}_\imath(h')\, \bar{\mathfrak{e}}^{*\lambda}_\imath(h) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{\mathfrak{e}^{*\lambda}_\imath(h') \bar{\mathfrak{e}}^{*\lambda}_\imath(h)}{\|\mathfrak{e}^{\odot\lambda}_\imath\|^2}.$$

Verify the first equality in (27) by substituting (26) into the formula (27). We get

$$\begin{aligned}\psi^*(h) &= \Big\langle \sum_{n\geq 0} \sum_{\imath^\lambda \vdash n} \frac{\langle \mathfrak{e}^{\odot\lambda}_\imath \mid \psi_n \rangle_\beta}{\|\mathfrak{e}^{\odot\lambda}_\imath\|^2_\beta} \mathfrak{e}^{*\lambda}_\imath(h') \mid \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{\mathfrak{e}^{*\lambda}_\imath(h') \bar{\mathfrak{e}}^{*\lambda}_\imath(h)}{\|\mathfrak{e}^{\odot\lambda}_\imath\|^2} \Big\rangle_{H^2_\beta} \\ &= \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \mathfrak{e}^{*\lambda}_\imath(h) \langle \mathfrak{e}^{\odot\lambda}_\imath \mid \psi_n \rangle_\beta = \langle \exp(h) \mid \psi \rangle_\beta \,.\end{aligned}$$

If $\omega^*(h') := \psi^*(h) \exp\langle h \mid h'\rangle [\exp\langle h' \mid h'\rangle]^{-1}$ then $\omega^*(h) = \psi^*(h)$ for $h = h' \in H$. Now, putting $\omega^*(h') := \big\langle \psi^*(\cdot) \mid \exp\langle h' \mid \cdot\rangle [\exp\langle h' \mid h'\rangle]^{-1} \exp\langle \cdot \mid h'\rangle \big\rangle_{H^2_\beta}$, we obtain

$$\begin{aligned}\psi^*(h) &= \omega^*(h) = \langle \omega^* \mid \exp(\cdot \mid h) \rangle_{H^2_\beta} \\ &= \big\langle \psi^*(\cdot) \mid \exp(h \mid \cdot) [\exp(h \mid h)]^{-1} \exp(\cdot \mid h) \big\rangle_{H^2_\beta} = \big\langle \psi^*(\cdot) \mid E(\cdot, h) \big\rangle_{H^2_\beta}.\end{aligned}$$

Hence, the second equality in (27) holds. Lemma 4 yields (28). □

Remark 1. Since $\phi_h = \sum \mathfrak{e}^*_i(h)\phi_i$ for all $h = \sum \mathfrak{e}^*_i(h)\mathfrak{e}_i$, a range of the embedding (21) coincides with $L^{2,1}_\chi$.

Lemma 6. *Denote* $\exp\langle h' \mid h\rangle := K(h', h)$. *The functions*

$$H \ni h \longmapsto (\Psi \circ K)(\mathfrak{u}, h) \quad \textit{and} \quad H \ni h \longmapsto (\Psi \circ E)(\mathfrak{u}, h)$$

with $\mathfrak{u} \in \mathfrak{U}$ *take values in* L^2_χ *and can be represented as follows*

$$(\Psi \circ K)(\mathfrak{u}, h) = \exp\left(\phi_h(\mathfrak{u})\right), \qquad (\Psi \circ E)(\mathfrak{u}, h) = \exp\left(2\, Re\, \phi_h(\mathfrak{u}) - \|h\|^2\right)$$

where the last exponential function has the power series expansion

$$\begin{aligned} \exp\left\{2\, Re\, \phi_h - \|h\|^2\right\} &= \sum_{m,n \geq 0} \frac{\|h\|^{m+n}}{m!n!} \mathfrak{h}_{n,m}\left(\phi_{h/\|h\|}, \bar{\phi}_{h/\|h\|}\right) \\ \mathfrak{h}_{n,m}(z, \bar{z}) &= \sum_{k=0}^{m \wedge n} (-1)^k k! \binom{m}{k} \binom{n}{k} z^{m-k} \bar{z}^{n-k} \end{aligned} \tag{29}$$

with coefficients in the form of complex Hermite polynomials $\mathfrak{h}_{n,m}(z, \bar{z})$, $z \in \mathbb{C}$.

Proof. Applying the transform Ψ to $K(h', h)$ in variable $h' \in H$, we obtain

$$(\Psi \circ K)(\mathfrak{u}, h) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \phi_\imath^\lambda(\mathfrak{u}) \mathfrak{e}_\imath^{*\lambda}(h) = \sum_{n \geq 0} \frac{1}{n!} \Big(\sum_{i \geq 0} \phi_i(\mathfrak{u}) \mathfrak{e}_i^*(h)\Big)^n = \exp\left(\phi_h(\mathfrak{u})\right).$$

Similarly, applying Ψ to $E(h', h)$ in variable $h' \in H$, we obtain

$$\begin{aligned} (\Psi \circ E)(\mathfrak{u}, h) &= \Big|\sum_{n \geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \phi_\imath^\lambda(\mathfrak{u}) \mathfrak{e}_\imath^{*\lambda}(h)\Big|^2 \left(\sum_{n \geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} |\mathfrak{e}_\imath^{*\lambda}(h)|^2\right)^{-1} \\ &= \exp\left(2\, \mathrm{Re}\, \phi_h(\mathfrak{u}) - \|h\|^2\right). \end{aligned}$$

By Lemma 4, $(\Psi \circ K)(\cdot, h)$ and $(\Psi \circ E)(\cdot, h)$ with $h \in H$ take values in L^2_χ. The expansion (29) follows from [13, n.12] where polynomials $\mathfrak{h}_{n,m}(z, \bar{z})$ were introduced. □

Theorem 2. *For any* $f = \sum f_n \in L^2_\chi$ *with* $f_n \in L^{2,n}_\chi$ *the entire function*

$$\hat{f}(h) := \langle \exp(h) \mid \Phi^* f \rangle_\beta \quad \textit{of variable} \quad h \in H$$

and its Taylor coefficients at zero $d_0^n \hat{f}$ *have the integral representations*

$$\begin{aligned} \hat{f}(h) &= \int \exp(\bar{\phi}_h) f \, d\chi = \int \exp\left(2\, Re\, \phi_h - \|h\|^2\right) f \, d\chi, \\ d_0^n \hat{f}(h) &= \int \bar{\phi}_h^n f_n \, d\chi, \end{aligned} \tag{30}$$

respectively. The Fourier transform $F\colon L^2_\chi \ni f \longmapsto \hat{f} \in H^2_\beta$ *provides the isometries*

$$L^2_\chi \overset{F}{\simeq} H^2_\beta \quad \textit{and} \quad L^{2,n}_\chi \overset{F}{\simeq} P^n_\beta(H).$$

Proof. Since $\Psi = \Phi \circ *^{-1}$, we obtain $\Psi^* = * \circ \Phi^*$. From (27) it follows that $\hat{f}(h) = \langle \exp(h) \mid \Phi^* f\rangle_\beta = \big\langle(\Psi^* \circ f)(\cdot) \mid K(\cdot,h)\big\rangle_{H^2_\beta} = \big\langle(\Psi^* \circ f)(\cdot) \mid E(\cdot,h)\big\rangle_{H^2_\beta}$. Thus,

$$\begin{aligned}\hat{f}(h) &= \big\langle(\Psi^* \circ f)(\cdot) \mid K(\cdot,h)\big\rangle_{H^2_\beta} = \big\langle(\Psi^* \circ f)(\cdot) \mid E(\cdot,h)\big\rangle_{H^2_\beta} \\ &= \big\langle f(\cdot) \mid (\Psi \circ E)(\cdot,h)\big\rangle_\chi = \int \exp\big(2\operatorname{Re}\phi_h - \|h\|_H^2\big) f\, d\chi\end{aligned}$$

by Lemma 6. On the other hand, according to the same claim

$$\hat{f}(h) = \big\langle(\Psi^* \circ f)(\cdot) \mid K(\cdot,h)\big\rangle_{H^2_\beta} = \big\langle f(\cdot) \mid (\Psi \circ K)(\cdot,h)\big\rangle_\chi = \int \exp\big(\bar{\phi}_h\big) f\, d\chi.$$

It particularly follows that for all $h = \alpha x$ with $x \in H$,

$$\hat{f}\,(\alpha x) = \int \exp\big(\bar{\phi}_{\alpha x}\big) f\, d\chi = \sum \alpha^n \int \frac{\bar{\phi}_x^n}{n!} f_n\, d\chi, \quad \alpha \in \mathbb{C}.$$

Using the n-homogeneity of derivatives, we find

$$d_0^n \hat{f}(\alpha x) = \frac{d^n}{d\alpha^n} \sum \alpha^n \int \frac{\bar{\phi}_x^n}{n!} f_n\, d\chi \,|_{\alpha=0} = \int \bar{\phi}_x^n f_n\, d\chi.$$

Finally, we notice that the isometry $L^2_\chi \overset{F}{\simeq} H^2_\beta$ holds, since the isometry Φ^* is surjective by Lemma 5. Similarly, we get $L^{2,n}_\chi \overset{F}{\simeq} P^n_\beta(H)$. □

Corollary 3. *For any $h \in H$ the Paley–Wiener map ϕ_h satisfies the equality*

$$\int \exp\big\{\, Re\,\phi_h\big\}\, d\chi = \exp\Big\{\frac{1}{4}\|h\|^2\Big\}.$$

Proof. It is enough to put $f \equiv 1$ and to replace h by $h/2$ in the formula (30). □

Corollary 4. *The isometry $*\colon \Gamma_\beta(H) \longrightarrow H^2_\beta$ has the factorization $* = F \circ \Phi$.*

Proof. In fact, $\Phi\colon \Gamma_\beta(H) \ni \psi \longmapsto \Phi\psi = f \in L^2_\chi$ and $F\colon L^2_\chi \ni f \longmapsto \hat{f} \in H^2_\beta$. □

Corollary 5. *For every $f \in L^2_\chi$ the Taylor expansion at zero of the function*

$$\hat{f}(h) = \sum \frac{1}{n!} d_0^n \hat{f}(h) \quad \textit{with} \quad f = \sum f_n \in L^2_\chi, \quad f_n \in L^{2,n}_\chi$$

has the coefficients

$$d_0^n \hat{f}(h) = \int f_n \bar{\phi}_h^n\, d\chi = \sum_{\imath^\lambda \vdash n} \hbar_\lambda s_\imath^\lambda[f_\imath\, \mathfrak{e}_\imath^*(h)], \quad f_\imath := \int f \bar{\phi}_\imath\, d\chi \tag{31}$$

with summation over all standard Young tabloids $[\imath^\lambda]$ such that $\imath^\lambda \vdash n$ where $s_\imath^\lambda = 0$ if the conjugate partition $\lambda^\intercal$ has $\lambda_1^\intercal > \eta(\lambda)$ and $s_\imath^\lambda[f_\imath\, \mathfrak{e}_\imath^*(h)] := s_\imath^\lambda(t_\imath)$ with $t_\imath = f_\imath\, \mathfrak{e}_\imath^*(h)$.

Proof. By the Frobenius formula [18, I.7] we find that $\phi_h^n(\mathfrak{u}) = \sum_{\imath^\lambda \vdash n} \hbar_\lambda s_\imath^\lambda(\mathfrak{u}, h)$, where $s_\imath^\lambda = 0$ if $\lambda_1^\intercal > \eta(\lambda)$, and $s_\imath^\lambda(\mathfrak{u}, h)$ is defined by (3), whereas $\hbar_\lambda$ by (25). Thus,

$$\exp \phi_h(\mathfrak{u}) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \hbar_\lambda s_\imath^\lambda(\mathfrak{u}, h) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \frac{n!}{\lambda!} \phi_\imath^\lambda(\mathfrak{u}) \mathfrak{e}_\imath^{*\lambda}(h). \tag{32}$$

Using (32) in combination with Theorem 1, we find

$$\hat{f}(h) = \int f(\mathfrak{u}) \exp \bar{\phi}_h(\mathfrak{u})\, d\chi(\mathfrak{u}) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\imath^\lambda \vdash n} \hbar_\lambda \bar{s}_\imath^\lambda[f_\imath\, \mathfrak{e}_\imath^*(h)]$$

where the derivative at zero may be defined as

$$d_0^n \hat{f}(h) = \sum_{\imath^\lambda \vdash n} \hbar_\lambda s_\imath^\lambda[f_\imath\, \mathfrak{e}_\imath^*(h)] \quad \text{with} \quad s_\imath^\lambda[f_\imath\, \mathfrak{e}_\imath^*(h)] := \int f(\mathfrak{u}) \bar{s}_\imath^\lambda(\mathfrak{u}, h)\, d\chi(\mathfrak{u}).$$

In fact, for zh with $z \in \mathbb{C}$ and $\imath^\lambda \vdash n$ with $\lambda_1^\intercal > \eta(\lambda)$ we find

$$s_\imath^\lambda[f_\imath\, \mathfrak{e}_\imath^*(zh)] = z^n s_\imath^\lambda[f_\imath\, \mathfrak{e}_\imath^*(h)].$$

Hence, the derivative $d_0^n \hat{f}(h) = (d^n/dz^n)\hat{f}(zh)|_{z=0}$ is a Taylor coefficient of $\hat{f}$.

Now, the Frobenius formula and Theorem 1 yield the first equality in (31). By Lemmas 5 and 6 the second formula in (31) also holds. □

Remark 2. In the finite-dimensional case $\mathfrak{U} = U(m)$, the Hardy space H_β^2 of entire analytic functions of variable $h \in \mathbb{C}^m$ has the following orthogonal basis $\left\{\mathfrak{e}^{*\lambda} = \mathfrak{e}_1^{*\lambda_1} \ldots \mathfrak{e}_m^{*\lambda_m} : \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Y}\right\}$. The Fourier transform

$$\hat{f}(h) = \int \exp(\bar{\phi}_h) f\, d\chi_m = \int \exp\left(2 \operatorname{Re} \phi_h - \|h\|^2\right) f\, d\chi_m, \quad h \in \mathbb{C}^m$$

provides the surjective isometry $F\colon L_{\chi_m}^2 \ni f \longmapsto \hat{f} \in H_\beta^2$, defined by mappings

$$F\colon \mathfrak{e}^{*\lambda} \mapsto \phi^\lambda \quad \text{such that} \quad \|\mathfrak{e}^{*\lambda}\|_{H_\beta^2}^2 = \|\phi^\lambda\|_{\chi_m}^2 = \frac{(m-1)!\lambda!}{(m-1+|\lambda|)!}$$

where the space $L_{\chi_m}^2$ with the Haar measure χ_m on $U(m)$ has the orthogonal basis $\left\{\phi^\lambda = \phi_1^{\lambda_1} \circ \pi_m^{-1} \ldots \phi_m^{\lambda_m} \circ \pi_m^{-1} : \lambda \in \mathbb{Y}\right\}$.

7. Intertwining Properties of Fourier Transform

The shift group on H_β^2 is defined as $T_a\psi^*(h) := \langle \mathcal{T}_a \exp(h) \mid \psi \rangle_\beta$ for all $\psi \in \varGamma_\beta(H)$, $a, h \in H$. By (27), $\langle \mathcal{T}_a \exp(h) \mid \psi \rangle_\beta = T_a\psi^*(h) = \left\langle T_a\psi^*(\cdot) \mid \exp\langle \cdot \mid h \rangle \right\rangle_{H_\beta^2}$. Hence,

$$T_a\psi^*(h) = \langle \mathcal{T}_a \exp(h) \mid \psi \rangle_\beta = \langle \psi^*(\cdot) \mid \exp\langle \cdot \mid h + a \rangle \rangle_{H_\beta^2} = \langle \psi^*(\cdot) \mid M_{a^*} \exp\langle \cdot \mid h \rangle \rangle_{H_\beta^2}$$

where $M_{a^*} \exp\langle \cdot \mid h \rangle := \exp a^*(\cdot) \exp\langle \cdot \mid h \rangle = \exp\langle \cdot \mid h + a \rangle$ is defined to be the multiplicative group onto the total set $\{\exp\langle \cdot \mid h \rangle : h \in H\}$ in H_β^2.

Comparing the above formulas, we obtain that M_{a^*} is adjoint to T_a on H^2_β. By virtue of adjoint relations, $\|T_a\psi^*\|_{H^2_\beta} = \|M_{a^*}\psi^*\|_{H^2_\beta}$. The isometry $H^2_\beta \simeq \Gamma_\beta(H)$ yields $\|T_a\psi^*\|_{H^2_\beta} = \|\mathcal{T}_a\psi\|_\beta$. According to (24), we have

$$\begin{aligned}&\|T_a\psi^*\|^2_{H^2_\beta} \le \exp\left(\|a\|^2\right)\|\psi^*\|^2_{H^2_\beta} \quad \text{and} \quad T_{a+b} = T_aT_b = T_bT_a\\ &\|M_{a^*}\psi^*\|^2_{H^2_\beta} \le \exp\left(\|a\|^2\right)\|\psi^*\|^2_{H^2_\beta} \quad \text{and} \quad M_{a^*+b^*} = M_{a^*}M_{b^*} = M_{b^*}M_{a^*}\end{aligned} \tag{33}$$

for $a, b \in H$. Thus, these groups are strongly continuous with densely defined closed generators $\partial^*_a\psi^* := \lim_{z\to 0}(T_{za}\psi^* - \psi^*)/z$ and $a^*\psi^* := \lim_{z\to 0}(M_{za^*}\psi^* - \psi^*)/z$.

Hence, the additive group $(H,+)$ on H^2_β is represented by $M_{a^*}: H^2_\beta \to H^2_\beta$ and the generator $dM_{za^*}/dz \mid_{z=0} = a^*$ of its 1-parameter subgroup M_{za^*} is strongly continuous with the dense domain $\mathfrak{D}(a^*) = \left\{\psi^* \in H^2_\beta : a^*\psi^* \in H^2_\beta\right\}$. On the other hand, the group $(H,+)$ can be represented as $M^\dagger_{a^*} = \Psi M_{a^*}\Psi^* : L^2_\chi \to L^2_\chi$. The generator of its strongly continuous subgroup

$$\mathbb{C} \ni z \longmapsto M^\dagger_{za^*}, \quad dM^\dagger_{za^*}/dz \mid_{z=0} = \bar\phi_a \quad \text{with} \quad \bar\phi_a = \Psi a^*\Psi^*$$

has the dense domain $\mathfrak{D}(\bar\phi_a) = \left\{f \in L^2_\chi : \bar\phi_a f \in L^2_\chi\right\}$ and is closed, since a^* is closed.

The group $(H,+)$ on L^2_χ can be also represented by $T^\dagger_a := \Psi T_a\Psi^* : L^2_\chi \to L^2_\chi$. From Lemmas 3 and 5 it follows that the generator of strongly continuous subgroup

$$\mathbb{C} \ni z \longmapsto T^\dagger_{z\mathfrak{a}}, \quad dT^\dagger_{za}/dz \mid_{z=0} = \partial^\dagger_a \quad \text{with} \quad \partial^\dagger_a := \Psi\partial^*_a\Psi^*$$

has the dense domain $\mathfrak{D}(\partial^\dagger_a) = \left\{f \in L^2_\chi : \partial^\dagger_a f \in L^2_\chi\right\}$ and is closed, since ∂^*_a is closed. By (27) $\hat f(h) = \langle\exp(h) \mid \Phi^* f\rangle_\beta = \left\langle(\Psi^*\circ f)(\cdot) \mid \exp\langle\cdot \mid h\rangle\right\rangle_{H^2_\beta}$. Hence, by Lemma 6,

$$T^\dagger_a \hat f(h) = \left\langle(\Psi^*\circ f)(\cdot) \mid T_a\exp\langle\cdot \mid h\rangle\right\rangle_{H^2_\beta} = \int f\exp\left(\bar\phi_{h+a}\right)\,d\chi.$$

Lemma 7. *The additive group $(H,+)$ on L^2_χ has two representations $a \mapsto M^\dagger_{a^*}$ and $a \mapsto T^\dagger_a$ which are adjoint, strongly continuous with closed densely defined generators $\bar\phi_a$ and $\partial^\dagger_a$, respectively. For every $f \in \mathfrak{D}(\bar\phi^m_a) = \left\{f \in L^2_\chi : \bar\phi^m_a f \in L^2_\chi\right\}$ with $m \in \mathbb{N}_0$,*

$$\partial^{*m}_a T_a F(f) = F\left(\bar\phi^m_a M^\dagger_{a^*} f\right), \quad a \in H. \tag{34}$$

For every $f \in \mathfrak{D}(\partial^{\dagger m}_a) = \left\{f \in L^2_\chi : \partial^{\dagger m}_a f \in L^2_\chi\right\}$ with $m \in \mathbb{N}_0$,

$$a^{*m} M_{a^*} F(f) = F\left(\partial^{\dagger m}_a T^\dagger_a f\right), \quad a \in H. \tag{35}$$

As a conclusion, $\partial^\dagger_{\mathbb{i}a} = -\mathbb{i}\partial^\dagger_a$. Moreover, the following commutation relations hold,

$$M^\dagger_{a^*}T^\dagger_b = \exp\langle a \mid b\rangle T^\dagger_b M^\dagger_{a^*}, \qquad \left(\bar\phi_a\partial^\dagger_b - \partial^\dagger_b\bar\phi_a\right)f = \langle a \mid b\rangle f, \tag{36}$$

for all f from the dense subspace $\mathfrak{D}(\bar\phi_a^2)\cap\mathfrak{D}(\partial_b^{\dagger 2})\subset L^2_\chi$ and nonzero $a,b\in H$.

Proof. Using that T_a and M_{a^*} are adjoint, we find that

$$\partial_a^{*m}T_a\hat f(h)=\int\frac{d^mM^\dagger_{za^*}f}{dz^m}\Big|_{z=0}\exp\bar\phi_h\,d\chi=\int(\bar\phi_a^m f)\exp\bar\phi_h\,d\chi,\quad m\geq 0$$

for all $f\in L^2_\chi$. This gives (34). Since $M_{a^*}\psi^*(h)=\big\langle\psi^*(\cdot)\mid M_{a^*}\exp\langle\cdot\mid h\rangle\big\rangle_{H^2_\beta}=\exp a^*(h)\,\psi^*(h)$, we obtain

$$\begin{aligned}a^{*m}M_{a^*}\hat f(h)&=\frac{d^mM_{za^*}\hat f(h)}{dz^m}\Big|_{z=0}=\int\frac{d^mT^\dagger_{za}f}{dz^m}\Big|_{z=0}\exp\bar\phi_h\,d\chi\\&=\int(\partial_a^{\dagger m}f)\exp\bar\phi_h\,d\chi\quad\text{with}\quad f\in\mathfrak{D}(\partial_a^{\dagger m}),\quad\psi^*=\Psi^*f.\end{aligned}\tag{37}$$

This together with the group property by applying F and F^{-1} yields (35).

Now, we prove the commutation relations. For any $f\in L^2_\chi$ and $h\in H$, we have

$$\begin{aligned}&M_{b^*}T_a\hat f(h)=\exp\langle h\mid b\rangle\hat f(h+a),\\&T_aM_{b^*}\hat f(h)=\exp\langle h+a\mid b\rangle\hat f(h+a)=\exp\langle a\mid b\rangle M_{b^*}T_a\hat f(h).\end{aligned}$$

For each $\hat f\in\mathfrak{D}(b^{*2})\cap\mathfrak{D}(\partial_a^2)$ and $t\in\mathbb{C}$ by differentiation, we obtain

$$\big(d^2/dt^2\big)T_{ta}M_{tb^*}\hat f\,|_{t=0}=\big(\partial_a^{*2}+2\partial_a^*b^*+b^{*2}\big)\hat f.\tag{38}$$

Subsequently, taking into account (38) together with $(d/dt)[\exp\langle ta\mid\bar tb\rangle M_{tb^*}T_{ta}]=[(d/dt)\exp\langle ta\mid\bar tb\rangle]M_{tb^*}T_{ta}+\exp\langle ta\mid\bar tb\rangle[(d/dt)M_{tb^*}T_{ta}]$, we find

$$\begin{aligned}\big(\partial_a^{*2}+2\partial_a^*b^*+b^{*2}\big)\hat f&=(d/dt)\big[(d/dt)\exp\langle ta\mid\bar tb\rangle M_{tb^*}T_{ta}\hat f\big]_{t=0}\\&=2\langle a\mid b\rangle\hat f+\big(\partial_a^{*2}+2b^*\partial_a^*+b^{*2}\big)\hat f.\end{aligned}$$

Hence, for each $\hat f$ from the dense subspace $\mathfrak{D}(b^{*2})\cap\mathfrak{D}(\partial_a^2)\subset H^2_\beta$, which includes all polynomials generated by finite sums $\Psi^*(f)=\bigoplus\psi_n\in\Gamma_\beta(H)$ with $\psi_n\in H_\beta^{\odot n}$,

$$T_aM_{b^*}=\exp\langle a\mid b\rangle M_{b^*}T_a,\quad\left(\partial_a^*b^*-b^*\partial_a^*\right)\hat f=\langle a\mid b\rangle\hat f.\tag{39}$$

Corollary 4 yields $F=*\circ\Phi^*$ and $F^{-1}=\Phi\circ*^{-1}$. The equality (37) for $m=0$ can be rewritten as $M_{b^*}\hat f(a)=\langle\exp(a)\mid T_b\Phi^*f\rangle_\beta$ with $f\in L^2_\chi$ or in another way $*\circ T_b=M_{b^*}\circ*$. Hence, $T_b^\dagger=\Phi T_b\Phi^*=\Phi\circ*^{-1}\circ M_{b^*}\circ*\circ\Phi^*=F^{-1}M_{b^*}F$ and $\partial_b^\dagger=F^{-1}b^*F$. Similarly, $M^\dagger_{a^*}=F^{-1}T_aF$ and $\bar\phi_a=F^{-1}\partial_a^*F$. Finally,

$$\begin{aligned}&M^\dagger_{a^*}T^\dagger_b=F^{-1}T_aM_{b^*}F=\exp\langle a\mid b\rangle F^{-1}M_{b^*}T_aF=\exp\langle a\mid b\rangle T^\dagger_bM^\dagger_{a^*},\\&\big(\bar\phi_a\partial_b^\dagger-\partial_b^\dagger\bar\phi_a\big)f=F^{-1}\left(\partial_a^*b^*-b^*\partial_a^*\right)Ff=\langle a\mid b\rangle f\end{aligned}$$

for all f from the dense subspace $\mathfrak{D}(\bar\phi_a^2)\cap\mathfrak{D}(\partial_b^{\dagger 2})\subset L^2_\chi$, which includes all functions generated by finite sums $\Phi\left(\bigoplus\psi_n\right)$ with $\psi_n\in H_\beta^{\odot n}$. □

8. Infinite-Dimensional Heisenberg Group

Our goal is to describe an irreducible representation on the space L^2_χ of the group $\mathcal{H}_\mathbb{C}$, defined by (1). We will use the appropriate generalization of Weyl's system which in our case is written in the form of L^2_χ-valued function of variable $h \in H$

$$W^\dagger(h) := W^\dagger(a,b) = \exp\Big\{\frac{1}{2}\langle a \mid b\rangle\Big\} T_b^\dagger M_{a^*}^\dagger.$$

For convenience, we will use the quaternion algebra $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbb{j}$ of numbers $\zeta = (\alpha_1 + \alpha_2\mathbb{i}) + (\alpha_1' + \alpha_2'\mathbb{i})\mathbb{j} = \alpha + \alpha'\mathbb{j}$ such that $\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = \mathbb{i}\mathbb{j}\mathbb{k} = -1$, $\mathbb{k} = \mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i}$, $\mathbb{k}\mathbb{i} = -\mathbb{i}\mathbb{k} = \mathbb{j}$, where $(\alpha, \alpha') \in \mathbb{C}^2$ with $\alpha = \alpha_1 + \alpha_2\mathbb{i}, \alpha' = \alpha_1' + \alpha_2'\mathbb{i} \in \mathbb{C}$ and $\alpha_\imath, \alpha_\imath' \in \mathbb{R}$ $(\imath = 1, 2)$ [26, 5.5.2]. Let us denote $\alpha' := \Im\zeta$ for all $\zeta = \alpha + \alpha'\mathbb{j} \in \mathbb{H}$.

Consider the Hilbert space $H \oplus H\mathbb{j}$ with $\mathbb{H}$-valued inner product

$$\langle h \mid h'\rangle = \langle a + b\mathbb{j} \mid a' + b'\mathbb{j}\rangle = \langle a \mid a'\rangle + \langle b \mid b'\rangle + [\langle a' \mid b\rangle - \langle a \mid b'\rangle]\,\mathbb{j}$$

where $h = a + b\mathbb{j}$ with a, $b \in H$. Hence,

$$\Im\langle h \mid h'\rangle = \langle a' \mid b\rangle - \langle a \mid b'\rangle, \qquad \Im\langle h \mid h\rangle = 0.$$

Theorem 3. *The representation of $\mathcal{H}_\mathbb{C}$ over L^2_χ in the Weyl–Schrödinger form*

$$S^\dagger\colon \mathcal{H}_\mathbb{C} \ni X(a,b,t) \longmapsto \exp(t)W^\dagger(h), \quad h = a + b\mathbb{j}$$

is well defined and irreducible. The Weyl system satisfies the relation

$$W^\dagger(h + h') = \exp\Big\{-\frac{\Im\langle h \mid h'\rangle}{2}\Big\} W^\dagger(h)W^\dagger(h') \tag{40}$$

which on any real subspace $\{\tau h\colon \tau \in \mathbb{R}\}$ transforms to the 1-parameter group

$$W^\dagger\left((\tau + \tau')h\right) = W^\dagger(\tau h)W^\dagger(\tau' h) = W^\dagger(\tau' h)W(\tau h) \tag{41}$$

with the densely defined generator on L^2_χ of the form $\mathfrak{p}_h^\dagger := \partial_b^\dagger + \bar\phi_a$. Moreover, the following commutation relations hold,

$$W^\dagger(h)W^\dagger(h') = \exp\big\{\Im\langle h \mid h'\rangle\big\}W^\dagger(h')W^\dagger(h) \quad \textit{where}$$
$$\Im\langle h \mid h'\rangle = -\big[\mathfrak{p}_h^\dagger, \mathfrak{p}_{h'}^\dagger\big] \quad \textit{with} \quad \big[\mathfrak{p}_h^\dagger, \mathfrak{p}_{h'}^\dagger\big] := \mathfrak{p}_h^\dagger\mathfrak{p}_{h'}^\dagger - \mathfrak{p}_{h'}^\dagger\mathfrak{p}_h^\dagger \tag{42}$$

on the dense subspace $\mathfrak{D}(\bar\phi_a^2) \cap \mathfrak{D}(\partial_b^{\dagger 2}) \subset L^2_\chi$.

Proof. Let us consider the auxiliary group $\mathbb{C} \times (H \oplus H\mathbb{j})$ with multiplication $(t,h)(t',h') = \big(t + t' - \frac{1}{2}\Im\langle h \mid h'\rangle,\, h + h'\big)$ for all $h = a + b\mathbb{j}$, $h' = a' + b'\mathbb{j} \in H \oplus H\mathbb{j}$. The mapping $G\colon X(a,b,t) \longmapsto \big(t - \frac{1}{2}\langle a \mid b\rangle,\, a + b\mathbb{j}\big)$ is a group isomorphism, since

$$\begin{aligned}
&G\big(X(a,b,t)X(a',b',t')\big) = G\big(X(a + a', b + b', t + t' + \langle a \mid b'\rangle)\big)\\
&\quad = \Big(t + t' + \langle a \mid b'\rangle - \frac{1}{2}\big(\langle a + a' \mid b + b'\rangle\big), (a + a') + (b + b')\mathbb{j}\Big)\\
&\quad = \Big(t + t' - \frac{1}{2}\big(\langle a \mid b\rangle + \langle a' \mid b'\rangle\big) + \frac{1}{2}\big(\langle a \mid b'\rangle - \langle a' \mid b\rangle\big), (a + a) + (b + b')\mathbb{j}\Big)
\end{aligned}$$

$$= \Big(t - \frac{1}{2}\langle a \mid b\rangle,\, a + b\mathbb{j}\Big)\Big(t' - \frac{1}{2}\langle a' \mid b'\rangle,\, a' + b'\mathbb{j}\Big) = G\left(X(a,b,t)\right) G\left(X(a',b',t')\right).$$

On the other hand, let us define the auxiliary Weyl system

$$W(h) = \exp\Big\{\frac{1}{2}\langle a \mid b\rangle\Big\} M_{b^*} T_a, \quad h = a + b\mathbb{j}. \tag{43}$$

Using group properties and the commutation relation (39), we obtain

$$\begin{aligned}\exp\Big\{-\frac{\Im\langle h \mid h'\rangle}{2}\Big\} W(h)W(h') &= \exp\Big\{\frac{\langle a \mid b'\rangle}{2} - \frac{\langle a' \mid b\rangle}{2}\Big\} W(h)W(h') \\ &= \exp\Big\{\frac{\langle a \mid b\rangle}{2} + \frac{\langle a' \mid b'\rangle}{2}\Big\} \exp\Big\{\frac{\langle a \mid b'\rangle}{2} - \frac{\langle a' \mid b\rangle}{2}\Big\} M_{b^*} T_a M_{b'^*} T_{a'} \\ &= \exp\Big\{\frac{1}{2}\langle a + a' \mid b + b'\rangle\Big\} M_{b^* + b'^*} T_{a+a'} = W(h + h'). \end{aligned} \tag{44}$$

Hence, the mapping $\mathbb{C} \times (H \oplus H\mathbb{j}) \ni (t,h) \longmapsto \exp(t)W(h)$ acts as a group isomorphism into the operator algebra over H^2_β. So, the representation

$$S\colon \mathcal{H}_{\mathbb{C}} \ni X(a,b,t) \longmapsto \exp(t)W(h) = \exp\Big\{t + \frac{1}{2}\langle a \mid b\rangle\Big\} M_{b^*} T_a$$

is also well defined over H^2_β, as a composition of group isomorphisms.

Let us check the irreducibility. Suppose the contrary. Assume there exist an element $h_0 \neq 0$ in H and an integer $n > 0$ such that

$$\exp\Big\{t + \frac{1}{2}\langle a \mid b\rangle\Big\} \exp\langle c \mid a\rangle\langle c + b \mid h_0\rangle^n = 0 \quad \text{for all} \quad a, b, c \in H.$$

But, this is only possible for $h_0 = 0$. It gives a contradiction. Finally, using that

$$\exp\Big\{t + \frac{1}{2}\langle a \mid b\rangle\Big\} T^\dagger_b M^\dagger_{a^*} = F^{-1}\Big(\exp\Big\{t + \frac{1}{2}\langle a \mid b\rangle\Big\} M_{b^*} T_a\Big) F,$$

we obtain that $S^\dagger = F^{-1} S\, F$ is irreducible. Applying F, F^{-1} to (44) we get (40).

Consider the Weyl system $W^\dagger$ on the space L^2_χ. By (40) we obtain the equality

$$\begin{aligned} W^\dagger(h)W^\dagger(h') &= \exp\Big\{\frac{\Im\langle h \mid h'\rangle}{2}\Big\} W^\dagger(h + h') = \exp\Big\{-\frac{\Im\langle h' \mid h\rangle}{2}\Big\} W^\dagger(h' + h) \\ &= \exp\Big\{-\Im\langle h' \mid h\rangle\Big\} \exp\Big\{\frac{\Im\langle h' \mid h\rangle}{2}\Big\} W^\dagger(h' + h) \\ &= \exp\Big\{-\Im\langle h' \mid h\rangle\Big\} W^\dagger(h')W^\dagger(h). \end{aligned}$$

Using this equality, we get (41) for any fixed $h = a + b\mathbb{j} \in H \oplus H\mathbb{j}$. The 1-parameter group $W^\dagger(\tau a, \tau b) = W^\dagger(\tau h)$ with real τ has the generator $\mathfrak{p}^\dagger_h = \mathfrak{p}^\dagger_{a,b}$, since

$$\mathfrak{p}^\dagger_{a,b} = \frac{d}{d\tau} W^\dagger(\tau h)\Big|_{\tau=0} = \frac{d}{d\tau} \exp\Big\{\frac{1}{2}\langle \tau a \mid \tau b\rangle\Big\} T^\dagger_{\tau b} M^\dagger_{\tau a^*}\Big|_{\tau=0} = \partial^\dagger_b + \bar{\phi}_a.$$

Taking into account the inequalities (33) and that F is isometric, we get

$$\|W^\dagger(\tau a,\tau b)f\|_\chi^2 \le \exp\big(\|\tau a\|^2+\|\tau b\|^2\big)\|f\|_\chi^2, \quad f\in L_\chi^2.$$

Hence, the group $W^\dagger(\tau a,\tau b)$ in variable $\tau\in\mathbb{R}$ is strongly continuous on L_χ^2 and therefore has the dense domain $\mathfrak{D}(\mathfrak{p}_h^\dagger)=\big\{f\in L_\chi^2\colon \mathfrak{p}_h^\dagger f\in L_\chi^2\big\}$. Moreover, its generator $\mathfrak{p}_h^\dagger$ is closed (see, e.g., [32]). Note also that $\mathfrak{p}_{\tau h}^\dagger=\tau\mathfrak{p}_h^\dagger$ for $\tau\in\mathbb{R}$.

Finally, applying the commutation relation (36) and commutability of group generators in different directions over the dense set $\mathfrak{D}(\bar\phi_a^2)\cap\mathfrak{D}(\partial_b^{\dagger 2})\subset L_\chi^2$, we have

$$\begin{aligned}-\Im\langle h\mid h'\rangle &= \langle a\mid b'\rangle-\langle a'\mid b\rangle=\bar\phi_a\partial_{b'}^\dagger-\bar\phi_{a'}\partial_b^\dagger+\partial_b^\dagger\bar\phi_{a'}-\partial_{b'}^\dagger\bar\phi_a\\ &=(\partial_b^\dagger+\bar\phi_a)(\partial_{b'}^\dagger+\bar\phi_{a'})-(\partial_{b'}^\dagger+\bar\phi_{a'})(\partial_b^\dagger+\bar\phi_a)=\big[\mathfrak{p}_h^\dagger,\mathfrak{p}_{h'}^\dagger\big].\end{aligned}$$

□

9. Heat Equation Associated with Weyl System

In what follows, we will consider the real Banach space c_0 and let ξ_n^* be the coordinate functional, i.e., $\xi_n^*(\xi)=\xi_n$ for $\xi\in c_0$. Since, the embedding $\mathcal{I}\colon l_2\hookrightarrow c_0$ is continuous, the Gelfand triple $l_1\xrightarrow{\mathcal{I}^*}l_2\hookrightarrow c_0$ with adjoint $\mathcal{I}^*$ holds. The mapping $Q\colon l_1\to c_0$ with $Q:=\mathcal{I}\circ\mathcal{I}^*$ is positive and $\langle Q\xi^*\mid Q\xi^*\rangle_{l_2}:=\xi^*(Q\xi^*)=\sum\xi_n^2=\|\xi\|_{l_2}^2$ where $\xi=Q\xi^*\in\mathscr{R}(Q)$ and $\xi^*\in l_1=c_0^*$. By the Aronszajn-Kolmogorov decomposition theorem (see e.g., [22, Prop.1]) the appropriative reproducing kernel Hilbert space can be determined as $\overline{\mathscr{R}(Q)}=l_2$.

Consider the abstract Wiener space defined by $\mathcal{I}\colon l_2\hookrightarrow c_0$. Given $\xi_1^*,\dots,\xi_n^*\in l^1=c_0^*$, we assign the family of cylinder sets $\Omega_n^c=\{\xi\in c_0\colon(\xi_1^*(\xi),\dots,\xi_n^*(\xi))\in\Omega_n\}$ with any Borel $\Omega_n\subset\mathbb{R}^n$ that are not a σ-field. Define the σ-additive extension $\mathfrak{w}$ of the Gaussian measure γ onto the Borel σ-algebra $\mathscr{B}(c_0)$, called futhure the Wiener measure, such that

$$\mathfrak{w}(\Omega_n^c):=\gamma(\Omega_n)\quad\text{with}\quad\gamma(\Omega_n):=(2\pi)^{-n/2}\int_{\Omega_n}\exp\big\{-\|\omega\|_{l_2}^2/2\big\}\,d\omega.$$

By Gross' theorem [10] there exists a smaller abstract Wiener space $\{w_0,\|\cdot\|_{w_0}\}$ such that injections $l_2\hookrightarrow w_0\hookrightarrow c_0$ are continuous and the increasing sequence of orthogonal projectors $p_n\colon l_2\to\mathbb{R}^n$ has the extension $(p_n^\sim)$ on w_0 that is convergent to the identity operator on w_0 and $\mathfrak{w}(w_0)=1$. The integral of any cylinder function $\upsilon\colon c_0\to\mathbb{R}$ such that $\upsilon=\rho\circ p_n^\sim$ is defined to be $\int_{\Omega_n^c}\upsilon\,d\mathfrak{w}=\int_{\Omega_n}\rho\,d\gamma$. The Fernique theorem [6], [15, Thm 3.1] implies that these exist $\varepsilon,\eta>0$ such that $\|\cdot\|_{w_0}$ satisfies the following conditions with a sufficiently large $K>0$,

$$\int_{c_0}\exp\big\{\varepsilon\|\xi\|_{w_0}^2\big\}d\mathfrak{w}(\xi)<\infty,\quad\mathfrak{w}\big(\|\xi\|_{w_0}\ge K\big)\le\exp\big\{-\eta K^2\big\}.$$

Let us go back to the Weyl system $W^\dagger$. Consider in L^2_χ the dense subspace $L^{+2}_\chi := \bigcup_{n\ge 0}\bigoplus_{m=0}^n L^{2,m}_\chi$. Let $a=b=\mathbb{i}\xi_m\mathfrak{e}_m$ with $\xi_m\in\mathbb{R}$. Then by Theorem 3

$$W^\dagger(\mathbb{i}\xi_m\mathfrak{e}_m,\mathbb{i}\xi_m\mathfrak{e}_m)=\exp\left\{-\xi_m^2/2\right\}T^\dagger_{\mathbb{i}\xi\mathfrak{e}_m}M^\dagger_{-\mathbb{i}\xi\mathfrak{e}^*_m}.$$

Theorem 4. *For any $f\in L^{+2}_\chi$ and $\xi=(\xi_m)\in c_0$ there exists the limit*

$$W^\dagger_\xi f=\lim_{n\to\infty}W^\dagger_{p_n^\sim(\xi)}f,\quad W^\dagger_{p_n^\sim(\xi)}:=\exp\left\{-\frac{\|p_n^\sim(\xi)\|^2_{w_0}}{2}\right\}\prod_{m=1}^n T^\dagger_{\mathbb{i}\xi_m\mathfrak{e}_m}M^\dagger_{-\mathbb{i}\xi_m\mathfrak{e}^*_m}$$

$\mathfrak{w}$-*almost everywhere on c_0 such that the 1-parameter Gaussian semigroup*

$$\mathfrak{G}^\dagger_r f=\frac{1}{\sqrt{4\pi r}}\int_{c_0}\exp\left\{-\frac{\|\xi\|^2_{w_0}}{4r}\right\}W^\dagger_\xi f\,d\mathfrak{w}(\xi),\quad r>0 \tag{45}$$

on the space L^{+2}_χ is generated by $-\sum\left(\partial^\dagger_m+\bar\phi_m\right)^2$ with $\partial^\dagger_m:=\partial^\dagger_{\mathfrak{e}_m}$. As a consequence, $w(r)=\mathfrak{G}^\dagger_r f$ is unique solution of the Cauchy problem

$$\frac{dw(r)}{dr}=-\sum\left(\partial^\dagger_m+\bar\phi_m\right)^2 w(r),\quad w(0)=f\in L^{+2}_\chi. \tag{46}$$

Proof. Note that $(M_{b^*}T_a)^*=T^*_aM^*_{b^*}=M_{a^*}T_b$. Hence, $(\partial^\dagger_a+\bar\phi_a)^*=\partial^\dagger_a+\bar\phi_a$ is self-adjoint for $a=b$, as a generator of the group $W^\dagger(\tau a,\tau a)=\exp\left\{\|\tau a\|^2/2\right\}$ $T^\dagger_{\tau a}M^\dagger_{\tau a^*}$ with $\tau\in\mathbb{R}$. Replacing $a=b$ by $\mathbb{i}\tau a$ with $\tau\in\mathbb{R}$, we obtain that

$$W^\dagger(\mathbb{i}\tau a,\mathbb{i}\tau a)=\exp\left\{-\frac{1}{2}\langle\tau a\mid\tau a\rangle\right\}T^\dagger_{\mathbb{i}\tau a}M^\dagger_{-\mathbb{i}\tau a^*}\quad\text{has the generator}\quad \mathbb{i}(\partial^\dagger_a+\bar\phi_a)$$

with self-adjoint $\partial^\dagger_a+\bar\phi_a$. By relations (36), $W^\dagger(\mathbb{i}\tau a,\mathbb{i}\tau a)$ is unitary.

Lemma 7 implies that $[M^\dagger_{-\mathbb{i}\xi_m\mathfrak{e}^*_m},T^\dagger_{\mathbb{i}\xi_k\mathfrak{e}_k}]=0$ and $[M^\dagger_{-\mathbb{i}\xi_m\mathfrak{e}^*_m},M^\dagger_{-\mathbb{i}\xi_k\mathfrak{e}^*_k}]=0$, as well as, $[T^\dagger_{\mathbb{i}\xi_m\mathfrak{e}_m},T^\dagger_{\mathbb{i}\xi_k\mathfrak{e}_k}]=0$ for any $m\ne k$. In view of the relations (36),

$$\left[\bar\phi_{\mathbb{i}\xi_m\mathfrak{e}_m},\partial^\dagger_{\mathbb{i}\xi_k\mathfrak{e}_k}\right]=0\quad\text{if}\quad m\ne k\quad\text{and}\quad\left[\bar\phi_{\mathbb{i}\xi_m\mathfrak{e}_m},\partial^\dagger_{\mathbb{i}\xi_m\mathfrak{e}_m}\right]=-\xi_m^2. \tag{47}$$

Check that (45) holds. Denote $W^\dagger_{p_n^\sim(\xi)}:=\prod_{m=1}^n W^\dagger(\mathbb{i}\xi_m\mathfrak{e}_m,\mathbb{i}\xi_m\mathfrak{e}_m)$ and $T^\dagger_{p_n^\sim(\xi)}:=\prod_{m=1}^n T^\dagger_{\mathbb{i}\xi_m\mathfrak{e}_m}$, as well as, $M^\dagger_{p_n^\sim(\xi)}:=\prod_{m=1}^n M^\dagger_{-\mathbb{i}\xi_m\mathfrak{e}^*_m}$ with $\xi=(\xi_m)$ $\in w_0$. Using (33) with the operator norm over H^2_β, we get the inequality

$$\ln\prod_{m=1}^n\|T_{\mathbb{i}\xi_m\mathfrak{e}_m}\|^2_{\mathscr{L}(H^2_\beta)}\le\sum_{m=1}^n\langle\xi_m\mathfrak{e}_m\mid\xi_m\mathfrak{e}_m\rangle^2=\sum_{m=1}^n\xi_m^2=\|p_n^\sim(\xi)\|^2_{l_2}.$$

The relation $T^\dagger_{\mathbb{i}\xi_m\mathfrak{e}_m}=\varPsi T_{\mathbb{i}\xi_m\mathfrak{e}_m}\varPsi^*$ implies that the left-hand side term above can be changed by $\ln\prod_{m=1}^n\|T^\dagger_{\mathbb{i}\xi_m\mathfrak{e}_m}\|^2_{\mathscr{L}(L^2_\chi)}$. For $M^\dagger_{p_n^\sim(\xi)}=\prod_{m=1}^n M^\dagger_{-\mathbb{i}\xi_m\mathfrak{e}^*_m}$ similarly.

Using the unitarity of groups $W^\dagger(\mathbb{i}\xi_m\mathfrak{e}_m,\mathbb{i}\xi_m\mathfrak{e}_m)$, we find by virtue of (47) that their product $W^\dagger_{p_n^\sim(\xi)}=\exp\left\{-\|p_n^\sim(\xi)\|^2_{l_2}/2\right\}T^\dagger_{p_n^\sim(\xi)}M^\dagger_{p_n^\sim(\xi)}$ is also unitary. Taking into account the continuity of $\mathcal{I}_0\colon l_2\looparrowright w_0$ and that $p_n^\sim$ converges to the identity mapping on w_0, as well as, that $\mathfrak{w}(w_0) = 1$, we obtain for all $f \in L^{+2}_\chi$, $n \geq 0$,

$$\|W^\dagger_{p^\sim_n(\xi)} f\|_\chi \leq \exp\big\{-\|p^\sim_n(\xi)\|^2_{l_2}/2\big\}\|f\|_\chi \leq \exp\big\{-\|\mathcal{I}_0\|^2\,\|\xi\|^2_{w_0}/2\big\}\|f\|_\chi.$$

The Lebesgue dominated convergence theorem implies that there exists $\lim \|W^\dagger_{p^\sim_n(\xi)} f\|_\chi$ $\mathfrak{w}$-almost everywhere in variable $\xi \in w_0$ for all $f \in L^{2,m}_\chi$ and $m > 0$. By completeness of $L^{2,m}_\chi$, the limit $W^\dagger_\xi f$ is well defined $\mathfrak{w}$-almost everywhere and

$$\|W^\dagger_\xi f\|_\chi \leq \exp\big\{-\|\mathcal{I}_0\|^2\,\|\xi\|^2_{w_0}/2\big\}\|f\|_\chi \quad \text{for all} \quad f \in L^{+2}_\chi, \quad \xi \in w_0. \tag{48}$$

The $\|\cdot\|_\chi$-norm of integrant in (45) is bounded by $\exp\big\{\varepsilon\|\xi\|^2_{w_0}\big\}$ with any $\varepsilon > 0$. By Fernique's theorem and (48), the integral (45) with the Wiener measure $\mathfrak{w}$ exists for all $f \in L^{+2}_\chi$. The equality $\mathfrak{w}(w_0) = 1$ implies that the integral (45) is absolutely convergent uniformly in variables $r > 0$ on the whole space c_0. It provides the C_0-property of $\mathfrak{G}_r$ in variables $r > 0$ on any finite sum $\bigoplus_{m=0}^n L^{2,m}_\chi$.

Prove that the semigroup $\mathfrak{G}_r$ is generated by $\sum \mathfrak{p}^{\dagger 2}_m$ with $\mathfrak{p}^\dagger_m := \mathbb{i}(\partial^\dagger_m + \bar{\phi}_m)$. By differentiation of $W^\dagger(\mathbb{i}\xi_m a, \mathbb{i}\xi_m a)$ at $\xi_m = 0$, we get that its generator coincides with $\mathfrak{p}^\dagger_m$. In fact, $W^\dagger(\mathbb{i}\xi_m a, \mathbb{i}\xi_m a) f = \exp\big\{\xi_m \mathfrak{p}^\dagger_m\big\} f$ for all $f \in \phi^{\mathbb{Y}}$. Applying the next formula for Gamma functions with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$

$$\begin{aligned}\prod_{m=1}^n \frac{1}{\sqrt{4\pi r}} \int \exp\left\{\frac{-\xi^2_m}{4r}\right\} \xi^{2\alpha_m}_m \, d\xi_m\Big|_{\xi_m = 2\sqrt{r} x_m} &= \prod_{m=1}^n \frac{(2\sqrt{r})^2}{\sqrt{\pi}} \int \exp\big\{-x^2_m\big\} x^{2\alpha_m}_m \, dx_m \\ &= 2^{2n} r^n \prod_{m=1}^n \Gamma\Big(\frac{2\alpha_m + 1}{2}\Big) = 2^n r^n \frac{(2\alpha - 1)!}{(\alpha - 1)!},\end{aligned}$$

we find that for any L^{+2}_χ-valued cylinder function $h_n = (W^\dagger_\xi f) \circ p^\sim_n$ we have

$$\begin{aligned}\mathfrak{G}^\dagger_r h_n &= \prod_{m=1}^n \frac{1}{\sqrt{4\pi r}} \int \exp\Big\{-\frac{\xi^2_m}{4r}\Big\} \exp\big\{\xi_m \mathfrak{p}^\dagger_m\big\} d\xi_m h_n \\ &= \sum_{\alpha \in \mathbb{N}^n_0} \prod_{m=1}^n \frac{\mathfrak{p}^{\dagger\alpha_m}_m}{\alpha_m!} \frac{1}{\sqrt{4\pi r}} \int \exp\Big\{-\frac{\xi^2_m}{4r}\Big\} \xi^{\alpha_m}_m \, d\xi_m h_n \\ &= \sum_{\alpha \in \mathbb{N}^n_0} 2^n r^n \prod_{m=1}^n \frac{(2\alpha_m - 1)!}{(\alpha_m - 1)!} \frac{\mathfrak{p}^{\dagger 2}_m}{(2\alpha_m)!} h_n = \exp\Big\{r \sum_{m=1}^n \mathfrak{p}^{\dagger 2}_m\Big\} h_n.\end{aligned}$$

Using (48), we obtain that $0 \leq r \longmapsto \mathfrak{G}^\dagger_r$ is the 1-parameter C_0-semigroup on any finite sum $\bigoplus_{m=0}^n L^{2,m}_\chi$ with densely defined closed generator $\sum_{m=1}^n \mathfrak{p}^{\dagger 2}_m$. Applying the known relation [32] between the initial problem (46) and the 1-parameter C_0-semigroup $\mathfrak{G}^\dagger_r$, we obtain that the function $w_n(r) = \mathfrak{G}^\dagger_r f_n$ for any $n \in \mathbb{N}$ solves this problem in the sense that $d\mathfrak{G}^\dagger_r f_n/dr|_{r=0} = \sum_{m=1}^n \mathfrak{p}^{\dagger 2}_m f_n$ for all $f_n \in \bigoplus_{m=0}^n L^{2,m}_\chi$. The theorem is proved. □

Taking into account the isometries $H^2_\beta \overset{\Psi}{\simeq} L^2_\chi$ and $P^n_\beta(H) \overset{\Psi}{\simeq} L^{2,n}_\chi$ from (28), defined by linearization, we can rewrite the Cauchy problem in polynomial form.

Consider the Weyl system $W(a,b) = \exp\{\langle a \mid b\rangle/2\} M_{b^*}T_a$ defined by (43) on the dense subspace of polynomials $P_\beta(H) := \sum_{n\geq 0} P^n_\beta(H)$ in H^2_β, consisting of all finite sums of n-homogenous polynomials $\psi^*(h) = \sum \psi^*_n(h)$ of variable $h \in H$ with components $\psi^*_n = \mathcal{P} \circ \psi_n \in P^n_\beta(H)$. Replacing a by τa and b by τb with real $\tau \in \mathbb{R}$, we get that $T_{\tau a}$ and $M_{\tau b^*}$ are generated by closed generators on $P_\beta(H)$,

$$\partial^*_a \psi^* = \lim_{\tau\to 0} \left(T_{\tau a}\psi^* - \psi^*\right)/\tau \quad \text{and} \quad a^*\psi^* = \lim_{\tau\to 0} \left(M_{\tau a^*}\psi^* - \psi^*\right)/\tau, \quad a,b \in H.$$

As a consequence, the 1-parameter Weyl system $W(\tau a, \tau b)$ has the generator

$$\frac{d}{d\tau} W(\tau a, \tau b)|_{\tau=0} = \frac{d}{d\tau} \exp\left\{\frac{1}{2}\langle a \mid b\rangle\right\}\Big|_{\tau=0} = b^* + \partial^*_a$$

densely defined on $P_\beta(H)$ such that $(\tau b)^* + \partial^*_{\tau a} = \tau(b^* + \partial^*_a)$ for real τ. Let $W_{p^\sim_n(\xi)} = \prod_{m=1}^n W(\mathbb{i}\xi_m \mathfrak{e}_m, \mathbb{i}\xi_m\mathfrak{e}_m)$, $T_{p^\sim_n(\xi)} = \prod_{m=1}^n T_{\mathbb{i}\xi_m\mathfrak{e}_m}$, $M_{p^\sim_n(\xi)} = \prod_{m=1}^n M_{-\mathbb{i}\xi_m\mathfrak{e}^*_m}$.

Corollary 6. *For all $\psi^* \in P_\beta(H)$ and $\xi = (\xi_m) \in c_0$ there exists the limit*

$$W_\xi\psi^* = \lim_{n\to\infty} W_{p^\sim_n(\xi)}\psi^*, \quad W_{p^\sim_n(\xi)} := \exp\left\{-\frac{\|p^\sim_n(\xi)\|^2_{w_0}}{2}\right\} \prod_{m=1}^n M_{-\mathbb{i}\xi_m\mathfrak{e}^*_m} T_{\mathbb{i}\xi_m\mathfrak{e}_m}$$

$\mathfrak{w}$-almost everywhere on c_0 such that the 1-parameter Gaussian semigroup

$$\mathfrak{G}_r\psi^* = \frac{1}{\sqrt{4\pi r}} \int_{c_0} \exp\left\{\frac{-\|\xi\|^2_{w_0}}{4r}\right\} W_\xi\psi^* d\mathfrak{w}(\xi), \quad r > 0$$

is generated by $-\sum(\mathfrak{e}^*_m + \partial^*_m)^2$. Thus, $w(r) = \mathfrak{G}_r\psi^*$ is unique solution of the problem

$$\frac{dw(r)}{dr} = -\sum \left(\mathfrak{e}^*_m + \partial^*_m\right)^2 w(r), \quad w(0) = \psi^* \in P_\beta(H)$$

in the space of Hilbert–Schmidt polynomials $P_\beta(H)$.

References

[1] Bargmann, V.: On a Hilbert space of analytic functions and a associated integral transform. Commun. Pure Appl. Math. **I**(14), 187–214 (1961)

[2] Bargmann, V.: On a Hilbert space of analytic functions and a associated integral transform. Commun. Pure Appl. Math. **II**(20), 1–101 (1967)

[3] Beltiţă, I., Beltiţă, D., Măntoiu, M.: On Wigner transforms in infinite dimensions. J. Math. Phys. **57**, 1–13 (2016)

[4] Bourbaki, N.: Integration II. Springer, Berlin (2004)

[5] Carando, D., Zalduendo, I.: Linearization of functions. Math. Ann. **328**, 683–700 (2004)

[6] Fernique, M.X.: Intégrabilité des vecteurs Gaussiens. C. R. Acad. Sci. Paris, Sér. A **270**, 1698–1699 (1970)

[7] Floret, K.: Natural norms on symmetric tensor products of normed spaces. Note Mat. **17**, 153–188 (1997)

[8] Fulton, W.: Young Tableaux: With Applications to Representation Theory and Geometry. Cambridge University Press, Cambridge (2008)

[9] Gromov, M., Milman, V.: A topological application of the isoperimetric inequality. Am. J. Math. **105**, 843–854 (1983)

[10] Gross, L.: Abstract Wiener spaces. In: Doebner, H.D. (ed.) Proceedings of 5th Berkeley Symposium on Mathematical Statistics and Probability, part I(1). California University Press, pp. 31–42 (1965)

[11] Hall, B.: Quantum Theory for Mathematicians, Graduate Texts in Mathematics, vol. 267. Springer, Berlin (2013)

[12] Holmes, I., Sengupta, A.N.: The Gaussian Radon transform in the classical Wiener space. Commun. Stoch. Anal. **8**(2), 247–268 (2014)

[13] Itô, K.: Complex multiple Wiener integral. Jpn. J. Math. **22**, 63–86 (1952)

[14] Klenke, A.: Probability Theory. A Comprehensive Course. Springer, Berlin (2008)

[15] Kuo, H.: Gaussian measures in Banach spaces. Lecture Notes, vol. 463. Springer, Berlin (1975)

[16] Lopushansky, O.: The Hilbert–Schmidt analyticity associated with infinite dimensional unitary groups. Results Math. **71**(1), 111–126 (2017)

[17] Lopushansky, O.: Paley–Wiener isomorphism over infinite-dimensional unitary groups. Results Math. **72**(4), 2101–2120 (2017)

[18] Macdonald, I.G.: Symmetric Functions and Hall Polynomial, 2nd edn. Oxford University Press, Oxford (2015)

[19] Neeb, K.-H.: Holomorphy and Convexity in Lie Theory. De Gruyter Expositions in Mathematics, vol. 28, Berlin (2000)

[20] Nelson, E.: Regular probability measures on functions space. Ann. Math. **69**, 630–643 (1959)

[21] Neretin, Y.A.: Hua-type integrals over unitary groups and over projective limits of unitary groups. Duke Math. J. **114**(2), 239–266 (2002)

[22] Niemi, H., Weron, A.: Dilation theorems for positive definite operator kernels having majorants. J. Funct. Anal. **40**(1), 54–65 (1981)

[23] Okada, S., Okazaki, Y.: Projective limit of infinite Radon measures. J. Aust. Math. Soc. **25**(A), 328–331 (1978)

[24] Olshanski, G.: The problem of harmonic analysis on the infinite-dimensional unitary group. J. Funct. Anal. **205**(2), 464–524 (2003)

[25] Pickrell, D.: Measures on infinite-dimensional Grassmann manifolds. J. Funct. Anal. **70**, 323–356 (1987)

[26] Procesi, G.: Lie Groups: An Approach Through Invariants and Representations. Springer, Berlin (2007)

[27] Rao, M.M.: Projective limits of probability spaces. Multivar. Anal. **1**(1), 28–57 (1971)

[28] Rudin, W.: Function Theory in the Unit Ball of $\mathbb{C}^n$. Springer, Berlin (2008)

[29] Stroock, D.W.: Probability Theory: An Analytic View. Cambridge University Press, Cambridge (2010)

[30] Tomas, E.: On Prohorov's criterion for projective limits. In: de Pagner, B. (ed.) Operator Theory: Advances and Applications, vol. 168, no. 15, pp. 251–261. Birkhauser, Basel (2006)

[31] Voiculescu, D.: Limit laws for Random matrices and free products. Invent. Math. **104**(1), 201–220 (1991)

[32] Vrabie, I.I.: Co-semigroups and Applications, North-Holland Mathematics Studies, vol. 191. Elsevier, Amsterdam (2003)

[33] Willard, S.: General Topology. Dover, New York (2004)

[34] Yamasaki, Y.: Projective limit of Haar measures on $O(n)$. Publ. Res. Inst. Math. Sci. Kyoto Univ. **8**, 141–149 (1972/73)

[35] Yamasaki, Y.: Lecture Notes on Measures on Infinite Dimensional Spaces. World Scientific Publishing, Singapore (1985)

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