Characterization of Inner Product Spaces by Strongly Schur-Convex Functions

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Abstract. Involving the notion of strongly Schur-convex functions we give a new characterization of inner product spaces among norm spaces. We also present a representation theorem for functions which generate strongly Schur-convex sums.

Mathematics Subject Classification. Primary 46C15; Secondary 26B25, 39B62.

Keywords. Inner product space, Schur-convex, strongly Schur-convex, Wright-convex, strongly Wright-convex.

1. Introduction

A rich collection of characterizations of inner product spaces among norm spaces can be found in Amir's book [8] (see also [1, Chpt. 11], [5–7, 9, 21]). The best known characterization was given in the classic paper of Jordan and von Neumann [12]. Namely, a norm space $(X, \|\cdot\|)$ is an inner product space if and only if the following equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

holds true for all $x, y \in X$. We call this equality the Jordan–von Neumann identity or the parallelogram law.

In the paper [20] the authors show a characterization of functions which generate strongly Schur-convex sums, obtaining equivalently a representation of such functions. In particular, they prove that the following three sentences are equivalent.

- (1) Function f generates strongly Schur-convex sums.
- (2) Function f is strongly Wright-convex.

- (3) There exists a convex function g and an additive function a such that $f = g + a + c \|\cdot\|^2$.

For details, see Theorem 7, p. 180 [20]; it should be mentioned, though, that the domains of investigated functions are subsets of an inner product space; i.e. the unitarity of a space is a sufficient condition for such a result. We will prove that it is also a necessary condition, which, together with the previously mentioned theorem from [20] will give a new characterization of inner product spaces among norm spaces. Moreover, we will obtain a representation theorem for functions generating strongly Schur-convex sums, and it will be a counterpart of the classical Hardy–Littlewood–Pólya majorization theorem (see [11]), works of Schur [22], Karamata [13], Ng [18] and Kominek [14]. The construction of presented theorems in this work was inspired by the works [3, 19].

It is worth noting that beside theoretical applications Schur-convex functions have also practical applications in data transmission in cellular networks (see [15]). Thus it seems that research on these functions can be very useful.

2. Main Result

In this paper X will be a space and X^n , where $n \geq 2$ is a natural number, will be a Cartesian product of n -copies of the space X (i.e. $X^n := \underbrace{X \times \cdots \times X}_{n\text{-times}}$).

We will start with recalling some definitions presented, for example, in the literature given in brackets and also discussed there, respectively.

Definition 2.1. [16, 20, 22] Let X be a real vector space. For $x, y \in X^n$ we say that x is majorized by y , written $x \preceq y$, if

$$x = y \cdot P$$

for a doubly stochastic matrix P (i.e. a matrix of degree n containing non-negative elements with all rows and columns summing up to 1).

Definition 2.2. [20] Let $(X, \|\cdot\|)$ be a real normed space and D be a convex subset of X . We say that a function $F : D^n \rightarrow \mathbb{R}$ is strongly Schur-convex with modulus $c > 0$ if

$$F(x) \leq F(y) - c (\|y\|^2 - \|x\|^2)$$

for all $x, y \in D$ such that $x \preceq y$ and $\|\cdot\|$ is the Cartesian norm in the space X^n (i.e. $\|x\| = \sqrt{\|x_1\|^2 + \cdots + \|x_n\|^2}$ for n -tuples $x = (x_1, \dots, x_n) \in X^n$).

Definition 2.3. [17] Let $(X, \|\cdot\|)$ be a real normed space and D be a convex subset of X . We say that a function $f : D \rightarrow \mathbb{R}$ is strongly Wright-convex with modulus $c > 0$ if

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) - 2ct(1-t)\|x - y\|^2$$

for all $x, y \in D$ and $t \in [0, 1]$.

Definition 2.4. Let $(X, \|\cdot\|)$ be a real normed space and D be a convex subset of X . We say that a function $f : D \rightarrow \mathbb{R}$ generates strongly Schur-convex sums with modulus $c > 0$ if for all natural numbers $n \geq 2$ the function $F : D^n \rightarrow \mathbb{R}$ defined by formula $F(x_1, \dots, x_n) := f(x_1) + \dots + f(x_n)$, is strongly Schur-convex with modulus c .

The following theorem gives a new characterization of inner product spaces. To be clear, the phrase “For all functions $f : D \rightarrow \mathbb{R}$ ” is understood as “For all subsets D of X and all functions f defined on D ”.

Theorem 2.5. Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent:

- (1) For all $c > 0$ and all functions $f : D \rightarrow \mathbb{R}$, f generates strongly Schur-convex sums with modulus c if and only if f is strongly Wright-convex with modulus c ;
- (2) $(X, \|\cdot\|)$ is an inner product space.

Proof. The implication (2) $\Rightarrow$ (1) is proved in [20] (Theorem 7, p. 180). Assume that (1) holds true. Immediately from the definition of functions f generating strongly Schur-convex sums with modulus c for each $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in D^n$ ($n \geq 2$) such that $x \prec y$, we have the following two equivalent inequalities

$$F(x) \leq F(y) - c \left(\|y\|^2 - \|x\|^2 \right),$$

$$f(x_1) - c\|x_1\|^2 + \dots + f(x_n) - c\|x_n\|^2 \leq f(y_1) - c\|y_1\|^2 + \dots + f(y_n) - c\|y_n\|^2.$$

The function $f(\cdot) := c\|\cdot\|^2$ satisfies the above inequality, and it means that it generates strongly Schur-convex sums with modulus c . Thus, from the adopted assumption, the function $c\|\cdot\|^2 : X \rightarrow \mathbb{R}$ is strongly Wright-convex with modulus c and, consequently, it satisfies the following inequality

$$c\|tx + (1-t)y\|^2 + c\|(1-t)x + ty\|^2 \leq c\|x\|^2 + c\|y\|^2 - 2ct(1-t)\|x - y\|^2,$$

for all $x, y \in X$ and $t \in (0, 1)$. Dividing this inequality by c and taking $t = \frac{1}{2}$ we obtain the inequality

$$2 \left\| \frac{x+y}{2} \right\|^2 \leq \|x\|^2 + \|y\|^2 - \frac{1}{2}\|x - y\|^2, \quad x, y \in X.$$

Multiplying the above inequality by 2 and using the homogeneous axiom of norm, we can write the result in the following form

$$\|x + y\|^2 + \|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X.$$

Now, putting the standard substitution (i.e. $u = x + y$ and $v = x - y$) in the last inequality, we get the reverse inequality and it means that the norm $\|\cdot\|$ satisfies the parallelogram law, which, together with the paper of Jordan-von Neumann [12], implies that $(X, \|\cdot\|)$ is an inner product space. The proof is finished. $\square$

Remark 2.6. We can replace the first condition in Theorem 2.5 with the following:

- (1') For all $c > 0$, a function $f : X \rightarrow \mathbb{R}$ generates strongly Schur-convex sums with modulus c if and only if f is strongly Wright-convex with modulus c .

A representation theorem for functions generating strongly Schur-convex sums looks as follows.

Theorem 2.7. *Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent:*

- (1) *For a $c > 0$ and a function $f : D \rightarrow \mathbb{R}$, f generates strongly Schur-convex sums with modulus c ;*
- (2) *There exists a convex function $g : D \rightarrow \mathbb{R}$ and an additive function $a : X \rightarrow \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c \|x\|^2, \quad x \in D.$$

Proof. Assuming (1) and using the definition of functions f generating strongly Schur-convex sums with modulus c (Definition 2.4), we conclude that for each $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in D^n$ ($n \geq 2$) such that $x \preceq y$ we receive the following inequality

$$F(x) \leq F(y) - c (\|y\|^2 - \|x\|^2),$$

which is equivalent to the inequality

$$f(x_1) - c\|x_1\|^2 + \dots + f(x_n) - c\|x_n\|^2 \leq f(y_1) - c\|y_1\|^2 + \dots + f(y_n) - c\|y_n\|^2.$$

It means that the function $h(\cdot) := f(\cdot) - c\|\cdot\|^2$ generates Schur-convex sums. Thus, from the theorem of Ng characterizing functions generating Schur-convex sums (see [14, 18]), there exists a convex function $g : D \rightarrow \mathbb{R}$ and an additive function $a : X \rightarrow \mathbb{R}$ such that

$$h(x) = g(x) + a(x), \quad x \in D,$$

and consequently, f is of the form

$$f(x) = g(x) + a(x) + c \|x\|^2, \quad x \in D.$$

To prove (2) $\Rightarrow$ (1), observe that the function h defined as before (i.e. $h(\cdot) := f(\cdot) - c\|\cdot\|^2$) is a sum of convex and additive functions. And using once more the aforementioned Ng's theorem, it generates Schur-convex sums, i.e.

$$h(x_1) + \dots + h(x_n) \leq h(y_1) + \dots + h(y_n), \quad x \preceq y.$$

Which is equivalent to the following inequalities

$$\begin{aligned} f(x_1) - c\|x_1\|^2 + \dots + f(x_n) - c\|x_n\|^2 &\leq f(y_1) - c\|y_1\|^2 + \dots + f(y_n) - c\|y_n\|^2, \\ f(x_1) + \dots + f(x_n) &\leq f(y_1) + \dots + f(y_n) \\ &- c (\|y_1\|^2 + \dots + \|y_n\|^2 - \|x_1\|^2 - \dots - \|x_n\|^2), \end{aligned}$$

for all $x \preceq y$. Finally, taking the function

$$F(x_1, \dots, x_n) := f(x_1) + \dots + f(x_n)$$

and the Cartesian norm $||| \cdot |||$ in the space X^n , we get the inequality

$$F(x) \leq F(y) - c (|||y|||^2 - |||x|||^2), \quad x \preceq y.$$

It proves that f generates strongly Schur-convex sums with modulus c . This finishes the proof. $\square$

Combining Theorems 2.5 and 2.7 we immediately obtain the following theorem. Notice that the implication (2) $\Rightarrow$ (1) in Theorem 2.8 is also proved in [17].

Theorem 2.8. *Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent:*

- (1) *For all $c > 0$ and all functions $f : D \rightarrow \mathbb{R}$, f is strongly Wright-convex with modulus c if and only if there exists a convex function $g : D \rightarrow \mathbb{R}$ and an additive function $a : X \rightarrow \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c\|x\|^2, \quad x \in D;$$

- (2) *$(X, \|\cdot\|)$ is an inner product space.*

Remark 2.9. We can replace the first condition in Theorem 2.8 with the following:

- (1') *For all $c > 0$, a function $f : X \rightarrow \mathbb{R}$ is strongly Wright-convex with modulus c if and only if there exists a convex function $g : X \rightarrow \mathbb{R}$ and an additive function $a : X \rightarrow \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c\|x\|^2, \quad x \in X.$$

Remark 2.10. In virtue of Remarks 2.6 and 2.9, taking (1') instead of (1) in Theorems 2.5 and 2.8, respectively, we get stronger implications (1') $\Rightarrow$ (2) than (1) $\Rightarrow$ (2), but the implications (2) $\Rightarrow$ (1') become weaker than (2) $\Rightarrow$ (1).

We end this work with two examples which show that in Theorem 2.5 neither the assumption that f generates strongly Schur-convex sums implies that f is strongly Wright-convex, nor conversely. The examples' construction is based on the ideas of the examples from [10, 19].

Example 2.11. Let $X = \mathbb{R}^2$, $\|x\| = |x_1| + |x_2|$, for $x = (x_1, x_2)$ and $f(x) = \|x\|^2$. Observe the following equivalent inequalities.

$$\begin{aligned} f(x_1) + \dots + f(x_n) &\leq f(y_1) + \dots + f(y_n) - (|||y|||^2 - |||x|||^2), \\ f(x_1) + \dots + f(x_n) &\leq f(y_1) + \dots + f(y_n) \\ &\quad - \left(\|y_1\|^2 + \dots + \|y_n\|^2 - \|x_1\|^2 - \dots - \|x_n\|^2 \right), \\ \|x_1\|^2 + \dots + \|x_n\|^2 &\leq \|y_1\|^2 + \dots + \|y_n\|^2 \end{aligned}$$

$$- \left(\|y_1\|^2 + \dots + \|y_n\|^2 - \|x_1\|^2 - \dots - \|x_n\|^2 \right),$$

which means that f generates Schur-convex sums with modulus $c = 1$. However, f is not strongly Wright-convex with modulus $c = 1$. Indeed, for $x = (1, 0)$, $y = (0, 1)$ and $t = \frac{1}{2}$ we have

$$f\left(\frac{x+y}{2}\right) = 1 > 0 = \frac{f(x) + f(y)}{2} - \frac{1}{4} \|x - y\|^2.$$

Example 2.12. Let $X = \mathbb{R}^2$, $\|x\| = |x_1| + |x_2|$, for $x = (x_1, x_2)$. We show that the function $f(x) = x_1^2 + x_2^2$ is strongly Wright-convex with modulus $c = \frac{1}{2}$. First of all, observe that f is strongly midconvex with modulus $c = \frac{1}{2}$ (i.e. $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \frac{c}{4} \|x - y\|^2$, with $c = \frac{1}{2}$). Indeed, for arbitrary $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we have

$$f\left(\frac{x+y}{2}\right) = \frac{1}{4} (x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2)$$

and

$$\begin{aligned} \frac{f(x) + f(y)}{2} - \frac{1}{8} \|x - y\|^2 &= \frac{3}{8} (x_1^2 + y_1^2 + x_2^2 + y_2^2) \\ &\quad + \frac{1}{4} (x_1y_1 + x_2y_2 - |x_1 - y_1| |x_2 - y_2|). \end{aligned}$$

Thus

$$\frac{f(x) + f(y)}{2} - \frac{1}{8} \|x - y\|^2 - f\left(\frac{x+y}{2}\right) = \frac{1}{8} (|x_1 - y_1| - |x_2 - y_2|)^2 \geq 0,$$

which means that f is strongly midconvex with modulus $c = \frac{1}{2}$. And consequently, f satisfies the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}t(1-t) \|x - y\|^2, \quad (2.1)$$

for all dyadic numbers $t \in (0, 1)$ (see [2] and also a generalization in [4]). Moreover, f is continuous, thus it is a strongly convex function with modulus $c = \frac{1}{2}$ (i.e. f satisfies (2.1) for all $t \in (0, 1)$). Now, interchanging t with $1 - t$ in (2.1) and adding the obtained inequality side by side to the inequality (2.1), we conclude that f is strongly Wright-convex with modulus $c = \frac{1}{2}$.

Suppose that f generates strongly Schur-convex sums with modulus $c = \frac{1}{2}$. Then, from Theorem 2.7, there exists a convex function g and an additive function a such that

$$f(x) = g(x) + a(x) + \frac{1}{2} \|x\|^2.$$

Hence the function

$$h(x) := f(x) - \frac{1}{2} \|x\|^2 = g(x) + a(x)$$

would be midconvex, but it is not. Indeed, for $x = (-1, 1)$ and $y = (1, 1)$ we get

$$h\left(\frac{x+y}{2}\right) = \frac{1}{2} > 0 = \frac{h(x) + h(y)}{2}.$$

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Received: February 5, 2020.

Accepted: April 7, 2020.

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