



On the Representation of Fields as Finite Sums of Proper Subfields

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Abstract. We study which fields F can be represented as finite sums of proper subfields. We prove that for any $n \geq 2$ every field F of infinite transcendence degree over its prime subfield can be represented as an unshortenable sum of n subfields, and every rational function field $F = K(x_1, \dots, x_n)$ can be represented as an unshortenable sum of $n + 1$ subfields. We also show that no subfield of the algebraic closure of a finite field is a finite sum of proper subfields, and no finite extension of the field \mathbb{Q} of rationals can be decomposed into a sum of two proper subfields.

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1. Introduction

In this paper we study the problem of which fields F can be represented as a finite sum of proper subfields, i.e., in the form

$$F = F_1 + \cdots + F_n, \tag{1.1}$$

where F_1, \dots, F_n are proper subfields of F and $F_1 + \cdots + F_n$ denotes the set of all sums $f_1 + \cdots + f_n$ with $f_1 \in F_1, \dots, f_n \in F_n$. We call the sum (1.1) *unshortenable* if no subfield F_i can be omitted in (1.1), i.e., $F \neq F_1 + \cdots + F_{i-1} + F_{i+1} + \cdots + F_n$ for every $i \in \{1, \dots, n\}$.

The problem of decomposability of fields has not been studied systematically before. Our motivation for studying this problem comes from the following three areas.

- *Representing fields (rings) as finite unions of proper subfields (subrings).* In 1959, Białynicki-Birula, Browkin and Schinzel [3] proved that no field can be represented as a finite union of proper subfields. Analogous results were obtained later for some wider classes of rings (e.g., skew fields [24] and simple Artinian rings [20]). It was also shown that the corresponding assertion does not hold for integral domains (e.g., [3, 5]), and furthermore, all rings that are unions of three proper subrings were described in [22]. In the context of these results it is natural to ask, how situation will change if the requirement of being finite unions is weakened to being finite sums.

- *Properties of rings which are sums of two proper subrings.* In 1962, Kegel [11] proved that if a ring R is a sum of two subrings R_1, R_2 (i.e., $R = R_1 + R_2$) and both the subrings are nilpotent, then so is R . For rings of the form $R = R_1 + R_2$, Kegel's theorem and its generalizations in [16, 21] initiated systematic studies of relationships between properties of R_1, R_2 and those of R , mainly concerning polynomial identities and radicals. In [19] the structure of R when R_1 is nil and R_2 reduced was studied, and in [17] fields which are sums of two Jacobson radical subrings were characterized. The main line of research in the area was inspired by the long-standing unanswered Beidar-Mikhalev's question [2] of whether R must be a PI-ring if so are R_1 and R_2 . Partial solutions to the problem were obtained, e.g., in [1, 10, 12–14, 18], and a full positive answer in [15]. The well-known Bokut's theorem [4] (according to which every algebra over a field can be embedded into a simple algebra which is a sum of three nilpotent subalgebras) shows that the cases of sums of two subrings and sums of more than two subrings are totally different. In the context of these results it is natural to ask for a characterization of fields which are sums of two proper subfields.

- *Representing subfields of $\overline{\mathbb{F}}_q$ as sums of proper subfields.* In 2011 Dai, Gong, Song and Ye [7], proved that no finite field can be represented as a sum of proper subfields. From [17, Theorem 2.2] it follows that no subfield of the algebraic closure $\overline{\mathbb{F}}_q$ of a finite prime field \mathbb{F}_q is a sum of two proper subfields. These results give rise to the question whether a subfield of $\overline{\mathbb{F}}_q$ can be a finite sum of proper subfields.

In this paper we approach the problem of decomposability of a field F (into a finite unshortenable sum of subfields) by combining it with the transcendence degree of F over its prime subfield F_0 . Taking into account the transcendence degree, we partition all possibilities into three disjoint cases and consider the problem separately for each case: when the transcendence degree is infinite, when it is finite but nonzero, and when it is equal to 0. We solve the problem completely for the first case as well as for “half” of the last case (for the subcase when F is of nonzero characteristic), and partially

for the middle case and the remaining “half” of the last case (when F is of characteristic zero).

The main results of particular sections of the paper are as follows. In Sect. 2 we show that no field F of transcendence degree 0 over its prime subfield F_0 and characteristic $q > 0$ can be represented as a finite sum of proper subfields. In Sect. 3 we prove that for any $n \geq 2$ every field F of infinite transcendence degree over its prime subfield F_0 can be represented as an unshortenable sum of n subfields. In Sect. 4 we show that for any integer $n \geq 2$ and field K the field $F = K(x_1, \dots, x_n)$ of rational functions in n indeterminates x_1, \dots, x_n is an unshortenable sum of $n + 1$ subfields. In Sect. 5 we show that, on the one hand, no finite extension of the field \mathbb{Q} of rational numbers can be decomposed into a sum of two proper subfields, and on the other hand, for any $n \geq 3$ there exists a finite extension F of \mathbb{Q} that can be represented as an unshortenable sum of n subfields. In Sect. 6, we close with some open questions.

Let F be a field. In this paper, an algebraic closure of F is denoted by \overline{F} . If F is an extension of a field K , then $[F : K]$ denotes the degree of F over K , and $\text{tr.deg}_K F$ denotes the transcendence degree of F over K . If A_1, \dots, A_n are nonempty subsets of F , then $A_1 + \dots + A_n$ (also written as $\sum_{i=1}^n A_i$) denotes the set sum of A_1, \dots, A_n , i.e., the set of all sums $a_1 + \dots + a_n$, where $a_i \in A_i$ for any $i \in \{1, \dots, n\}$.

The composite of subfields F_1, \dots, F_n of a field F is denoted by $F_1 \cdots F_n$. Obviously, if $F = F_1 + \dots + F_n$, then $F = F_1 \cdots F_n$; we will use this observation freely in the sequel.

For a set A the symbol $|A|$ denotes the cardinality of A . A finite field of s elements is denoted by \mathbb{F}_s . The symbol \mathbb{N} stands for the set of positive integers.

2. No Subfield of $\overline{\mathbb{F}_q}$ is a Finite Sum of Proper Subfields

Let F be a field of transcendence degree 0 over its prime subfield F_0 and characteristic $q > 0$, or equivalently, let F be a subfield of the algebraic closure $\overline{\mathbb{F}_q}$ of a finite prime field \mathbb{F}_q . In this section we prove that F cannot be represented as a finite sum of its proper subfields. In our proof we use the well-known description of subfields of $\overline{\mathbb{F}_q}$ via Steinitz numbers (see, e.g., [6, Section 2.3] or [23, Section 9.8]). For convenience of the reader, as well as to fix notation, below we briefly present the connection between Steinitz numbers and subfields of $\overline{\mathbb{F}_q}$.

Let \mathbb{P} be the set of all prime numbers. A *Steinitz number* S is a formal product

$$S = \prod_{p \in \mathbb{P}} p^{\alpha_p},$$

where $\alpha_p \in \{0, 1, 2, \dots\} \cup \{\infty\}$. In what follows, the exponent α_p will be denoted by $S(p)$. Using the unique decomposition into prime powers, we can view any positive integer as a Steinitz number; therefore Steinitz numbers are sometimes called supernatural numbers or generalized natural numbers.

Let S and T be Steinitz numbers. We say that S divides T and write $S|T$ if $S(p) \leq T(p)$ for all $p \in \mathbb{P}$; in this case we denote by T/S the Steinitz number

$$\prod_{p \in \mathbb{P}} p^{T(p)-S(p)}, \tag{2.1}$$

with the convention $\infty - \infty = 0$ for the subtraction in the exponent of (2.1).

Let q be a prime number. For a Steinitz number S , let

$$\mathbb{F}_{q^S} = \bigcup_{d|S} \mathbb{F}_{q^d},$$

where each d is a positive integer,

i.e., the union is over all $d \in \mathbb{N}$ which divide S . Then by defining

$$\varphi(S) = \mathbb{F}_{q^S}$$

we obtain a bijection φ from the set of Steinitz numbers onto the set of subfields of $\overline{\mathbb{F}}_q$. Furthermore we have the following result (see, e.g., [6, Theorems 2.4 and 2.10] or [23, Theorem 9.8.4]).

Theorem 2.1 (Steinitz, 1910). *Let q be a prime number.*

- (1) *For any Steinitz numbers S and T ,*
 - (a) *$S|T$ if and only if $\mathbb{F}_{q^S} \subseteq \mathbb{F}_{q^T}$.*
 - (b) *If $S|T$, then $\mathbb{F}_{q^S} \subseteq \mathbb{F}_{q^T}$ is a finite field extension if and only if T/S is an ordinary integer, and in this case $[\mathbb{F}_{q^T} : \mathbb{F}_{q^S}] = T/S$.*
- (2) *For any Steinitz numbers S_1, \dots, S_n ,*
 - (a) *$\mathbb{F}_{q^{S_1}} \cdots \mathbb{F}_{q^{S_n}} = \mathbb{F}_{q^V}$, where*

$$V(p) = \max\{S_1(p), \dots, S_n(p)\} \text{ for all } p \in \mathbb{P};$$

- (b) *$\mathbb{F}_{q^{S_1}} \cap \cdots \cap \mathbb{F}_{q^{S_n}} = \mathbb{F}_{q^U}$, where*

$$U(p) = \min\{S_1(p), \dots, S_n(p)\} \text{ for all } p \in \mathbb{P}.$$

In our proof that no subfield of $\overline{\mathbb{F}}_q$ is a finite sum of proper subfields we will also need the following lemma. For nonempty subsets $\mathcal{B}_1, \dots, \mathcal{B}_n$ of a field F we define $\mathcal{B}_1 \cdots \mathcal{B}_n$ to be the set product of $\mathcal{B}_1, \dots, \mathcal{B}_n$, i.e., the set of all products $b_1 \cdots b_n$, where $b_i \in \mathcal{B}_i$ for any $i \in \{1, \dots, n\}$.

Lemma 2.2. *Let q be a prime number. Let K and L be subfields of the field $\overline{\mathbb{F}}_q$ and let $M = K \cap L$. If \mathcal{B}_1 and \mathcal{B}_2 are bases over M of K and L , respectively, then $\mathcal{B}_1 \mathcal{B}_2$ is a basis of KL over M .*

Proof. Since the extension $M \subseteq L$ is algebraic, \mathcal{B}_2 spans KL over K and thus $\mathcal{B}_1 \mathcal{B}_2$ spans KL over M . To prove that $\mathcal{B}_1 \mathcal{B}_2$ is linearly independent over M , it suffices to show that for any finite subsets $A \subseteq \mathcal{B}_1$ and $B \subseteq \mathcal{B}_2$ the set $AB = \{ab \mid a \in A, b \in B\}$ is linearly independent over M . Denote

$$P = M(A), \quad Q = M(B), \quad k = [P : M], \quad l = [Q : M].$$

Since both A and B are linearly independent over M , we can extend A to a basis \mathcal{A} of P over M , and B to a basis \mathcal{B} of Q over M . Let S, T, U be Steinitz numbers such that

$$P = \mathbb{F}_{q^S}, \quad Q = \mathbb{F}_{q^T}, \quad M = \mathbb{F}_{q^U}.$$

Theorem 2.1(1) implies that $k = S/U$ and $l = T/U$. Since $P \cap Q = M$, we deduce from Theorem 2.1(2b) that k and l are relatively prime, and thus \mathcal{AB} is a basis of PQ over M . Since $AB \subseteq \mathcal{AB}$, it follows that AB is linearly independent over M . □

We are now in a position to prove the main result of this section. In the proof, for given subfields K_1, \dots, K_n of a field F and $i \in \{1, \dots, n\}$, the composite of the $n - 1$ fields $K_1, \dots, K_{i-1}, K_{i+1}, \dots, K_n$, with K_i omitted, will be denoted by $(K_1 \cdots K_n)^{(i)}$, i.e.,

$$(K_1 \cdots K_n)^{(i)} = K_1 \cdots K_{i-1} K_{i+1} \cdots K_n.$$

We will apply analogous notation to elements $a_1, \dots, a_n \in F$ and to nonempty subsets $\mathcal{B}_1, \dots, \mathcal{B}_n \subseteq F$, i.e.,

$$(a_1 \cdots a_n)^{(i)} = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$$

and

$$(\mathcal{B}_1 \cdots \mathcal{B}_n)^{(i)} = \mathcal{B}_1 \cdots \mathcal{B}_{i-1} \mathcal{B}_{i+1} \cdots \mathcal{B}_n.$$

Theorem 2.3. *A subfield F of the algebraic closure $\overline{\mathbb{F}_q}$ of a finite prime field \mathbb{F}_q cannot be represented as a finite sum of proper subfields.*

Proof. Suppose, for a contradiction, that there exists a subfield $F \subseteq \overline{\mathbb{F}_q}$ such that F is a finite sum of its proper subfields. Hence there exists a minimal n such that

$$F = F_1 + \cdots + F_n \tag{2.2}$$

for some proper subfields F_1, \dots, F_n of F .

We show first that the field F can be built from some special “bricks” K_1, \dots, K_n . Namely, we show that there exist proper subfields K_1, \dots, K_n of F such that

- (i) $F = \sum_{i=1}^n (K_1 \cdots K_n)^{(i)}$;
- (ii) $F_j \cap F_m = \bigcap_{i=1}^n K_i$ for any $j \neq m$.

Let $I = \{1, \dots, n\}$ and let S (respectively, S_i) be the Steinitz number corresponding to the field F (respectively, F_i , where $i \in I$). For any $i \in I$, since F_i is a proper subfield of F , by Theorem 2.1(1) there exists $p_i \in \mathbb{P}$ such that $S_i(p_i) < S(p_i)$. Denote

$$\beta_i = S(p_i) \quad \text{and} \quad U_i = \{S_1(p_i), \dots, S_n(p_i)\}.$$

From (2.2) it follows that $F = F_1 \cdots F_n$ and thus part (2a) of Theorem 2.1 implies that β_i is the largest element of U_i . Since $S_i(p_i) < \beta_i$, the set $U_i \setminus \{\beta_i\}$ is

nonempty. Let α_i be the largest element of $U_i \setminus \{\beta_i\}$ and let L_i be the subfield of \mathbb{F}_q corresponding to the Steinitz number T_i defined by (for all $p \in \mathbb{P}$)

$$T_i(p) = \begin{cases} \alpha_i & \text{if } p = p_i, \\ S(p) & \text{if } p \neq p_i. \end{cases}$$

It is easy to see that $S_i \mid T_i$ and $T_i \mid S$, and since furthermore $T_i(p_i) = \alpha_i < \beta_i = S(p_i)$, we deduce that $T_i \neq S$. Hence $F_i \subseteq L_i \subsetneq F$ and thus (2.2) implies that

$$F = L_1 + \cdots + L_n, \tag{2.3}$$

where L_1, \dots, L_n are proper subfields of F .

If we would have $p_i = p_j$ for some different $i, j \in I$, then $T_i = T_j$, so $L_i = L_j$, which together with (2.3) contradicts the minimality of n . Hence the primes p_1, \dots, p_n are distinct and for any $i \in I$ it follows from the definition of T_i that

$$T_i = p_1^{\beta_1} \cdots p_{i-1}^{\beta_{i-1}} p_i^{\alpha_i} p_{i+1}^{\beta_{i+1}} \cdots p_n^{\beta_n} W,$$

where $\alpha_j < \beta_j$ for any $j \in I$, and W is the Steinitz number defined by

$$W(p) = \begin{cases} 0 & \text{if } p \in \{p_1, \dots, p_n\}, \\ S(p) & \text{if } p \in \mathbb{P} \setminus \{p_1, \dots, p_n\}. \end{cases}$$

Now for any $i \in I$ we define K_i to be the subfield of \mathbb{F}_q corresponding to the Steinitz number

$$p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\beta_i} p_{i+1}^{\alpha_{i+1}} \cdots p_n^{\alpha_n} W$$

(which arises from T_i by interchanging β_i and α_i). From Theorem 2.1(2) we deduce that

$$L_i = K_1 \cdots K_{i-1} K_{i+1} \cdots K_n \quad \text{for any } i \in I$$

and

$$K_j \cap K_m = \bigcap_{i=1}^n K_i \quad \text{for any different } j, m \in I,$$

which together with (2.3) shows that the fields K_1, \dots, K_n have the desired properties (i) and (ii).

To complete the proof, let us denote $M = \bigcap_{i=1}^n K_i$. For any $i \in I$, let \mathcal{B}_i be a basis of K_i over M such that $1 \in \mathcal{B}_i$. From (ii) and Lemma 2.2 it follows that for any different $i, j \in I$, $\mathcal{B}_i \mathcal{B}_j$ is a basis of $K_i K_j$ over M . Hence for any pairwise different $i, j, m \in I$, since $(K_i K_j) \cap K_m = M$ by (ii), Lemma 2.2 implies that $\mathcal{B}_i \mathcal{B}_j \mathcal{B}_m$ is a basis of $K_i K_j K_m$ over M . Using (ii) and Lemma 2.2 repeatedly, we deduce that

$$\text{for any } i \in I \text{ the set } (\mathcal{B}_1 \cdots \mathcal{B}_n)^{(i)} \text{ is a basis of } (K_1 \cdots K_n)^{(i)} \text{ over } M \tag{2.4}$$

and

$$\mathcal{B}_1 \cdots \mathcal{B}_n \text{ is a basis of } F = K_1 \cdots K_n \text{ over } M. \tag{2.5}$$

Furthermore, since each of the bases \mathcal{B}_j contains 1, for any $i \in I$ we have that

$$(\mathcal{B}_1 \cdots \mathcal{B}_n)^{(i)} \subseteq \mathcal{B}_1 \cdots \mathcal{B}_n. \tag{2.6}$$

Now, for any $i \in I$ we choose an element $a_i \in \mathcal{B}_i \setminus M$ and set $a = a_1 \cdots a_n$. It follows from (2.5) and (2.6) that

$$\text{the set } \{a\} \cup \bigcup_{i \in I} (\mathcal{B}_1 \cdots \mathcal{B}_n)^{(i)} \text{ is linearly independent over } M. \tag{2.7}$$

On the other hand, (i) and (2.4) imply that $a = c_1 + \cdots + c_n$, where each c_i is a linear combination over M of some elements of the basis $(\mathcal{B}_1 \cdots \mathcal{B}_n)^{(i)}$, and thus a is a linear combination of some elements of $\bigcup_{i \in I} (\mathcal{B}_1 \cdots \mathcal{B}_n)^{(i)}$. This and (2.7) imply that there exists $i \in I$ such that

$$a \in (\mathcal{B}_1 \cdots \mathcal{B}_n)^{(i)} \subseteq (K_1 \cdots K_n)^{(i)}$$

and thus $a_i = a(a_1^{-1}a_2^{-1} \cdots a_n^{-1})^{(i)} \in (K_1 \cdots K_n)^{(i)}$. Therefore,

$$a_i \in K_i \cap (K_1 \cdots K_n)^{(i)} = M,$$

and this contradiction completes the proof. □

As an immediate consequence of Theorem 2.3 we obtain the following result, which was proved in [7, Appendix].

Corollary 2.4. *A finite field F is not a sum of its proper subfields.*

Corollary 2.4 was applied in [7] in studying trace representation and linear complexity of binary eth power residue sequences of period p .

3. Every Field F with $\text{tr.deg}_{F_0} F = \infty$ is a Sum of n Proper Subfields

In the previous section we have proved that no field of prime characteristic and transcendence degree 0 over its prime subfield is a finite sum of proper subfields. In this section we show that for infinite transcendence degrees (independently of characteristic) the situation is completely different. Namely, we prove that for any $n \geq 2$ every field F of infinite transcendence degree over its prime subfield F_0 can be represented as an unshortenable sum of n subfields. Our proof of the result is based on Proposition 3.2, whose proof in turn applies the following lemma.

Lemma 3.1. *Let $L \subseteq M \subseteq E$ be field extensions and let S be a subset of E such that S is algebraically independent over M . Then $L(S) \cap M = L$.*

Proof. Obviously $L \subseteq L(S) \cap M$. To prove the opposite inclusion, consider any $c \in L(S) \cap M$. Since $c \in L(S)$, there exist distinct $a_1, a_2, \dots, a_m \in$

S , and distinct $b_1, b_2, \dots, b_n \in S$, and polynomials $f \in L[x_1, x_2, \dots, x_m]$, $g \in L[x_1, x_2, \dots, x_n]$ such that

$$c = \frac{f(a_1, a_2, \dots, a_m)}{g(b_1, b_2, \dots, b_n)}. \quad (3.1)$$

Let k be the number of all pairs a_i, b_j such that $a_i = b_j$, say $a_1 = b_1$, $a_2 = b_2, \dots, a_k = b_k$. Then

$$a_1, \dots, a_m, b_{k+1}, \dots, b_n \text{ are distinct elements of } S. \quad (3.2)$$

Let us consider the following polynomial belonging to $M[x_1, \dots, x_m, y_{k+1}, \dots, y_n]$:

$$h(x_1, \dots, x_m, y_{k+1}, \dots, y_n) = f(x_1, \dots, x_m) - cg(x_1, \dots, x_k, y_{k+1}, \dots, y_n).$$

Since S is algebraically independent over M and by (3.1) we have

$$h(a_1, \dots, a_m, b_{k+1}, \dots, b_n) = 0,$$

(3.2) implies that $h = 0$, i.e.,

$$cg(x_1, \dots, x_k, y_{k+1}, \dots, y_n) = f(x_1, \dots, x_m). \quad (3.3)$$

Since both g and f have coefficients in L and $g \neq 0$, it follows from (3.3) that $ca = b$ for some $a \in L \setminus \{0\}$ and $b \in L$. Hence $c = a^{-1}b \in L$, which completes the proof. \square

The following result will play a crucial role in the proof of the main result of this section.

Proposition 3.2. *Let F be a field and let K be a subfield of F such that $\text{tr.deg}_K F$ is infinite and $|K| \leq \text{tr.deg}_K F$. Then there exist proper subfields P, Q of F such that $F = P + Q$, $K \subseteq P \cap Q$, $\text{tr.deg}_{P \cap Q} Q$ is infinite and $|P \cap Q| \leq \text{tr.deg}_{P \cap Q} Q$.*

Proof. Let T be a transcendence basis of F over K . By hypothesis T is infinite and thus there exists an infinite sequence T_1, T_2, \dots of pairwise disjoint subsets of T such that

$$T = \bigcup_{i \in \mathbb{N}} T_i \quad \text{and} \quad |T_i| = |T| \text{ for any } i \in \mathbb{N}.$$

For any $i \in \mathbb{N}$, let A_i be the subfield of F consisting of all elements of F which are algebraic over $K(T_1 \cup T_2 \cup \dots \cup T_i)$. Since the field $K(T_1 \cup \dots \cup T_i)$ is infinite, we have $|A_i| = |K(T_1 \cup \dots \cup T_i)|$, and since $|K| \leq |T| = |T_1 \cup \dots \cup T_i|$ and the set T is infinite, we also have $|K(T_1 \cup \dots \cup T_i)| = |T|$. Thus $|A_i| = |T|$ and obviously $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$. Furthermore, $\bigcup_{i \in \mathbb{N}} A_i$ is the set of all elements of F which are algebraic over $K(T)$ and thus

$$F = \bigcup_{i \in \mathbb{N}} A_i. \quad (3.4)$$

For any $i \in \mathbb{N}$ we have $|A_i| = |T|$, hence $|A_i| = |T_{i+1}|$ and thus there exists a bijection $\varphi_i : A_i \rightarrow T_{i+1}$. Let

$$B_i = \{a - \varphi_i(a) \mid a \in A_i\}.$$

For any $a \in A_i$ we have $a = (a - \varphi_i(a)) + \varphi_i(a) \in B_i + T_{i+1}$ and thus

$$A_i \subseteq B_i + T_{i+1}. \tag{3.5}$$

Now we define subfields P_i and Q_i of F as follows. We set

$$P_1 = K \quad \text{and} \quad Q_1 = K$$

and for any integer $i \geq 1$ we define

$$P_{i+1} = P_i(B_i) \quad \text{and} \quad Q_{i+1} = Q_i(T_{i+1}).$$

Let

$$P = \bigcup_{i \in \mathbb{N}} P_i \quad \text{and} \quad Q = \bigcup_{i \in \mathbb{N}} Q_i. \tag{3.6}$$

Since $P_1 \subseteq P_2 \subseteq P_3 \subseteq \dots$ and $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \dots$, it follows that P and Q are subfields of F . Moreover, (3.4) and (3.5) imply that

$$F = P + Q$$

Below we show that $T_1 \not\subseteq P$ and $T_1 \not\subseteq Q$, which will imply that P and Q are proper subfields of F .

From the construction it is clear that $Q = K(T_2 \cup T_3 \cup \dots) = K(T \setminus T_1)$ and thus $T_1 \not\subseteq Q$. To show that $T_1 \not\subseteq P$, we will apply Lemma 3.1 to some $i \in \mathbb{N}$, taking in Lemma 3.1 $L = P_i, M = A_i, E = F, S = B_i$. To apply Lemma 3.1, we show first that for any $i \in \mathbb{N}$,

$$(*) \ P_i \subseteq A_i \quad \text{and} \quad (**) \ B_i \text{ is algebraically independent over } A_i.$$

To prove (*), observe that $P_1 = K \subseteq A_1$, and assuming (*) we obtain

$$P_{i+1} = P_i(B_i) \subseteq A_i(B_i) \subseteq A_i(A_i + T_{i+1}) = A_i(T_{i+1}) \subseteq A_{i+1}(T_{i+1}) = A_{i+1}.$$

Hence (*) follows by induction.

We now focus on proving (**). Observe that T_{i+1} is algebraically independent over $K(T_1 \cup \dots \cup T_i)$ and all elements of A_i are algebraic over $K(T_1 \cup \dots \cup T_i)$. Hence T_{i+1} is algebraically independent over A_i and thus

$$\forall_{a \in A_i} \varphi_i(a) \text{ is transcendental over } A_i(T_{i+1} \setminus \{\varphi_i(a)\}). \tag{3.7}$$

Since $\varphi_i : A_i \rightarrow T_{i+1}$ is a bijection, it follows that

$$T_{i+1} \setminus \{\varphi_i(a)\} = \{\varphi(c) \mid c \in A_i, c \neq a\}.$$

Hence the definition of B_i implies that

$$A_i(T_{i+1} \setminus \{\varphi_i(a)\}) = A_i(B_i \setminus \{a - \varphi_i(a)\}), \tag{3.8}$$

and from (3.7) and (3.8) we deduce that

$$\forall_{a \in A_i} a - \varphi_i(a) \text{ is transcendental over } A_i(B_i \setminus \{a - \varphi_i(a)\}). \tag{3.9}$$

Since every element of B_i is of the form $a - \varphi_i(a)$ for some $a \in A_i$, $(**)$ is an immediate consequence of (3.9).

We are now in a position to show that $T_1 \not\subseteq P$. Suppose for a contradiction that there exists $t \in T_1$ such that $t \in P$. Since clearly $t \notin K$, from the definition of P it follows that $t \in P_{i+1}$ for some $i \in \mathbb{N}$. Hence $t \in P_i(B_i)$. Since $T_1 \subseteq A_i$, it follows that $t \in P_i(B_i) \cap A_i$. Now $(*)$, $(**)$ and Lemma 3.1 imply that

$$t \in P_i(B_i) \cap A_i = P_i.$$

We have shown that if $t \in P_{i+1}$, then $t \in P_i$. Repeating this reasoning we finally get $t \in P_1 = K$, a contradiction. Hence $T_1 \not\subseteq P$.

We have proved above that P and Q are proper subfields of F such that $F = P + Q$. Since $P_1 = Q_1 = K$, we have that $K \subseteq P \cap Q$. Furthermore, since

$$P \cap Q \subseteq Q = K(T \setminus T_1), \tag{3.10}$$

it follows that

$$|P \cap Q| \leq |K(T \setminus T_1)| = |T|. \tag{3.11}$$

Since obviously T_1 is algebraically independent over $K(T \setminus T_1)$, from (3.10) it also follows that T_1 is algebraically independent over $P \cap Q$ and thus $\text{tr.deg}_{P \cap Q} F \geq |T_1| = |T|$. Hence, since $F = PQ$, we obtain that

$$|T| \leq \text{tr.deg}_{P \cap Q} F \leq \text{tr.deg}_{P \cap Q} P + \text{tr.deg}_{P \cap Q} Q,$$

and thus

$$|T| \leq \text{tr.deg}_{P \cap Q} P \text{ or } |T| \leq \text{tr.deg}_{P \cap Q} Q.$$

Eventually interchanging the symbols P and Q in (3.6), we can assume that $|T| \leq \text{tr.deg}_{P \cap Q} Q$, and from (3.11) we obtain that $|P \cap Q| \leq \text{tr.deg}_{P \cap Q} Q$, which completes the proof. \square

As an immediate consequence of Proposition 3.2 we obtain the following result (Corollary 3.3) on rational function fields. The task of proving this result was posed in 1974 by Dlab, Formanek and Ringel in [8], and its solution by Pelling appeared in [9]. Pelling proved even more, namely that any field having countably infinite degree of transcendence over its prime field is a sum of two proper subfields (which also is an immediate consequence of Proposition 3.2). We would like to stress that although Pelling’s solution suggested to us an idea of how to prove Proposition 3.2, our method significantly differs from that of Pelling.

Corollary 3.3 (Dlab, Formanek, Pelling, Ringel; 1976). *Let K be a countable field and let $F = K(x_1, x_2, \dots)$ be the rational function field over K in countably many indeterminates. Then*

$$\overline{F} = F_1 + F_2$$

for some proper subfields F_1, F_2 of \overline{F} .

Before we come to our next result, we make an important observation on Proposition 3.2. If F is a field and K is its subfield satisfying the hypothesis of Proposition 3.2, then a representation of F in the form $F = P + Q$ with P, Q satisfying all the conditions listed in Proposition 3.2 will be called a *decomposition of F over K* . Denoting $K_1 = P \cap Q$ we see from Proposition 3.2 that $\text{tr.deg}_{K_1} Q$ is infinite and $|K_1| \leq \text{tr.deg}_{K_1} Q$, and thus we can apply Proposition 3.2 once again to obtain a decomposition $Q = P_1 + Q_1$ of Q over K_1 . Next we can apply Proposition 3.2 to obtain a decomposition $Q_1 = P_2 + Q_2$ of Q_1 over $K_2 = P_1 \cap Q_1$, and so on. Hence we can repeat the decomposition procedure any finite number of times; this property will play a crucial role in the proof of the following theorem, which is the main result of this section.

Theorem 3.4. *Let F be a field of infinite transcendence degree over its prime subfield. Then for any integer $n \geq 2$ there exist subfields F_1, \dots, F_n of F such that $F = F_1 + \dots + F_n$ and the sum is unshortenable.*

Proof. Let $n \geq 2$ and let L be the prime subfield of F . Then obviously $|L| \leq \aleph_0$ and thus $|L| \leq \text{tr.deg}_L F$. Hence we can apply Proposition 3.2 to obtain a decomposition $F = F_1 + U_1$ of F over L . As noted in the paragraph preceding this theorem, we can continue, obtaining a decomposition $U_1 = F_2 + U_2$ of U_1 over $K_1 = F_1 \cap U_1$, next a decomposition $U_2 = F_3 + U_3$ of U_2 over $K_2 = F_2 \cap U_2$, and so on. We repeat the decomposition procedure $n - 1$ times, finally obtaining a decomposition $U_{n-2} = F_{n-1} + U_{n-1}$ of U_{n-2} over $K_{n-2} = F_{n-2} \cap U_{n-2}$. Hence

$$F = F_1 + U_1 = F_1 + (F_2 + U_2) = F_1 + F_2 + (F_3 + U_3),$$

and continuing this way we obtain that

$$F = F_1 + F_2 + F_3 + \dots + F_{n-1} + U_{n-1}.$$

Hence after denoting $F_n = U_{n-1}$ we have that

$$F = F_1 + F_2 + \dots + F_n, \tag{3.12}$$

where F_1, F_2, \dots, F_n are proper subfields of F .

It remains to show that the sum (3.12) is unshortenable. For a contradiction, suppose that some subfield F_i can be omitted in the sum (3.12), i.e.,

$$F_1 + \dots + F_n = F_1 + \dots + F_{i-1} + F_{i+1} + \dots + F_n \text{ for some } i. \tag{3.13}$$

If $i = 1$, then (3.13) implies $F_1 + \dots + F_n = F_2 + \dots + F_n$, i.e., $F = U_1$, which is a contradiction, since U_1 is a proper subfield of F . Hence $i \geq 2$ and from (3.13) and the modularity of the lattice of additive subgroups of F we obtain

$$\begin{aligned} (*) F_2 + \dots + F_n &= (F_1 + \dots + F_{i-1} + F_{i+1} + \dots + F_n) \cap (F_2 + \dots + F_n) \\ &= [F_1 + (F_2 + \dots + F_{i-1} + F_{i+1} + \dots + F_n)] \cap (F_2 + \dots + F_n) \\ &= F_1 \cap (F_2 + \dots + F_n) + F_2 + \dots + F_{i-1} + F_{i+1} + \dots + F_n \\ &= F_1 \cap U_1 + F_2 + \dots + F_{i-1} + F_{i+1} + \dots + F_n. \end{aligned}$$

Since by the construction we have

$$F_1 \cap U_1 \subseteq F_2 \cap U_2 \subseteq \dots \subseteq F_{n-1} \cap U_{n-1} = F_{n-1} \cap F_n,$$

it follows that

$$F_i \cap U_i \subseteq F_j \quad \text{for any } i \leq j \leq n.$$

Hence (*) yields

$$F_2 + \dots + F_n = F_2 + \dots + F_{i-1} + F_{i+1} + \dots + F_n. \quad (3.14)$$

If $i = 2$, then (3.14) implies $F_2 + \dots + F_n = F_3 + \dots + F_n$, i.e., $U_1 = U_2$, which is a contradiction, since U_2 is a proper subfield of U_1 . Thus $i \geq 3$. Continuing this way we obtain

$$F_{n-1} + F_n = F_{n-1} \quad \text{or} \quad F_{n-1} + F_n = F_n,$$

i.e., $U_{n-2} = F_{n-1}$ or $U_{n-2} = U_{n-1}$, which is impossible, since both F_{n-1} and U_{n-1} are proper subfields of U_{n-2} . This contradiction completes the proof. \square

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5. *Let F be an uncountable field. Then for any $n \geq 2$ there exist subfields F_1, \dots, F_n of F such that $F = F_1 + \dots + F_n$ and the sum is unshortenable.*

Corollary 3.5 applies, for instance, to the field \mathbb{R} of reals, the field \mathbb{C} of complex numbers, and the field of rational functions $\mathbb{R}(T)$, where T is any set of indeterminates.

4. Every Field $K(x_1, \dots, x_n)$ with $n \geq 2$ is a Sum of $n + 1$ Proper Subfields

In this section we study the problem of decomposability for fields F which are purely transcendental extensions of a field K , of finite transcendence degree ≥ 2 over K . Any such a field F is isomorphic to the rational function field $K(x_1, \dots, x_n)$ in $n \geq 2$ indeterminates x_1, \dots, x_n . We show that for any integer $n \geq 2$ and field K the field $K(x_1, \dots, x_n)$ can be represented in the form

$$K(x_1, \dots, x_n) = F_1 + F_2 + \dots + F_{n+1}, \quad (4.1)$$

where F_1, F_2, \dots, F_{n+1} are subfields of $K(x_1, \dots, x_n)$ and the sum (4.1) is unshortenable. As we will see, this result is an immediate consequence of the following general observation which also will be used in Sect. 5.

Proposition 4.1. *Let F be a field. If there exist a subfield M of F , an integer $n \geq 2$ and elements $a_1, \dots, a_n \in F$ such that*

$$F = M(a_1, \dots, a_n), \quad [F : M] = 2^n \quad \text{and} \quad a_i^2 \in M \quad \text{for any } i \in \{1, \dots, n\},$$

then F is an unshortenable sum of $n + 1$ subfields.

Proof. By hypothesis there exist a subfield M of F , an integer $n \geq 2$ and a set $U = \{a_1, \dots, a_n\} \subseteq F$ such that $F = M(U)$, $[F : M] = 2^n$ and $a_i^2 \in M$ for any $i \in \{1, \dots, n\}$. Since $a_1^2, \dots, a_n^2 \in M$, the field $M(U)$ consists of linear combinations over M of 1 and all possible products $b_1 \cdots b_k$, where $k \leq n$ and b_1, \dots, b_k are distinct elements of U . If some a_i does not appear in the product $b_1 \cdots b_k$, then $b_1 \cdots b_k \in M(U \setminus \{a_i\})$ and thus

$$M(U) = M(U \setminus \{a_1\}) + M(U \setminus \{a_2\}) + \cdots + M(U \setminus \{a_n\}) + M(a_1 a_2 \cdots a_n). \tag{4.2}$$

Hence $F = M(U)$ is a sum of $n + 1$ subfields.

It remains to show that the sum (4.2) is unshortenable. Aiming for a contradiction, suppose first that some subfield $M(U \setminus \{a_i\})$ can be omitted in (4.2). Without loss of generality we can assume that $i = 1$. Then

$$M(U \setminus \{a_1\}) \subseteq M(U \setminus \{a_2\}) + \cdots + M(U \setminus \{a_n\}) + M(a_1 \cdots a_n). \tag{4.3}$$

We claim that (4.3) implies

$$M(a_2) \subseteq M(a_1) + M(a_1 a_2). \tag{4.4}$$

This is obvious if $n = 2$, so assume that $n \geq 3$ and denote $V = U \setminus \{a_n\}$. Then (4.3) can be rewritten as

$$\begin{aligned} M(V \setminus \{a_1\})(a_n) &\subseteq M(V \setminus \{a_2\})(a_n) \\ &+ \cdots + M(V \setminus \{a_{n-1}\})(a_n) + M(V) + M(a_1 \cdots a_n) \end{aligned}$$

and thus for any $b \in M(V \setminus \{a_1\})$ there exist $c, d \in M(V \setminus \{a_2\}) + \cdots + M(V \setminus \{a_{n-1}\})$, $e \in M(V)$ and $f, g \in M$ such that $ba_n = c + da_n + e + f + ga_1 \cdots a_n$. Since $(b - d - ga_1 \cdots a_{n-1})a_n = c + e + f \in M(V) = M(U \setminus \{a_n\})$ and $a_n \notin M(U \setminus \{a_n\})$, it follows that $b - d - ga_1 \cdots a_{n-1} = 0$ and thus $b = d + ga_1 \cdots a_{n-1} \in M(V \setminus \{a_2\}) + \cdots + M(V \setminus \{a_{n-1}\}) + M(a_1 \cdots a_{n-1})$. Hence

$$M(V \setminus \{a_1\}) \subseteq M(V \setminus \{a_2\}) + \cdots + M(V \setminus \{a_{n-1}\}) + M(a_1 \cdots a_{n-1}), \tag{4.5}$$

i.e., assuming containment (4.3) for $U = \{a_1, \dots, a_n\}$, we obtain the analogous containment (4.5) for $V = \{a_1, \dots, a_{n-1}\}$. Continuing in this way, we obtain (4.4), which proves our claim.

By (4.4), $a_2 = g + m_1 + m_2 a_1 a_2$ for some $g \in M(a_1)$ and $m_1, m_2 \in M$. Since $(1 - m_2 a_1) a_2 = g + m_1 \in M(a_1)$ and $a_2 \notin M(a_1)$, it must be $1 - m_2 a_1 = 0$, and since obviously $m_2 \neq 0$, we obtain $a_1 = m_2^{-1} \in M$, a contradiction.

We have shown that no subfield $M(U \setminus \{a_i\})$ can be omitted in the sum (4.2). Analogously one can show that also the subfield $M(a_1 a_2 \cdots a_n)$ cannot be omitted in (4.2). Hence the sum (4.2) is unshortenable. \square

Now we are ready to prove the main result of this section.

Theorem 4.2. *Let K be a field. Then for any $n \geq 2$ there exist subfields F_1, \dots, F_{n+1} of the rational function field $K(x_1, \dots, x_n)$ such that $K(x_1, \dots, x_n) = F_1 + \cdots + F_{n+1}$ and the sum is unshortenable.*

Proof. It suffices to apply Proposition 4.1 to the fields $F = K(x_1, \dots, x_n)$ and $M = K(x_1^2, \dots, x_n^2)$, and elements $a_1 = x_1, \dots, a_n = x_n$. \square

We close this section with the following remark on representing rational function fields $K(x_1, \dots, x_n)$ as finite sums of proper subfields.

Remark 4.3. a) Let n and m be integers such that $n \geq 2$ and $3 \leq m \leq n + 1$. Then for any field K we have

$$K(x_1, \dots, x_n) = \left(K(x_m, x_{m+1}, \dots, x_n) \right) (x_1, x_2, \dots, x_{m-1}).$$

Hence from Theorem 4.2 it follows that the rational function field $K(x_1, \dots, x_n)$ is an unshortenable sum of m subfields.

b) If K is a field of infinite transcendence degree over its prime subfield, then Theorem 3.4 implies that for any integers $n \geq 1$ and $l \geq 2$ the rational function field $K(x_1, \dots, x_n)$ is an unshortenable sum of l subfields.

5. Some Fields $F \subseteq \overline{\mathbb{Q}}$ are Finite Sums of Proper Subfields and Some Aren't

In Sect. 2 we have proved that no field of prime characteristic and transcendence degree 0 over its prime subfield is a finite sum of proper subfields. In this section we show that situation is different for fields whose characteristic and transcendence degree over prime subfield both are equal to 0, i.e., for subfields of the algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} .

Let F be a subfield of $\overline{\mathbb{Q}}$. If $[F : \mathbb{Q}]$ is a prime number, then clearly F cannot be represented as a sum of two (or more than two) proper subfields. However, it is not so evident that F cannot be represented as a sum of two proper subfields in the case where $[F : \mathbb{Q}]$ is finite. This property of finite extensions of \mathbb{Q} is an immediate consequence of the following general result.

Proposition 5.1. *Let F be a field and let F_1, F_2 be proper subfields of F such that $F = F_1 + F_2$. Then $[F_1 : F_1 \cap F_2] = \infty$ and $[F_2 : F_1 \cap F_2] = \infty$.*

Proof. Denote $M = F_1 \cap F_2$. Let g_1, g_2, \dots, g_k be any k elements of F_2 that are linearly independent over M and let $a \in F_1 \setminus F_2$. Since $F = F_1 + F_2$, for any $i \in \{1, 2, \dots, k\}$ there exist $h_i \in F_1$ and $y_i \in F_2$ such that

$$ag_i = h_i + y_i. \quad (5.1)$$

We claim that the $k+1$ elements $1, h_1, h_2, \dots, h_k \in F_1$ are linearly independent over M . From the claim it follows that for any k , if $[F_2 : M] \geq k$, then $[F_1 : M] \geq k + 1$, and since assumptions on F_1 and F_2 are symmetric, this in turn implies that $[F_2 : M] \geq k + 2$, and the result follows by induction.

To prove the claim, let $m_0, m_1, \dots, m_k \in M$ be such that

$$m_0 + m_1 h_1 + \dots + m_k h_k = 0. \quad (5.2)$$

From (5.1) and (5.2) we obtain

$$a \sum_{i=1}^k m_i g_i = \sum_{i=1}^k m_i (a g_i) = \sum_{i=1}^k m_i h_i + \sum_{i=1}^k m_i y_i = -m_0 + \sum_{i=1}^k m_i y_i \in F_2. \tag{5.3}$$

Since $\sum_{i=1}^k m_i g_i \in F_2$ and $a \notin F_2$, (5.3) implies that $\sum_{i=1}^k m_i g_i = 0$, and since g_1, \dots, g_k are linearly independent over M , it must be $m_1 = m_2 = \dots = m_k = 0$. Hence also $m_0 = 0$ by (5.2), which proves the claim. \square

Immediately from Proposition 5.1 we obtain the aforementioned property of finite extensions of \mathbb{Q} .

Corollary 5.2. *If F is a subfield of $\overline{\mathbb{Q}}$ such that $F = F_1 + F_2$ for some proper subfields F_1, F_2 of F , then $[F : \mathbb{Q}] = \infty$.*

By Corollary 5.2, no field which is a finite extension of \mathbb{Q} can be decomposed into a sum of two proper subfields. As the following theorem shows, the situation is different when we consider decompositions of finite extensions of \mathbb{Q} into a sum of more than two proper subfields.

Theorem 5.3. *For every integer $n \geq 3$ there exists a field F such that F is a finite extension of \mathbb{Q} and F is an unshortenable sum of n subfields.*

Proof. Let $n \geq 3$. Denote $m = n - 1$ and let $U = \{a_1, a_2, \dots, a_m\}$ be any set of complex numbers such that $[\mathbb{Q}(U) : \mathbb{Q}] = 2^m$ and $a_i^2 \in \mathbb{Q}$ for any $i \in \{1, 2, \dots, m\}$ (for example, we can take $a_i = \sqrt{p_i}$, where p_i is the i th prime number). It follows from Proposition 4.1 that the field $F = \mathbb{Q}(U)$ is an unshortenable sum of $n = m + 1$ subfields. \square

6. Open Questions

We conclude the paper with four open questions for further study.

From Theorem 3.4 it follows that if F is a field of infinite transcendence degree over its prime subfield F_0 , then F is a sum of two proper subfields. We do not know whether the opposite implication holds.

Problem 1. *Is it true that a field F is a sum of two proper subfields if and only if the transcendence degree of F over its prime subfield F_0 is infinite?*

In Sect. 4 we have shown that for any integer $n \geq 2$ and field K the rational function field $K(x_1, \dots, x_n)$ is an unshortenable sum of $n+1$ subfields. We do not know whether the result can be extended to any field F of finite transcendence degree $n \geq 2$ over its prime subfield F_0 .

Problem 2. *Is it true that every field F of finite transcendence degree $n \geq 2$ over its prime subfield F_0 is an unshortenable sum of $n + 1$ subfields?*

By Theorem 4.2, for any $n \geq 2$ and prime field K the field $K(x_1, \dots, x_n)$ is an unshortenable sum of $n + 1$ subfields. We suspect that corresponding result for $n = 1$ fails, which leads to the following more general question.

Problem 3. *Is it true that no field F of transcendence degree 1 over its prime subfield F_0 is a sum of two proper subfields?*

Clearly, a negative answer to Problem 3 implies a negative answer to Problem 1.

In Sect. 5 we have shown that for any $n \geq 3$ there exists a subfield F of $\overline{\mathbb{Q}}$ which is an unshortenable sum of n subfields. We do not know whether an analogous result holds for $n = 2$.

Problem 4. *Does there exist a subfield F of $\overline{\mathbb{Q}}$ such that $F = F_1 + F_2$ for some proper subfields F_1, F_2 of F ?*

Obviously, a positive answer to Problem 4 implies a negative answer to Problem 1.

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