Lineability of Linearly Sensitive Functions

Artur Bartoszewicz, Małgorzata Filipczak, and
Małgorzata Terepeta

Abstract. In the paper we will focus on lineability of some subsets of $\mathbb{R}^{[0,1]}$ which are called linearly sensitive. A function f is called linearly sensitive with respect to the property (or condition) (P) if f has the property (P) and for any $a \neq 0$ the function $f + a \cdot \text{id}$ does not have the property (P) . We discuss some general method of proving $\mathfrak{c}$ -lineability and use this method to examine lineability of the family of all continuous functions linearly sensitive to the Luzin (N) -property, the family of functions linearly sensitive to the Świątkowski condition and the family of functions linearly sensitive to the strong Świątkowski condition.

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1. Introduction

Let κ be a cardinal number. We say that a subset M of a linear space V is κ -lineable if $M \cup \{0\}$ contains a linear subspace of dimension κ . If, additionally, V is an algebra, then in a similar way one can define an algebraability of subset of V . For a vector topological space V , a subset $M \subset V$ is called spaceable (dense lineable) if $M \cup \{0\}$ contains a closed infinitely dimensional subspace (dense subspace) of V . The lineability (algebraability) problem of subsets of function spaces has been studied during previous years by many authors (see [1, 2, 16]) and also recently (see [3, 6]). In the present paper we will focus on $\mathfrak{c}$ -lineability of some subsets of $\mathbb{R}^{[0,1]}$.

The common way of proving κ -lineability (in particular $\mathfrak{c}$ -lineability) of M is to construct a set of cardinality κ of linearly independent elements of M and to show that any linear combination of them belongs to M .

There are some subsets of $\mathbb{R}^{[0,1]}$ which are extremely nonlinear in certain senses. For example, it is well known that a sum of two Darboux functions can be any arbitrary function in $\mathbb{R}^{[0,1]}$. The same can be said about the class of perfectly everywhere surjective functions. On the other hand, the maximal additive class for Darboux functions consists of constant functions only, which means that for any nonconstant function f there exists a Darboux function g such that $f + g$ is not Darboux.

The classes of linearly sensitive functions with respect to the property (P) are also extremely nonlinear in a very strong sense.

Definition 1. Let (P) be a property of functions in $\mathbb{R}^{[0,1]}$. We say that f is linearly sensitive with respect to (P) if f has the property (P) and for any real $a \neq 0$ the function $f + a \cdot \text{id}$ does not have the property (P) .

2. The Main Method

In the present paper we consider

- continuous functions linearly sensitive with respect to the Luzin (N) -property,
- functions linearly sensitive with respect to the Świątkowski condition,
- functions linearly sensitive with respect to the strong Świątkowski condition.

We will prove that all these classes are $\mathfrak{c}$ -lineable. The fundamental method to prove it is contained in the following theorem:

Theorem 2. Let $M \subset \mathbb{R}^{[0,1]}$. Assume that there exists a sequence $([a_n, b_n])_{n \in \mathbb{N}}$ of closed disjoint intervals contained in $[0, 1]$ and a sequence $(F_n)_{n \in \mathbb{N}}$ of functions $F_n : [a_n, b_n] \rightarrow \mathbb{R}$ such that for any $A \subset \mathbb{N}$ and any bounded nonzero sequence of reals $(\alpha_i)_{i \in A}$, the function

$$G_{\alpha_i, A}(x) = \begin{cases} \alpha_i F_i(x) & \text{for } x \in [a_i, b_i], i \in A \\ 0 & \text{for } x \notin \bigcup_{i \in A} [a_i, b_i]. \end{cases} \quad (1)$$

belongs to M . Then M is $\mathfrak{c}$ -lineable.

An easy proof of Theorem 2 is based on the well known theorem on the cardinality of independent families of sets in $\mathcal{P}(X)$ for an arbitrary set X .

Theorem 3 (Fichtenholz-Kantorovich [8]). *For any set X there exists a family $\mathcal{B} \subset \mathcal{P}(X)$ of cardinality $2^{|X|}$ such that for any finite sequences $B_1, \dots, B_n \in \mathcal{B}$ and $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$*

$$B_1^{\varepsilon_1} \cap \dots \cap B_n^{\varepsilon_n} \neq \emptyset$$

(where $B^1 = B$ and $B^0 = X \setminus B$). Moreover all the sets in $\mathcal{B}$ can be chosen with the cardinality $|X|$.

Such a family $\mathcal{B}$ is called an independent one.

Proof of Theorem 2. Let $\mathcal{B} = \{B_\xi : \xi < \mathfrak{c}\}$ be an independent family in $\mathcal{P}(\omega)$ (where $\omega = \{0, 1, 2, \dots\}$). For any $\xi < \mathfrak{c}$ we put

$$f_\xi(x) = \begin{cases} F_i(x) & \text{for } x \in [a_i, b_i], i \in B_\xi \\ 0 & \text{otherwise.} \end{cases}$$

By (1), any function f_ξ for $\xi < \mathfrak{c}$ belongs to M . Moreover, $\{f_\xi : \xi < \mathfrak{c}\}$ is a family of linearly independent functions in M . Indeed, suppose that $f_\xi = c_1 f_{\xi_1} + \dots + c_k f_{\xi_k}$ for some $k \in \mathbb{N}$, $\xi_i < \mathfrak{c}$ and nonzero c_i , $i = 1, \dots, k$. Observe that $f_\xi \neq 0$ on $[a_n, b_n]$, $n \in B_\xi$. The sum $c_1 f_{\xi_1} + \dots + c_k f_{\xi_k}$ is nonzero on $[a_n, b_n]$ with $n \in B_{\xi_1} \cup \dots \cup B_{\xi_k}$. From the Fichtenholz-Kantorovich theorem it follows that there exists $n_0 \in B_\xi \setminus (B_{\xi_1} \cup \dots \cup B_{\xi_k}) = B_\xi^1 \cap B_{\xi_1}^0 \cap \dots \cap B_{\xi_k}^0 \neq \emptyset$.

To finish the proof it is sufficient to show that all the linear combinations of functions $\{f_\xi : \xi < \mathfrak{c}\}$ also belong to M . Take $f = c_1 f_{\xi_1} + c_2 f_{\xi_2}$ for some $\xi_1, \xi_2 < \mathfrak{c}$ and nonzero $c_1, c_2 \in \mathbb{R}$. Then

$$f(x) = \begin{cases} c_1 F_n(x) & \text{for } x \in [a_n, b_n], n \in B_{\xi_1} \setminus B_{\xi_2} \\ c_2 F_n(x) & \text{for } x \in [a_n, b_n], n \in B_{\xi_2} \setminus B_{\xi_1} \\ (c_1 + c_2) F_n(x) & \text{for } x \in [a_n, b_n], n \in B_{\xi_1} \cap B_{\xi_2} \\ 0 & \text{otherwise.} \end{cases}$$

Since f is of the form (1), $f \in M$. In the same way we can show that any linear combination of functions $\{f_\xi : \xi < \mathfrak{c}\}$ belongs to M . $\square$

Hence to prove $\mathfrak{c}$ -lineability of the classes of linearly sensitive functions described above it is sufficient to show that for these classes the assumptions of Theorem 2 are fulfilled.

A very similar method was used to prove $2^{\mathfrak{c}}$ -lineability (or even algebraability) in the papers [4, 5, 14].

3. Applications

In this section we will use the presented method to prove $\mathfrak{c}$ -lineability of families of functions linearly sensitive with respect to some known properties of real functions. We will start with the classic Luzin (N)-property introduced by Nikolai Luzin at the beginning of the twentieth century [10]. Recall that a function fulfills the Luzin (N)-property if the image of any null set has measure

zero. The concept of the Luzin (N) -property plays an essential role in the theory of absolute continuity. A continuous function may not have Luzin (N) -property (for instance the Cantor function), but if a continuous function is of bounded variation and has the Luzin (N) -property then it is absolutely continuous (Banach-Zaretsky theorem).

In 1928 S. Mazurkiewicz showed that there exists a real continuous function f which fulfills the Luzin (N) -property such that $f + \text{id}$ does not have this property. Two years later Mazurkiewicz generalized this construction and proved the following theorem.

Theorem 4 (Mazurkiewicz [13]). *There exists a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ fulfilling the Luzin (N) -property such that $f + a \cdot \text{id}$ does not satisfy the Luzin (N) -property for any $a \neq 0$.*

In fact, without any set-theoretic assumptions, Mazurkiewicz constructed a graph Q of some function as an intersection of a sequence of closed subsets of $\mathbb{R}^2$. The projections of Q on Ox and Oy are null sets while after transformation by adding $a \cdot \text{id}$ the projection of Q on Oy is a set of positive measure. Making use of the Tietze theorem, the set Q was complemented so that Mazurkiewicz obtained a graph of desired function defined on $[0, 1]$.

We will use the above theorem to examine lineability of the family $\mathcal{A}_N$ of all continuous functions linearly sensitive to the Luzin (N) -property.

Theorem 5. *The family $\mathcal{A}_N$ is $\mathfrak{c}$ -lineable.*

Proof. Let $F : [0, 1] \rightarrow \mathbb{R}$ be a function from Mazurkiewicz's theorem. Take an arbitrary sequence of intervals $\{[a_n, b_n] : n \in \mathbb{N}\}$ such that

$$0 \leq a_1 < b_1 < a_2 < b_2 < \dots < 1$$

For any $n \in \mathbb{N}$ we divide the interval $[a_n, b_n]$ into three segments: $[a_n, c_n]$, $[c_n, d_n]$ and $[d_n, b_n]$. By squeezing and translating the function F to the interval $[c_n, d_n]$, and adding two linear pieces, one can define a continuous function $F_n : [a_n, b_n] \rightarrow \mathbb{R}$ such that $F_n(a_n) = F_n(b_n) = 0$. For example F_n can be defined by the formula

$$F_n(x) = \begin{cases} \frac{1}{n} \cdot F(0) \cdot \frac{x-a_n}{c_n-a_n} & \text{for } x \in [a_n, c_n] \\ \frac{1}{n} \cdot F\left(\frac{x-c_n}{d_n-c_n}\right) & \text{for } x \in [c_n, d_n] \\ \frac{1}{n} \cdot F(1) \cdot \frac{b_n-x}{b_n-d_n} & \text{for } x \in [d_n, b_n]. \end{cases}$$

It is easy to check that F_n has the Luzin (N) -property.

On the other hand, for any $a \neq 0$ the function $F_a(x) := F(x) + ax$ does not have (N) -property. Thus, there is a null set $X \subset [0, 1]$ such that $F_a(X)$ is not a null set. Obviously, the set $Y := (d_n - c_n)X + c_n$ has the measure zero and it is contained in $[c_n, d_n]$. Simultaneously,

$$(F_n + a \cdot id)(Y) = \frac{1}{n} \cdot F_a(X) + a \cdot c_n,$$

so $(F_n + a \cdot id)(Y)$ is not a null set, which proves that F_n is linearly sensitive to (N) -property.

Now, fix a set $A \subset \mathbb{N}$ and a bounded sequence $(\alpha_i)_{i \in A}$ of nonzero numbers, and consider the function $G_{\alpha_i, A}$

$$G_{\alpha_i, A}(x) = \begin{cases} \alpha_i F_i(x) & \text{if } x \in [a_i, b_i], i \in A \\ 0 & \text{if } x \in [0, 1] \setminus \bigcup_{i \in A} [a_i, b_i]. \end{cases}$$

Observe that $G_{\alpha_i, A}$ is continuous. It is evident when A is a finite set. If A is infinite, the continuity of $G_{\alpha_i, A}$ at $\sup A$ is the result of boundedness of $(\alpha_i)_{i \in A}$.

To complete the proof it is enough to show that $G_{\alpha_i, A}$ is linearly sensitive with respect to (N) -property. It is not difficult to see that $G_{\alpha_i, A}$ has (N) -property. Indeed, suppose that a set $X \subset [0, 1]$ has measure zero. Since

$$X = \bigcup_{i \in A} (X \cap [a_i, b_i]) \cup \left([0, 1] \setminus \bigcup_{i \in A} [a_i, b_i] \right)$$

and functions F_i have (N) -property, the set

$$G_{\alpha_i, A}(X) = \bigcup_{i \in A} \alpha_i F_i(X \cap [a_i, b_i]) \cup \{0\}$$

has the measure zero, too.

Finally, fix a nonzero number a and choose any $i \in A$. We know that the function $F_i(x) + ax$ does not have (N) -property, so there is a null set $Y \subset [a_n, b_n]$ such that the image of Y under $F_i + a \cdot id$ is not a null set. Since $\alpha_i \neq 0$, $G_{\alpha_i, A}(Y)$ is not a null set either. $\square$

Observe that we have proved the lineability on the maximum level, because there are only $\mathfrak{c}$ continuous real functions on $[0, 1]$.

The Świątkowski condition was introduced in 1978 by Mańk and Świątkowski in the paper [11] connected with the sufficient conditions for the monotonicity of a function. They considered Darboux functions, Baire one functions and functions fulfilling some new condition, later called the Świątkowski condition (in [15]). They showed that those three conditions are independent and the sum of two functions with the Świątkowski condition may not satisfy this condition. It is worth noting that any continuous function satisfies the Świątkowski condition.

Definition 6 ([11]). We say that f satisfies the Świątkowski condition if for all $x_1 < x_2$ with $f(x_1) \neq f(x_2)$ there is a point $x \in (x_1, x_2)$ such that f is continuous at x and $f(x)$ lies between $f(x_1)$ and $f(x_2)$.

FIGURE 1. $y = F(x)$, $F \in \mathcal{A}_{\mathcal{S}}$

Theorem 7. *The family $\mathcal{A}_{\mathcal{S}}$ of all Baire one functions linearly sensitive to the Świątkowski condition is $\mathfrak{c}$ -lineable.*

Proof. Let $\tilde{F} : [0, 1] \rightarrow \mathbb{R}$ be defined by the formula

$$\tilde{F}(x) = \begin{cases} -1 & \text{if } x = 0 \\ 1 & \text{if } x = \frac{1}{2k}, k = 1, 2, 3, \dots \\ x^2 & \text{if } x = \frac{1}{2k-1}, k = 2, 3, \dots \\ 2 - 2x & \text{if } x = \frac{1}{2} + \frac{1}{2k}, k = 1, 2, 3, \dots \\ 2x - 2 & \text{if } x = \frac{1}{2} + \frac{1}{2k-1}, k = 1, 2, 3, \dots \\ \text{affine} & \text{otherwise} \end{cases}$$

and

$$F(x) = \begin{cases} \tilde{F}(x) & \text{for } x \in [0, 1] \\ \tilde{F}(-x) & \text{for } x \in [-1, 0). \end{cases}$$

Evidently, F (its graph is on Fig. 1) is a Baire one function as it has only three discontinuity points: at $-\frac{1}{2}$, at 0 and at $\frac{1}{2}$. Fix $x_1 < x_2$. If $F|_{[x_1, x_2]}$ is continuous then, by Darboux property, for any y lying between points $F(x_1)$ and $F(x_2)$ there is a point $x \in (x_1, x_2)$ such that $F(x) = y$. Of course, F is continuous at x . If $x_1 \in [-1, -\frac{1}{2})$ and $x_2 \notin [-1, -\frac{1}{2})$ then $F((x_1, x_2)) \supset F((x_1, -\frac{1}{2})) = (-1, 1)$, and again every value between $F(x_1)$ and $F(x_2)$ is achieved at some continuity point $x \in (x_1, x_2)$. If $x_1 \in [-\frac{1}{2}, 0]$ and $x_2 \in [0, \frac{1}{2}]$ then $F((x_1, x_2)) \supset F((x_1, 0)) = (0, 1)$. Therefore, there is a point $x \in (x_1, x_2)$ such that F is continuous at x and $F(x)$ lies between $F(x_1)$ and $F(x_2)$ even if x_1 (or x_2) is equal to 0. Finally, if $x_2 \in (\frac{1}{2}, 1]$ and $x_1 \notin (\frac{1}{2}, 1]$ then again $F((x_1, x_2)) \supset (-1, 1)$. Thus, F satisfies the Świątkowski condition.

FIGURE 2. $y = F(x) + ax$ for $a > 0$

Now, fix $a > 0$. Note that the parabola $y = x^2 + ax$ has the vertex at $-\frac{a}{2}$ (see Fig. 2). Therefore, for k such that $\frac{1}{2k-1} < \frac{a}{2}$, $x_1 = -\frac{1}{2k-1}$ and $x_2 = 0$ there is no $x \in (x_1, x_2)$ such that $F(x) < F(x_1)$. Hence $F + a \cdot \text{id}$ does not satisfy the Świątkowski condition. If $a < 0$ the situation is symmetrical.

Analogously as in the proof of Theorem 5, we fix a sequence of intervals $\{[a_n, b_n] : n = 1, 2, \dots\}$ such that

$$0 \leq a_1 < b_1 < a_2 < b_2 < \dots < 1.$$

and observe that for any n the function $F_n : [a_n, b_n] \rightarrow \mathbb{R}$ defined by the formula

$$F_n(x) = F\left(\frac{2(x - a_n)}{b_n - a_n} - 1\right)$$

is a Baire one function linearly sensitive to the Świątkowski condition and $F_n(a_n) = F_n(b_n) = 0$.

It remains to prove that the function

$$G_{\alpha_i, A}(x) = \begin{cases} \alpha_i F_i(x) & \text{if } x \in [a_i, b_i], i \in A \\ 0 & \text{if } x \in [0, 1] \setminus \bigcup_{i \in A} [a_i, b_i] \end{cases}$$

is a Baire one function linearly sensitive to Świątkowski condition, for any set $A \subset \mathbb{N}$ and any bounded sequence $(\alpha_i)_{i \in A}$ of nonzero numbers. Since $G_{\alpha_i, A}$ has at most countably many discontinuity points, it is a Baire one function. It is also easy to check that for any $a \neq 0$ the function $G_{\alpha_i, A} + a \cdot \text{id}$ does

not fulfill the Świątkowski condition. The only remaining point concerns the Świątkowski condition of the function $G_{\alpha_i, A}$.

Assume that $0 \leq x_1 < x_2 \leq 1$ and $|G_{\alpha_i, A}(x_1)| < |G_{\alpha_i, A}(x_2)|$. Since $|G_{\alpha_i, A}(x_2)| > 0$, there exists $n \in N$ such that $x_2 \in (a_n, b_n)$. If x_1 and x_2 belong to the same interval then the thesis follows from the fact that F_n fulfills the Świątkowski condition. Suppose that $x_1 \notin (a_n, b_n)$. If $G_{\alpha_i, A}$ is continuous on $[a_n, x_2]$ then $G_{\alpha_i, A}$ takes all values between 0 and $G_{\alpha_i, A}(x_2)$. In particular, it takes values between $G_{\alpha_i, A}(x_1)$ and $G_{\alpha_i, A}(x_2)$.

If there is a discontinuity point of $G_{\alpha_i, A}$ between a_n and x_2 , then $G_{\alpha_i, A}((a_n, x_2)) = \alpha_n F_n((a_n, x_2)) = (-|\alpha_n|, |\alpha_n|)$. Moreover, $G_{\alpha_i, A}|_{(a_n, a_n + \frac{b_n - a_n}{4})}$ is continuous,

$$G_{\alpha_i, A}\left(\left(a_n, a_n + \frac{b_n - a_n}{4}\right)\right) = (-|\alpha_n|, |\alpha_n|)$$

and $0 \leq |G_{\alpha_i, A}(x_1)| < |G_{\alpha_i, A}(x_2)| \leq |\alpha_n|$. If $|G_{\alpha_i, A}(x_2)| < |G_{\alpha_i, A}(x_1)|$ the considerations are analogous. $\square$

In the paper [17] Wódka proved that the family of all functions satisfying the Świątkowski condition is not $\mathfrak{c}^+$ -lineable, which means that even in the wider class (not only Baire one) of functions satisfying the Świątkowski condition the obtained result is on the best possible level.

Let us look closer to the sensitivity of Darboux functions. Kirchheim and Natkaniec [9] proved that if the union of fewer than $\mathfrak{c}$ of the first category subsets of $\mathbb{R}$ is a set of the first category ($\text{add}(\mathcal{M}) = \mathfrak{c}$), then there exists a Darboux function F such that $F + f$ is not Darboux for any continuous and nowhere constant function f . Based on the general method one can prove that the family of linearly sensitive functions with respect to the Darboux property is $\mathfrak{c}$ -lineable. On the other hand, if F is a Baire one Darboux function and f is continuous, then $F + f$ is a Baire one Darboux function (see [7]). Consequently, there is no Baire one function sensitive to the Darboux property.

Now we will consider a property which is a bit stronger than the Darboux property. It was introduced by Maliszewski [12] in 1995 as a modification of the Świątkowski condition.

Definition 8. We say that f satisfies the strong the Świątkowski condition if for all $x_1 < x_2$ with $f(x_1) \neq f(x_2)$ and each y between $f(x_1)$ and $f(x_2)$, there is a point $x \in (x_1, x_2)$ such that $f(x) = y$ and f is continuous at x .

Theorem 9. The family $\mathcal{A}_{s\acute{S}}$ of all Baire one functions linearly sensitive to strong Świątkowski condition is $\mathfrak{c}$ -lineable.

FIGURE 3. $y = F(x)$

Proof. Firstly consider the function $F: [-2, 2] \rightarrow \mathbb{R}$

$$F(x) = \begin{cases} 0 & \text{for } x \in [-2, -1] \cup [1, 2] \\ x + 1 & \text{for } |x| = 1 - \frac{1}{2^k} \text{ or } x \in [-\frac{1}{2}, \frac{1}{2}] \\ -x + 1 & \text{for } |x| = 1 - \frac{1}{2^{k-1}} \\ \text{affine} & \text{on intervals } \left[1 - \frac{1}{k}, 1 - \frac{1}{k+1}\right], \left[\frac{1}{k-1} - 1, \frac{1}{k} - 1\right] \text{ for } k > 2. \end{cases}$$

It is clear that F is a Darboux Baire one function. It satisfies the strong Świątkowski condition because F is discontinuous only at -1 and 1 , and

$$F(1) = F(-1) = 0 = \min \{F(x) : x \in [-2, 2]\}.$$

Indeed, take any $x_1 < x_2$ and y between $F(x_1)$ and $F(x_2)$. Since F is Darboux, there is $x \in (x_1, x_2)$ such that $F(x) = y$. Since $F(x) > 0$, F is continuous at x (see Fig. 3).

However, for any $a > 0$ the function $F + a \cdot \text{id}$ does not satisfy the strong Świątkowski condition. Indeed, the value $-a$ is taken only at discontinuity point of $F + a \cdot \text{id}$ (Fig. 4). For $a < 0$ the situation is analogous. Thus, F is linearly sensitive with respect to the strong Świątkowski condition.

Now, fix a sequence $\{[a_n, b_n] : n = 1, 2, \dots\}$ with $0 \leq a_1 < b_1 < a_2 < b_2 < \dots < 1$. For any $n \in \mathbb{N}$ the function $F_n : [a_n, b_n] \rightarrow \mathbb{R}$ defined by the formula

$$F_n(x) = F\left(\frac{4(x - a_n)}{b_n - a_n} - 2\right)$$

is Baire one and $F_n(a_n) = F_n(b_n) = 0$. It is not difficult to check that F_n is linearly sensitive to the strong Świątkowski condition.

FIGURE 4. $y = F(x) + ax$ for $a > 0$

For a fixed set $A \subset \mathbb{N}$ and a bounded sequence $(\alpha_i)_{i \in A}$ of nonzero numbers, consider the function $G_{\alpha_i, A}$

$$G_{\alpha_i, A}(x) = \begin{cases} \alpha_i F_i(x) & \text{if } x \in [a_i, b_i], i \in A \\ 0 & \text{if } x \in [0, 1] \setminus \bigcup_{i \in A} [a_i, b_i]. \end{cases}$$

It is clear that $G_{\alpha_i, A}$ is a Baire one function. To observe that $G_{\alpha_i, A}$ fulfills the strong Świątkowski condition fix $x_1, x_2 \in [0, 1]$. If both points belong to the same interval $[a_i, b_i]$ (or at least one of them belongs to $[0, 1] \setminus \bigcup_{i \in A} [a_i, b_i]$) then the existence of an appropriate point x follows from the fact that $\alpha_i F_i$ satisfies the strong Świątkowski condition. So let us assume that $x_1 \in [a_{i_1}, b_{i_1}]$ and $x_2 \in [a_{i_2}, b_{i_2}]$ for $i_1 \neq i_2$. Then any value between $G_{\alpha_i, A}(x_1)$ and 0 is achieved at continuity point of $\alpha_{i_1} F_{i_1}$, and any value between $G_{\alpha_i, A}(x_2)$ and 0 is achieved at continuity point of $\alpha_{i_2} F_{i_2}$.

Finally, consider the function $G_{\alpha_i, A} + a \cdot \text{id}$ for some nonzero a . The fact that this function does not satisfy the strong Świątkowski condition follows from the analogous property of functions $F_i + a \cdot \text{id}$ for $i \in A$. $\square$

Again, it is the lineability on the maximum level. Moreover, according to the paper [17], the last result is the best possible in the class of all functions linearly sensitive with respect to the strong Świątkowski condition.

Observe, that in the case of Luzin (N)-property we did not need the formula for a function. We could use any function which is sensitive with respect to (N)-property. When we consider the Świątkowski condition, the form of the function plays a crucial role. It is not enough to use any function linearly sensitive with respect to Świątkowski condition.

Indeed, the function H defined by the formula

$$H(x) = \begin{cases} 0 & \text{for } x \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \text{ and } |x| = \frac{1}{2k}, k \in \mathbb{N} \\ -2 & \text{for } x = 0 \\ x^2 - 1 & \text{if } |x| = \frac{1}{2k-1}, k \in \mathbb{N} \\ \text{affine} & \text{on the intervals} \end{cases}$$

is linearly sensitive with respect to the Świątkowski condition (see Figs. 5, 6).

FIGURE 5. $y = H(x)$

FIGURE 6. $y = H(x) + ax$, for $a > 0$

FIGURE 7. Functions of type H on two disjoint intervals $[a_n, b_n]$ and $[a_{n+1}, b_{n+1}]$

However, for two disjoint intervals the sum of functions obtained from H by squeezing and translating (and connected by $y = 0$ on $[b_n, a_{n+1}]$) may not satisfy the Świątkowski condition as we can observe on Fig. 7.

It is enough to take $x_1 = \frac{a_n + b_n}{2}$ and $x_2 = \frac{a_{n+1} + b_{n+1}}{2}$. Then the obtained function has the values: -1 at x_1 and -2 at x_2 , and the Świątkowski condition is not satisfied. So the function must be chosen in such a way that for any disjoint intervals the sum of such functions must still fulfill the Świątkowski condition.

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Artur Bartoszewicz and Małgorzata Terepeta
Institute of Mathematics
Lodz University of Technology
ul. Wólczańska 215
90-924 Lodz
Poland
e-mail: artur.bartoszewicz@p.lodz.pl

Małgorzata Terepeta
e-mail: malgorzata.terepeta@p.lodz.pl

Małgorzata Filipczak
Faculty of Mathematics and Computer Science
Łódź University
ul. Stefana Banacha 22
90-238 Lodz
Poland
e-mail: malgorzata.filipczak@wmii.uni.lodz.pl

Małgorzata Terepeta
Center of Mathematics and Physics, Lodz University of Technology
al. Politechniki 11
90-924 Lodz
Poland

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