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On the Mountain Pass Solutions to Boundary Value Problems on the Sierpinski Gasket

Marek Galewski

Abstract. We investigate the continuous dependence on parameters for the mountain pass solutions of second order elliptic equations with Dirichlet type boundary conditions considered on the Sierpinski gasket.

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1. Introduction

Let V denote the Sierpiński gasket in $\mathbb{R}^{N-1}$, $N \geq 2$, V_0 its intrinsic boundary, Δ the weak Laplacian on V and μ the restriction to V of the normalized $\log N/\log 2$-dimensional Hausdorff measure, so that $\mu(V) = 1$. We assume that

A *$a \in L^1(V,\mu)$, with $a \leq 0$ almost everywhere in V or else*

$$\int_V |a\,(y)|\, d\mu < \frac{1}{(2N+3)^2}. \tag{1.1}$$

Denote by $\Sigma \subset \mathbb{R}$ a fixed closed interval and

$$L_\Sigma := \left\{ w \in L^2(V,\mu) : w\,(y) \in \Sigma \text{ for a.e. } y \in V \right\}.$$

The aim of this note is investigate the dependence on parameters for mountain pass solutions of solutions of problems on the fractal setting. Namely, we are interested in the following problem subject to a parameter $w \in L_\Sigma$

$$\begin{cases} \Delta u(y) + a(y)u(y) = f(y, u(y), w(y)) & \text{for a.e. } y \in V \backslash V_0, \\ u|_{V_0} = 0, \end{cases} \tag{1.2}$$

where $f : V \times \mathbb{R} \times \Sigma \to \mathbb{R}$ is a continuous function with some growth conditions which we will formulate later.

The scheme which we apply can be summarized as follows: corresponding to a sequence of parameters there exists a sequence of solutions. Supposing that the sequence of parameters is convergent we arrive at the limit of a subsequence selected from a sequence of solutions; this limit is a solution to the considered problem and it corresponds to the limit of a sequence of parameters. Any solution we investigate is reached via mountain pass approach apart from the limit solution which need not have mountain geometry. It seems the outcome would not put any new information on the solution. However, we prove that it is a nontrivial solution to the problem under consideration whose localization is known.

The approach towards investigation of a dependence on a functional parameter for solutions of ODE in case of coercive action functional originates for example from [14]. It was further generalized to the case of solutions obtained through mountain pass technique in [4] but the method applied there is rather complicated while it provides that the limit solution is also reached by the mountain geometry. Our approach does not involve the investigation of action functional or operators for its dependence on a parameter. Instead we show that it suffice to investigate the equations itself and apply some version of the iterative technique. We have already undertaken investigations concerning dependence on parameters for mountain pass solutions in [10] however only for discrete problem. We used some results in the area of differential equations, [12,20] in order to combine the critical point approach using the mountain pass lemma in the discrete setting with some iterative technique. Now since we work in an infinite dimensional Banach space, the approach must be suitably modified. We would like to note that our results are new also for differential equations considered on classical domains.

The Sierpiński gasket has its origin in a paper by Sierpiński [17]. In two dimensions, this fractal domain can be described as the subset of the plane obtained from an equilateral triangle by removing first the open middle inscribed equilateral triangle of $1/4$ of the area, then removing the corresponding open triangle from each of the three constituent triangles and iterating this procedure. It can also be viewed as the closure of the set of vertices arising in this construction.

The background contributions to the theory of boundary value problems for nonlinear elliptic equations on fractals are [5,8,9]. For the complete analysis we refer to [13,18]. Recently existence and multiplicity results were obtained by a number of variational methods and critical point theory in [1,2,6,16].

We recall some basic tools which we use from the critical point theory, see [15,21]. Let E be a real reflexive Banach space and $J : E \to \mathbb{R}$. We say that J satisfies Palais–Smale condition (PS) condition for short- if for any sequence

$(u_n) \subset E$, such that $(J(u_n))$ is bounded and $J^{'}(u_n) \to 0$ as $n \to \infty$, there exists a convergent subsequence.

Lemma 1 (Mountain pass lemma). *Assume that* $J \in C^1(E, \mathbb{R})$ *and that* J *satisfies (PS) condition. Suppose also that*

1. $J(0) = 0$;
2. *there exist* $\rho > 0$ *and* $\alpha > 0$ *such that* $J(u) \geq \alpha$ *for all* $u \in E$ *with* $\|u\| = \rho$;
3. *there exist* u_1 *in* E *with* $\|u_1\| > \rho$ *such that* $J(u_1) < \alpha$.

Then J *has a critical value* $c \geq \alpha$. *Moreover,* c *can be characterized as*

$$\inf_{g \in \Gamma} \max_{t \in [0,1]} J(u(t)),$$

where $\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = u_1\}$.

For checking the (PS) condition the following result is of use

Theorem 1. *Let* $J \in C^1(E, \mathbb{R})$. *Suppose that*

$$J^{'}(u) = Lu + K(u),$$

where L *is an invertible linear continuous operator and* K *is a compact operator. Suppose also that any (PS) sequence for* J *is bounded in* E. *Then,* J *satisfies the (PS) condition.*

2. Fractal and Variational Framework

In this section we describe the functional setting of problem (1.2) following [6] where we also refer to some more detailed description.

Denote now by $C(V)$ the space of continuous functions on V and by

$$C_0(V) := \{u \in C(V) : u_{|V_0} = 0\};$$

endowed with supremum norm $\|\cdot\|_\infty$.

For $u : V \to \mathbb{R}$ and $n \in \mathbb{N}$, set

$$W_n(u) := \left(\frac{N+2}{N}\right)^n \sum_{\substack{x,y \in V_n \\ |x-y| = 2^{-n}}} (u(x) - u(y))\,(u(x) - u(y))$$

and define $W(u) := \lim_{n \to \infty} W_n(u)$. Put

$$H_0^1(V) := \{u \in C_0(V) : W(u) < \infty\}.$$

Then $H_0^1(V)$ is a dense linear subset of $L^2(V, \mu)$ endowed with a norm

$$\|u\| := \sqrt{W(u)}.$$

Such a norm is generated by the inner product $\mathcal{W}(u,v) := \lim_{n\to\infty} \mathcal{W}_n(u,v)$, where

$$\mathcal{W}_n(u,v) := \left(\frac{N+2}{N}\right)^n \sum_{\substack{x,y\in V_n \\ |x-y|=2^{-n}}} (u(x)-u(y))\,(v(x)-v(y))$$

for any $u,v \in H_0^1(V)$, $n \in \mathbb{N}$, and $H_0^1(V)$ is a real Hilbert space.

For every $u \in H_0^1(V)$, the following inequality holds

$$\|u\|_\infty \le (2N+3)\|u\|_{H_0^1(V)}, \tag{2.1}$$

and the following embedding is compact

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty). \tag{2.2}$$

Now, let Z be a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$. Then, in [9] it is defined a linear, bijective, self-adjoint operator $\Delta : Z \to L^2(V,\mu)$, the *weak Laplacian* on V, such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v d\mu, \tag{2.3}$$

for every $(u,v) \in Z \times H_0^1(V)$. Indeed, if $H^{-1}(V)$ denotes the closure of $L^2(V,\mu)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{\substack{h\in H_0^1(V) \\ \|h\|=1}} \left| \int_V u\,(y)\, h\,(y)\; d\mu \right|, \quad u \in L^2(V,\mu),$$

then the relation (2.3) uniquely defines a function $\Delta u \in H^{-1}(V)$ for every $u \in H_0^1(V)$.

Remark 1. Following [3] we observe that the norm

$$\|u\|_1 := \left(\mathcal{W}(u,u) - \int_V a(y) u^2\,(y)\, d\mu\right)^{1/2},$$

is equivalent to $\sqrt{W(u)}$ in $H_0^1(V)$. We have

$$\sqrt{1-(2N+3)^2 \int_V |a\,(y)|\, d\mu}\|u\| \le \|u\|_1 \le \sqrt{\left(1+(2N+3)^2 \int_V |a\,(y)|\, d\mu\right)}\|u\|.$$

3. The Assumptions and Auxiliary Lemmas

Let $F(y,u,w) = \int_0^u f(y,s,w)ds$ for $(y,u,w) \in V \times \mathbb{R} \times \Sigma$ and observe that the integration is with respect to the second variable which is real while $(y,w) \in V \times \Sigma$ are held fixed. We will employ the following assumptions.

H1 There exist constants $c > 0$ and $r > 2$ such that

$$|f(y,u,w)| \le c(1+|u|^{r-1}), \quad \text{for all } y \in V,\ u \in \mathbb{R},\ w \in \Sigma;$$

H2 $\lim_{x\to 0} \frac{f(y,x,w)}{|x|} = 0$ uniformly for all $y \in V$, $w \in \Sigma$;

H3 there exists a constant $\theta > 2$ such that

$$uf(y,u,w) \le \theta F(y,u,w) < 0, \quad \text{for all } y \in V,\ u \in \mathbb{R}\setminus\{0\},\ w \in \Sigma;$$

H4 there exist constants $c_1, c_2 > 0$ such that

$$F(y,u,w) \le -c_1|u|^\theta + c_2, \quad \text{for all } y \in V,\ u \in \mathbb{R},\ w \in \Sigma.$$

Concerning the above assumptions a few remarks are in order.

Remark 2. Note that **H2** implies assumption employed in [9]:
H2' *there are positive constants M_1, β such that*

$$\max_{y\in V, |v|\le M_1, w\in\Sigma} |f(y,v,w)| \le \frac{M_1}{2(\beta+1)(2N+3)^2}. \tag{3.1}$$

Moreover, instead of **H3** we may impose a slightly weaker assumption
H3' there exist a constant $\theta > 2$ and a constant $M > 0$ such that

$$uf(y,u,w) \le \theta F(y,u,w) < 0, \quad \text{for all } y \in V,\ |u| > M,\ w \in \Sigma.$$

We would obtain same results as here however with some minor technical changes.

Remark 3. It is well known, see for example [19], that when functions f, F do not depend on an additional parameter then from **H3** it follows that there are constants $c_1, c_2 > 0$ such that

$$F(y,u) \le -c_1|u|^\theta + c_2, \quad \text{for all } y \in V, u \in \mathbb{R}.$$

This is not the case with parameter dependence. Let $\theta = 4$ and define $f(u,w) = -4u^3 \exp(-w^2)$ which satisfies **H1**–**H3** and does not satisfy **H4**. We see that $F(u,w) = u^4 \exp(-w^2)$ and for any positive constants $c_1, c_2 > 0$ there exist $\overline{u}, \overline{w} \in \mathbb{R}$ such that

$$F(\overline{u},\overline{w}) \ge -c_1|\overline{u}|^4 + c_2. \tag{3.2}$$

When $c_1 > 1$, $c_2 > 0$ we take $\overline{u} = \left(\frac{c_2}{c_1-1}+1\right)^{\frac{1}{4}}$, $\overline{w} = 0$ so that to obtain (3.2). When $c_1 \le 1$, $c_2 > 0$ we can find $n \in \mathbb{N}$ such that $\frac{1}{c_1} < n$. We obtain (3.2) putting

$$\overline{u} = (c_2 n + 1)^{\frac{1}{4}}, \qquad \overline{w} = \left(\ln\left(c_1 - \frac{1}{n}\right)^{-1}\right)^{\frac{1}{2}}.$$

Assume that $w \in L_\Sigma$ is fixed to the end of this section if it is not said otherwise. We can derive the existence result for (1.2) from the main result contained in [9]. We put however here a different proof leading to the (PS) condition.

Theorem 2. *The action functional $I_w : H_0^1(V) \to \mathbb{R}$ given by*

$$I_w(u) = \frac{1}{2}\|u\|_1^2 + \int_V F(y,u(y),w(y))d\mu$$

is continuously differentiable and its critical points correspond to weak solutions to (1.2). For any $h \in H_0^1(V)$

$$I_w'(u)(h) = \mathcal{W}(u,h) - \int_V a(y)u(y)h(y)d\mu + \int_V f(y,u(y),w(y))h(y)d\mu.$$

We have the following simple proof of Proposition 2.24 from [9] which we restate below in an equivalent form.

Proposition 1. *If* $(u_n) \subset H_0^1(V)$ *is a Palais–Smale sequence for the functional* I_w *and if it is bounded, then* (u_n) *is a Cauchy sequence, i.e. it is convergent.*

Proof. Consider a continuously differentiable functional $J_1 : H_0^1(V) \to R$ defined as

$$J_1(u) = \int_V F(y,u(y),w(y))d\mu.$$

Then for any $h \in H_0^1(V)$

$$J_1'(u)(h) = \int_V f(y,u(y),w(y))h(y)d\mu.$$

Take a sequence $(u_n) \subset H_0^1(V)$ weakly convergent to some u_0. Then (u_n) is also uniformly convergent. By the Lebesgue Dominated Convergence Theorem we reach that $J_1^{'}$ is a compact operator. Since the weak Laplacian $(-\Delta) : H_0^1(V) \to H^{-1}(V)$ defines a linear invertible operator, we see that the assumptions of Theorem 1 are now fulfilled. □

Lemma 2. *Assume that* ***H2***, ***H3*** *are satisfied. Then functional* I_w *satisfies the (PS) condition.*

Proof. Suppose that $|I_w(u_n)| \le b$ and $\lim_{n\to\infty} I_w^{'}(u_n) = 0$. Since for any n

$$I_w^{'}(u_n)u_n = \|u_n\|_1 + \int_V f(y,u_n(y),w(y))u_n(y)d\mu,$$

then we see that for sufficiently large n

$$\begin{aligned} &b + \frac{1}{\theta}\|u_n\|_1 \ge I_w(u_n) - \frac{1}{\theta}I_w^{'}(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_1 + \frac{1}{\theta}\int_V (\theta F(y,u_n(y),w(y)) - f(y,u_n(y),w(y))\,u_n(y))d\mu. \end{aligned}$$

By **H3**, we have

$$\int_V (\theta F(y,u_n(y),w(y)) - f(y,u_n(y),w(y))u_n(y))\,d\mu \ge 0$$

Since $\frac{1}{2} - \frac{1}{\theta} > 0$ we see that (u_n) is bounded in $H_0^1(V)$. Thus we get the assertion by Proposition 1. □

Similarly to the proof in [9] we have

Theorem 3. *Assume that* ***H2****,* ***H3****,* ***H4*** *are satisfied. Then problem* (1.2) *has at least one nontrivial weak solution.*

Now we adapt a result on continuity of the Niemytskij operator taken from [11] to the case of our fractal setting.

Theorem 4. *If for any sequences* $(u_k) \subset L^2(V,\mu)$, $(w_k) \subset L_\Sigma$, *convergent to* $\overline{u} \in L^2(V,\mu)$, $\overline{w} \in L^2(V,\mu)$, *respectively, there exists a function* $h \in L^2(V,\mu)$ *such that*

$$|f(y,u_k(y),w_k(y))| \leq h(y), \qquad \text{for } k \in \mathbb{N} \text{ and a.e. } y \in V, \tag{3.3}$$

then the Niemytskij operator induced by f

$$N_f : L^2(V,\mu) \times L^2(V,\mu) \ni (u(\cdot), w(\cdot)) \longmapsto f(\cdot, u(\cdot), w(\cdot)) \in L^2(V,\mu),$$

is well defined and continuous.

Proof. Operator N_f is well defined by (3.3). By the assumptions both sequences have subsequences convergent almost everywhere, we denote them by the same symbol. By the continuity of f we obtain that for *a.e.* $y \in V$

$$f(y,u_k(y),w_k(y)) \to f(y,\overline{u}(y),\overline{w}(y))$$

Thus $|f(y,\overline{u}(y),\overline{w}(y))| \leq h(y)$ for *a.e.* $y \in V$. Using (3.3) we obtain

$$|f(y,u_k(y),w_k(y)) - f(y,\overline{u}(y),\overline{w}(y))|^2 \leq 4h^2(y)$$

for *a.e.* $y \in V$. Now by the Lebesgue Dominated Convergence Theorem we obtain

$$\lim_{k\to\infty} \int_\Omega |f(y,u_k(y),w_k(y)) - f(y,\overline{u}(y),\overline{w}(y))|^2 \, d\mu = 0.$$

□

4. Dependence on Parameters for (1.2)

We begin with providing some estimations on solutions which lead to the fact that the limit solution is also non-trivial. We underline that the limit solution need not be obtained via the mountain geometry and therefore it must be proved that this is nontrivial.

Theorem 5. *Assume that conditions* ***H1****–****H4*** *hold. Then, there exist constants* C_1, $C_2 > 0$ *such that for any* $w \in L_\Sigma$ *problem* (1.2) *has at least one nonzero solution* u_w *satisfying*

$$C_1 \leq ||u_w||_1 \leq C_2. \tag{4.1}$$

Proof. Let us fix $w \in L^2(V,\mu)$. By Theorem 3 there exists $0 \neq u_w^* \in E$ and $c^* > 0$ such that

$$I_w(u_w) = \inf_{g\in\Gamma} \max_{t\in[0,1]} I_w(g(t)) = c^*. \tag{4.2}$$

We will only prove that there is a constant $\alpha > 0$ such that

$$I_w(u_w) \geq \alpha \quad \text{for any } w \in L_\Sigma \tag{4.3}$$

by recalling part of a original proof from [9]. Indeed, from Remark 2 there are positive constants M_1, β such that (3.1) holds. Denote

$$\gamma := \frac{2N+3}{\sqrt{1-(2N+3)^2 \int_V |a\,(y)|\, d\mu}}.$$

If $u \in H_0^1(V)$ is such that $\|u\|_1 = \frac{M_1}{\gamma}$ we see by (2.1) and Remark 1 that $|u\,(y)| \leq M_1$ and obviously

$$\int_V |F(y, u\,(y), w\,(y))|\, d\mu \leq \int_V \int_0^{M_1} |f(y, s, w\,(y))|\, ds d\mu \leq \frac{(M_1)^1}{2\,(\beta+1)\,(2N+3)^2}$$

and therefore

$$I_w(u) \geq \frac{\beta}{(\beta+1)} \frac{(M_1)^1}{2\,(2N+3)^2} := \alpha > 0. \tag{4.4}$$

From (4.4) and Lemma 1 we see that (4.3) holds.

We will show that (4.1) holds. From **H1** and **H2** for a given $0 < \varepsilon < \frac{1}{2\gamma^2}$ there exists a positive constant c_ε, independent of w, such that

$$|f(y,u,w)| \leq \varepsilon|u| + c_\varepsilon |u|^{r-1} \tag{4.5}$$

for $y \in V$, $u \in \mathbb{R}$. Since u_w is a weak solution of (1.2) we can write

$$\|u_w\|_1^2 + \int_V f(y, u_w(y), w\,(y)) u_w(y) d\mu = 0$$

and we see by (4.5) and relations mentioned above that

$$\|u_w\|_1^2 \leq \varepsilon \int_V |u\,(y)\,|^2 d\mu + c_\varepsilon \int_V |u\,(y)\,|^r d\mu$$
$$\varepsilon\gamma^2 \|u_w\|_1^2 + c_\varepsilon^r \gamma \|u_w\|_1^r \leq \frac{1}{2} \|u_w\|_1^2 + c_\varepsilon \gamma^r \|u_w\|_1^r .$$

Therefore we may put

$$C_1 = \sqrt[r-2]{\frac{1}{2c_\varepsilon(2N+3)^r \gamma^r}} \tag{4.6}$$

and we see that $||u_w||_1 \geq C_1$.

From **H4** it follows that for $c_3 = c_1 \gamma^\theta$

$$\int_V F(y, u_w(y), w\,(y)) d\mu \leq -c_3 \|u_w\|_1^\theta + c_2. \tag{4.7}$$

By (4.2), (4.7) and since u_w is a solution to (1.2) we obtain

$$\frac{1}{\theta}(\frac{\theta}{2}-1)\|u_w\|_1^2 = \left(I_w(u_w) - \int_V F(y,u_w,w)d\mu\right) + \frac{1}{\theta}\int_V f(y,u_w,w)u_w(y)d\mu$$
$$\leq I_w(u_w) = \inf_{g\in\Gamma}\max_{t\in[0,1]} I_w(g(t)) \leq \max_{t\in[0,1]} I_w\left(t\frac{u_w}{||u_w||_1}\right)$$
$$\leq \max_{t\in[0,1]}\left(\frac{1}{2}t^2 - c_3t^\theta + c_2\right).$$

A continuous function $r(t) = \frac{1}{2}t^2 - c_3t^\theta$ achieves its maximum on $[0,1]$ at some $t_0 \in [0,1]$. Since $r(0) = 0$, it follows that $\max_{t\in[0,1]}(r(t)+c_2) \geq c_2 > 0$ and we put

$$C_2 = \frac{\theta(\frac{1}{2}t_0^2 - c_3t_0^\theta + c_2)}{\frac{\theta}{2}-1}. \tag{4.8}$$

□

Now we prove the result concerning the existence of a limit solution.

Theorem 6. *Assume that conditions* ***H1–H4*** *are satisfied. Let* $(w_k) \subset L_\Sigma$ *be a convergent sequence of parameters with* $\lim_{k\to\infty} w_k = w_0$*. For any sequence* (u_k) *of nontrivial solutions to* (1.2) *corresponding to* (w_k)*, there exist a subsequence* $(u_{k_i}) \subset H_0^1(V)$ *and an element* $u_0 \in H_0^1(V)$ *such that* $\lim_{i\to\infty} u_{k_i} = u_0$ *(strongly) and that* u_0 *is a non-zero solution to* (1.2) *corresponding to* w_0*. Moreover,* $C_1 \leq ||u_0||_1 \leq C_2$*.*

Proof. We define sequence $(u_n) \in H_0^1(V)$ taking u_n as a solution to (1.2) corresponding $w = w_n$. We note that $C_1 \leq \|u_n\|_1 \leq C_2$ for $n = 1, 2, \ldots$, where C_1, C_2 are determined by (4.6) and (4.8). Since $\|u_n\|_1 \leq C_2$, the sequence (u_n) can be assumed weakly convergent, up to a subsequence which we still denote (u_n). So there exists $u_0 \in H_0^1(V)$ such that $u_n \rightharpoonup u_0$ and $\|u_0\|_1 \leq C_2$. By (2.2) we may assume that $u_n \to u_0$ in $L^2(V,\mu)$. Since u_n is a weak solution to (1.2) we see that for any $h \in H_0^1(V)$ it holds:

$$\mathcal{W}(u_u,h) - \int_V a(y)u_n(y)h(y)d\mu + \int_V f(y,u_n(y),w_n(y))h(y)d\mu = 0.$$

Now we see by Theorem 4 that $f(\cdot,u_n(\cdot),w_n(\cdot)) \to f(\cdot,u_0(\cdot),w_0(\cdot))$ in $L^2(V,\mu)$. We observe that also $\mathcal{W}(u_u,h) \to \mathcal{W}(u_0,h)$ and (by the weak convergence)

$$\int_V a(y)u_n(y)h(y)d\mu \to \int_V a(y)u_0(y)h(y)d\mu$$

for any $h \in H_0^1(V)$. Thus we obtain that for any $h \in H_0^1(V)$ it holds:

$$\mathcal{W}(u_0,h) - \int_V a(y)u_0(y)h(y)d\mu + \int_V f(y,u_0(y),w_0(y))h(y)d\mu = 0$$

which implies that u_0 solves (1.2) for w_0.

Now we show that $u_n \to u_0$ in $H_0^1(V)$. Indeed, we see that

$$\mathcal{W}(u_0, u_0) - \int_V a(y)u_0(y)u_0(y)d\mu + \int_V f(y, u_0(y), w_0(y))u_0(y)d\mu = 0,$$

$$\mathcal{W}(u_n, u_n) - \int_V a(y)u_n^2(y)d\mu + \int_V f(y, u_n(y), w_n(y))u_n(y)\, d\mu = 0.$$

Since by Theorem 4 it follows that

$$\int_V f(y, u_n(y), w_n(y))u_n(y)\, d\mu \to \int_V f(y, u_0(y), w_0(y))u_0(y)d\mu \tag{4.9}$$

and since obviously $\int_V a(y)u_n^2(y)d\mu \to \int_V a(y)u_0(y)u_0(y)d\mu$ we obtain the following relation

$$\lim_{n\to\infty} \mathcal{W}(u_n, u_n) = \mathcal{W}(u_0, u_0).$$

Since also $u_n \rightharpoonup u_0$ we see by the well known properties of a real Hilbert space that $u_n \to u_0$.

Now we will prove that u_0 is nontrivial. Recall that $\|u_n\|_1 \geq C_1$. Since $u_n \to u_0$ in $H_0^1(V)$, we see that $\|u_0\|_1 \geq C_1$. □

5. Examples and Some Applications

We provide some examples and a remark on the type of convergence of the sequence of parameters. We finish our investigations with an application to a certain optimization problem.

Example 1. Let $g \in C(V)$ be such that $\inf_{y\in V} |g(y)| > 0$ and let $\Sigma = [-1, 1]$. Define $f : V \times \mathbb{R}\times\Sigma \to \mathbb{R}$ by

$$f(y, u, w) = -u^3(y)\left(1 + w^2(y)\right)|g(y)|.$$

Then

$$F(y, u, w) = -\frac{1}{4}u^4(y)\left(1 + w^2(y)\right)|g(y)|$$

and $\theta = 4$, $c_1 = \frac{1}{4}\inf_{y\in V}|g(y)|$, $c_2 = 0$, $c = 2\|g\|_\infty$, $r = 4$. Thus assumptions **H1**–**H4** are satisfied.

In case parameter w is involved in f in a linear manner, we can consider the case of weakly convergent sequences of parameters. The following theorem holds in that case which retains the assertion of Theorem 6

Theorem 7. *Assume that conditions **H1**–**H4** are satisfied and that $f(y, u, w) = f_1(y, u)w + f_2(y, u)$ for all $(y, u, w) \in V \times \mathbb{R} \times L_\Sigma$, where $f_i : V \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ are continuous. Let $(w_k) \subset L_\Sigma$ be a weakly convergent sequence of parameters such that $\lim_{k\to\infty} w_k = w_0$. For any sequence (u_k) of nontrivial solutions to (1.2) corresponding to (w_k), there exist a subsequence $(u_{k_i}) \subset H_0^1(V)$ and an element $u_0 \in H_0^1(V)$ such that $\lim_{i\to\infty} u_{k_i} = u_0$ (strongly)* *and that* u_0 *is a non-zero solution to (1.2) corresponding to* w_0*. Moreover,* $C_1 \leq ||u_0||_1 \leq C_2$*.*

Proof. The only change in the proof arises in demonstrating that (4.9) holds. By Theorem 4 it follows that $f_1(\cdot, u_n(\cdot))u_n(\cdot) \to f_1(\cdot, u_0(\cdot))u_0(\cdot)$ in $L^2(V, \mu)$. Since $w_k \rightharpoonup w_0$ we see that

$$\int_V f_1(y, u_n(y))u_n(y)\, w_n(y) d\mu \to \int_V f_1(y, u_0(y))u_0(y)w_0(y)d\mu.$$

This proves that (4.9) holds. The other parts of the proof follow exactly as the proof of Theorem 6 and thus are omitted. □

As far as the example is concerned we modify a bit Example 1 as follows:

Example 2. Let $g_i \in C(V)$ be such that $\inf_{y \in V} |g_i(y)| > 0$ and let $\Sigma = [-1, 1]$. Let $f : V \times \mathbb{R} \times \Sigma \to \mathbb{R}$ be given by

$$f(y, u, w) = -u^3(y)\, |g_1(y)| - u^3(y)\, |g_2(y)|\, w(y).$$

We observe that all calculations employed in Example 1 remain valid.

Now we consider an optimal control problem of minimizing the action functional

$$J_0 = \int_V f_0(y, u(y), w(y))\, d\mu$$

where the admissible pairs satisfy (1.2) and where

H5 $f_0 : V \times \mathbb{R} \times \Sigma \to \mathbb{R}$ *is continuous and for any fixed* $(y, u) \in V \times \mathbb{R}$ *function* $w \to f_0(y, u, w)$ *is convex.*

We construct a set $A \subset H_0^1(V) \times L_\Sigma \subset H_0^1(V) \times L^2(V, \mu)$ consisting of pairs (u_w, w) chosen as follows: we fix a function $w \in L_\Sigma$ and next using Theorem 5 we take u_w as a solution to (1.2) corresponding to w. We note that since the functions from L_Σ are pointwisely equibounded we get $\lim_{k\to\infty} w_k = \overline{w}$ weakly in $L^2(V, \mu)$, up to a subsequence, for any sequence $(w_k) \subset L_\Sigma$. Moreover, there is some $d > 0$ such that $\|u_w\|_\infty \leq d$ when $(u_w, w) \in A$.

Theorem 8. *Assume that conditions* ***H1–H5*** *hold. Then there exists a pair* $(\overline{u}_{\overline{w}}, \overline{w}) \in H_0^1(V) \times L^2(V, \mu)$ *such that* $J_0(\overline{u}_{\overline{w}}, \overline{w}) = \inf_{(u_w, w)\in A} J_0(u_w, w)$*.*

Proof. Since by the remarks preceding the theorem functional J_0 is bounded from below on A, we may choose a minimizing sequence (u_{w_k}, w_k), that is

$$\lim_{k\to\infty} J(u_{w_k}, w_k) = \inf_{(u_w, w)\in A} J_0(u_w, w).$$

Then sequence (w_k) can be assumed to be weakly convergent in $L^2(V, \mu)$ to a certain $\overline{w}$, possibly up to a subsequence. By Theorem 5 sequence (u_{w_k}) can be chosen so that it is norm convergent, again possibly up to a subsequence. Therefore, by convexity of J_0 with respect to the second variable

$$\inf_{(u_w, w)\in A} J_0(u_w, w) = \lim_{k\to\infty}\inf J_0(u_{w_k}, w_k) \geq J_0(\overline{u}_{\overline{w}}, \overline{w}) \geq \inf_{(u_w, w)\in A} J_0(u_w, w).$$

Therefore $(\overline{u}_{\overline{w}}, \overline{w})$ solves our optimization problem. □

Example 3. We conclude with the example of the integrand f_0 for which the assumptions of Theorem 8 hold. We put

$$f_0\,(y, u, w) = h\,(y)\,u^3 w^2,$$

where $h \in C\,(V, \mu)$ is positive on V.

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Marek Galewski
Institute of Mathematics
Lodz University of Technology
Wolczanska 215
90-924 Lodz
Poland
e-mail: marek.galewski@p.lodz.pl

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