




Prescribed Schouten Tensor in Locally Conformally Flat Manifolds

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Abstract. We consider the pseudo-Euclidean space (\mathbb{R}^n, g) , with $n \geq 3$ and $g_{ij} = \delta_{ij}\varepsilon_i$, where $\varepsilon_i = \pm 1$, with at least one positive ε_i and non-diagonal symmetric tensors $T = \sum_{i,j} f_{ij}(x)dx_i \otimes dx_j$. Assuming that the solutions are invariant by the action of a translation $(n - 1)$ - dimensional group, we find the necessary and sufficient conditions for the existence of a metric \bar{g} conformal to g , such that the Schouten tensor \bar{g} , is equal to T . From the obtained results, we show that for certain functions h , defined in \mathbb{R}^n , there exist complete metrics \bar{g} , conformal to the Euclidean metric g , whose curvature $\sigma_2(\bar{g}) = h$.

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1. Introduction

In recent years, problems involving the Ricci curvature have aroused great interest. Among the problems studied, we highlight the Einstein manifolds, Ricci solitons gradient, prescribed Ricci and Schouten tensor, prescribed curvature tensor, and Einstein field equation. For more details see [1, 3, 6, 10, 14, 19]. In [5], Deturck and Yang considered the following problem:

Given a Riemannian manifold (M^n, g_0) , with $n \geq 3$, and a symmetric tensor of order 2, T , defined in M^n , find a Riemannian metric g such that

$$Ric_g + \lambda K_g g = T \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a constant and Ric_g and K_g are the Ricci tensor and scalar curvature of g , respectively. They have shown that when T is non-singular,

the problem (1.1) has always a local solution. In this case, because the problem (1.1) admits local solutions, it makes sense to consider the following: find necessary and sufficient conditions so that problem (1.1) has a global solution; further, once global solutions are found, under what conditions are they complete? When $\lambda = 0$ and $\lambda = -\frac{1}{2}$, the problem (1.1) is known in the literature as the prescribed Ricci and Einstein tensor, respectively. This problem has been studied for a particular family of tensors (see [16–20]). Recently, Pulemotov studied the problem (1.1) for $\lambda = 0$ in homogeneous manifolds (see [21]). When $\lambda = \frac{-1}{2(n-1)}$, the problem (1.1) is equivalent to the prescribed Schouten tensor, because the Schouten tensor of a metric g is defined by

$$A_g = \frac{1}{n-2} \left(Ric_g - \frac{K}{2(n-1)}g \right).$$

Problem (1.1) has also been studied locally by Robert Bryant for any value of λ , proving that the problem always has local solutions when the components of the tensor are analytic functions. Motivated by the work of Deturck and Yang [5], our goal is to find global solutions to the following problem:

Given a $(0, 2)$ symmetrical tensor T defined in a manifold (M^n, g_0) , with $n \geq 3$, is there a metric g such that

$$A_g = T? \tag{1.2}$$

This problem corresponds to studying a system of nonlinear second order partial differential equations. The importance of Schouten tensors in conformal geometry can be seen in the following decomposition of the Riemann curvature tensor

$$R_g = W_g + A_g \odot g,$$

where R_g is the Riemann curvature tensor, \odot is the Kulkarni–Nomizu product, and W_g is the Weyl tensor of g (see [2]). Because the Weyl tensor is conformally invariant, i. e., $g^{-1}W_g$ is invariant in a given conformal class, in a conformal class the Schouten tensor is important, especially when g is locally conformally flat ($W_g = 0$). Therefore, if g is locally conformally flat, the Riemann curvature tensor is determined by the Schouten tensor. From the Schouten tensor, curvatures that extend the concept of the scalar curvature can be defined. This study was first conducted by Jeff Viaclovsky in [22]. For an integer $1 \leq k \leq n$ and σ_k - or k -scalar curvature, the Schouten curvature is defined by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot A_g),$$

where $(g^{-1} \cdot A_g)$ is defined locally by $(g^{-1} \cdot A_g)_{ij} = \sum_k g^{ik}(A_g)_{kj}$ and σ_k and the k -th symmetrical elementary function. Thus, we define $\sigma_k(g)$ as being the k -th elementary symmetric function of the auto-values of the operator $g^{-1}A_g$, to $1 \leq k \leq n$, where $\sigma_0(g) = 1$. Considering the eigenvalues of the Schouten tensor A_g ($\lambda_1, \lambda_2, \dots, \lambda_n$) with respect to the metric g , to $1 \leq k \leq n$, the k th polymorphic elementary symmetric functions σ_k are given by $\sigma_k(A_g) =$

$\sigma_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$. When $k = 1$, a $\sigma_1(g)$ -scalar curvature is exactly the scalar curvature (less than one constant). Thus, $\sigma_1(g)$ is constant if and only if (M^n, g) has constant scalar curvature. In [11], the authors considered the problem of classifying compact Riemannian manifolds locally conformally flat with $\sigma_k(g)$ constant for some $k \geq 2$.

In [22], Viaclovsky also noted that $\sigma_2(g)$ has still a variational structure. For $k > 2$, $\sigma_k(g)$ has a variational structure, if and only if the manifold considered is locally conformally flat. From this work of Viaclovsky and the work of Chang, Gursky, and Yang in [4], an intensive investigation started for the variational problem related to the Schouten curvature $\sigma_k(g)$, seeking to find a metric g , in the class of $[g_0]$, satisfying

$$\sigma_k(g) = c, \tag{1.3}$$

where

$$g \in [g_0] \cap \Gamma_k^+.$$

and Γ_k^+ is a convex open cone (the Garding cone) defined by

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in R^n \mid \sigma_j(\Lambda) > 0, \forall j \leq k\}.$$

Here, $g \in \Gamma_k^+$ represents the Schouten tensor $A_g(x) \in \Gamma_k^+$, for any $x \in M$; an important fact is that $g \in \Gamma_k^+$ guarantees that equation (1.3) is elliptic.

Several authors have recently studied subjects related to the Schouten curvature, see for example [7, 8], and [11]. In [7], the authors consider the problem $\frac{\sigma_2}{\sigma_1} = f$, where f is a given differentiable function.

In [12], Simon et al. showed that, if $(M^n; g)$ is a compact locally conformally flat manifold with nonzero curvature $\sigma_k(A_g)$ for some $2 \leq k \leq n$ and A_g defined as semi-positive, then (M^n, g) is a space form of positive sectional curvature. In [13], Simon et al. studied the extreme properties of the Schouten function defined in the quotient of the Riemannian metric space by the group of diffeomorphisms.

In [15], the authors considered the pseudo-Euclidean space (\mathbb{R}^n, g) , with $n \geq 3$ and $g_{ij} = \delta_{ij}\varepsilon_i, \varepsilon_i = \pm 1$, and tensors of the form $T = \sum_i \varepsilon_i f_i(x) dx_i^2$, and found necessary and sufficient conditions for the existence of a metric \bar{g} , conformal to g , such that $A_{\bar{g}} = T$. The solution to this problem was explicitly given for special cases of the tensor T , including a case where the metric \bar{g} is complete in \mathbb{R}^n . Similar problems were considered for locally conformally flat manifolds. As an application of these results, the authors considered the problem of finding metrics \bar{g} , conformal to g , such that $\sigma_2(\bar{g})$ or $\frac{\sigma_2(\bar{g})}{\sigma_1(\bar{g})}$ are equal to a certain function.

In this work we will consider the pseudo-Euclidean space (\mathbb{R}^n, g) , with $n \geq 3$, coordinates $x = (x_1, \dots, x_n)$, and metric g , where $g_{ij} = \delta_{ij}\varepsilon_i$, with $\varepsilon_i = \pm 1$, with at least one positive ε_i , and a non-diagonal tensor of order 2 of the form $T = \sum_{i,j} f_{ij}(x) dx_i \otimes dx_j$, where $f_{ij}(x), 1 \leq i, j \leq n$, are differentiable

functions. We want to find metrics $\bar{g} = \frac{1}{\varphi^2}g$, such that the Schouten tensor of the metric \bar{g} is T , that is, we want to solve the following problem:

$$\begin{cases} A_{\bar{g}} = T \\ \bar{g} = \frac{1}{\varphi^2}g. \end{cases} \tag{1.4}$$

To obtain solutions to the problem (1.4), let us assume that the metric \bar{g} is invariant by the action of a $(n - 1) - dimensional$ translation group. In this case, we find necessary conditions on the tensor T , so that the problem admits solution (Lemma 2.2). For this special class of metrics, we find necessary and sufficient conditions for the problem to have solutions (Theorem 2.3). As a consequence of the Theorems (2.3) and (2.5) we obtain complete metrics in Euclidean space \mathbb{R}^n , with prescribed Schouten tensors. The results obtained were extended to locally conformally flat manifolds (Theorem 2.6).

As applications of these results, we show explicit solutions for a second order nonlinear partial differential equation in \mathbb{R}^n . The geometric interpretation of this result is equivalent to finding conformal metrics in \mathbb{R}^n with $\sigma_2(\bar{g})$ prescribed. In particular, by considering $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we find examples of complete metrics \bar{g} , conformal to the Euclidean metric, such that $\sigma_2(\bar{g}) = f$.

2. Main Results

Let $\varphi_{,x_i x_j}$ and f_{ij,x_k} denote the second order derivatives of φ with respect to $x_i x_j$ and the derivative of f_{ij} with respect to x_k , respectively.

Theorem 2.1. *Let (\mathbb{R}^n, g) , with $n \geq 3$, be the pseudo-Euclidean space, with coordinates $x = (x_1, \dots, x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, and $T = \sum_{i,j=1}^n f_{ij}(x)dx_i \otimes dx_j$ be a non-diagonal tensor of order 2, where $f_{ij}(x)$ are differentiable functions. Then, there exists a positive function φ such that the metric $\bar{g} = \frac{1}{\varphi^2}g$ satisfies $A_{\bar{g}} = T$ if and only if the functions f_{ij} and φ satisfy the following set of equations:*

$$\begin{cases} 2\varphi\varphi_{x_i x_i} - \sum_{k=1}^n \epsilon_k (\varphi_{x_k})^2 \epsilon_i - 2\varphi^2 f_{ii} = 0 & \forall i : 1, \dots, n, \\ \varphi_{x_i x_j} - f_{ij}\varphi = 0 & \forall i \neq j. \end{cases} \tag{2.1}$$

In an attempt to find solutions to the system (2.1) we will consider the solutions $\bar{g} = \frac{1}{\varphi^2}g$ to be invariant under the action of a group $(n - 1)$ -dimensional, where $\xi = \sum_{i=1}^n a_i x_i$, with $a_i \in \mathbb{R}$, is a basic invariant of the group.

Initially, we will determine the necessary conditions on the tensor T .

Lemma 2.2. *Consider $\varphi = \varphi(\xi)$, where $\xi = \sum_i^n a_i x_i$, with $a_i \in \mathbb{R}$. If $\varphi = \varphi(\xi)$ is the solution of the system (2.1), then $f_{ij} = a_i a_j f(\xi) \forall i \neq j = 1, \dots, n$, and $f_{ii} = f_{ii}(\xi)$, $\forall i = 1, \dots, n$, where f is a differentiable function.*

From Lemma (2.2), we can state one of the main theorems of this section. We consider $\sum_{k=1}^n \epsilon_k a_k^2 \neq 0$ and, without loss of generality, we assume that $\sum_{k=1}^n \epsilon_k a_k^2 = \epsilon = \pm 1$, and the case where $\sum_{k=1}^n \epsilon_k a_k^2 = 0$ will be dealt with later.

Theorem 2.3. Let (\mathbb{R}^n, g) , with $n \geq 3$, be the pseudo Euclidean space, with coordinates $x = (x_1, \dots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$. Consider the non-diagonal tensor of order 2 $T = \sum_{i=1}^n f_{ij}(\xi)dx_i \otimes dx_j$, where $f_{ii} = f_{ii}(\xi)$, $f_{ij} = a_i a_j f(\xi)$, $\xi = \sum_{i=1}^n a_i x_i$, $a_i \neq 0, \forall i = 1, \dots, n$, and $\sum_{k=1}^n a_k^2 \epsilon_k = \epsilon$. Then, there is a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $A_{\bar{g}} = T$, with $\varphi = \varphi(\xi)$, if and only if the components of the tensor satisfy the following equations

$$\begin{cases} \epsilon_i(fa_i^2 - f_{ii}) = \epsilon_j(fa_j^2 - f_{jj}) & \forall i, j = 1, \dots, n \\ 2\epsilon\epsilon_i(fa_i^2 - f_{ii}) > 0 \\ 2\epsilon\epsilon_i(fa_i^2 - f_{ii}) + \frac{\epsilon_0\epsilon_i\epsilon(f'a_i^2 - f'_{ii})}{\sqrt{2\epsilon\epsilon_i(fa_i^2 - f_{ii})}} = f & \forall i = 1, \dots, n, \end{cases} \tag{2.2}$$

and

$$\varphi(\xi) = Ke^{\epsilon_0 \int \sqrt{2\epsilon\epsilon_i(fa_i^2 - f_{ii})}d\xi},$$

where $\epsilon_0 = \pm 1$ and K is a positive constant.

As a direct consequence of the Theorem (2.3), we obtain the following example in the Riemannian case.

Example. Consider the Euclidean space (\mathbb{R}^n, g) $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$. Given the tensor $T = \sum_{i=1}^n f_{ii}dx_i^2 + \sum_{i \neq j=1}^n f_{ij}dx_i \otimes dx_j$, where $f_{ii} = \left(\frac{2-n}{2n}\right)g - \frac{g'}{2n\sqrt{g}}$ and $f_{ij} = \frac{1}{n}\left(g - \frac{g'}{2\sqrt{g}}\right)$, $a_i = a_j = a$, $\epsilon = 1$, $\epsilon_0 = -1$, and g is a positive function. Then, there is a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $A_{\bar{g}} = T$, where:

$$\varphi(\xi) = Ke^{-\int \sqrt{g}d\xi}. \tag{2.3}$$

Moreover, if $|\int \sqrt{g}d\xi| \leq C_1$, C_1 is a positive constant, then the metric \bar{g} is complete in \mathbb{R}^n .

The case where $\xi = a_k x_k$ for some fixed k , will be treated in the next Theorem. Note that, in this case, in (2.1), $\varphi_{x_i} = \varphi' a_i$, $\varphi_{x_i x_i} = \varphi'' a_i^2 \forall i = 1, \dots, n$, and $\varphi_{x_i x_j} = 0, \forall i \neq j$. Then, for the second equation, we have $f_{ij}\varphi = 0$, which is equivalent to the given tensor T being diagonal, because, in this case, $f_{ij} = 0 \forall i \neq j$. Without loss of generality, let us consider $\xi = a_1 x_1$ and $\epsilon_1 a_1^2 = \epsilon$.

Theorem 2.4. Let (\mathbb{R}^n, g) , with $n \geq 3$, be the pseudo Euclidean space, with coordinates $x = (x_1, \dots, x_n)$, and metric $g_{ij} = \delta_{ij}\epsilon_i$. Consider $T = \sum_{i=1}^n f_{ii}(\xi)dx_i^2$, where $\xi = a_1 x_1$, $\epsilon_1 a_1^2 = \epsilon = \pm 1$. Then, there is a metric $\bar{g} = \frac{1}{\varphi^2}g$, with $\varphi = \varphi(\xi)$, such that $A_{\bar{g}} = T$, if and only if the components of the tensor satisfy the equations:

$$\begin{aligned} \epsilon_i f_{ii} &= \epsilon_j f_{jj} & \forall i \neq j = 2, \dots, n \\ -2\epsilon\epsilon_i f_{ii} &> 0 & \forall i = 2, \dots, n \\ f_{11} &= -2a_1^2 \epsilon\epsilon_i f_{ii} - \frac{\epsilon\epsilon_0\epsilon_i f'_{ii} a_i^2}{\sqrt{-2\epsilon\epsilon_i f_{ii}}} + \epsilon_i \epsilon_1 f_{ii} \end{aligned}$$

and

$$\varphi(\xi) = Ke^{\epsilon_0 \int \sqrt{-2\epsilon\epsilon_i f_{ii}} d\xi}$$

where K is a positive constant and $\epsilon_0 = \pm 1$.

As a consequence of Theorem (2.4), we will show an example in the Riemannian case.

Example. Consider the Euclidean space (\mathbb{R}^n, g) $n \geq 3$, with coordinates $x = (x_1, \dots, x_n)$. Given the tensor $T = f_{11}dx_1^2 + \sum_{i=2}^n g(\xi)dx_i^2$, where $f_{11} = g(1 + 2a_1^2) + \frac{g'a_1^2}{\sqrt{-2g}}$, $\epsilon = 1$, $\epsilon_0 = -1$, and $f_{ii} = f_{jj} = g \ \forall i \neq j = 2, \dots, n$, where g is a differentiable function smaller than zero. Then, there is a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $A_{\bar{g}} = T$, where

$$\varphi(\xi) = Ke^{-\int \sqrt{-2gd\xi}}. \tag{2.4}$$

Note that in the above theorem, conditions (2.4) are checked. Because the $g < 0$ function is arbitrary, it can be chosen so that $\int \sqrt{-2gd\xi}$ is limited, and, in this case, the metric \bar{g} will be complete in \mathbb{R}^n .

Considering now the case where $\xi = \sum a_i x_i$ and $\sum a_i^2 \epsilon_i = 0$, we obtain the following result.

Theorem 2.5. *Let (\mathbb{R}^n, g) , with $n \geq 3$, be the pseudo Euclidean space, with coordinates $x = (x_1, \dots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$. Consider $T = \sum_{i,j} f_{ij}(\xi) dx_i \otimes dx_j$, where $f_{ii} = f_{ii}(\xi)$, $f_{ij} = a_i a_j f(\xi)$, and $\xi = \sum_{i=1}^n a_i x_i$, with $\sum_{i=1}^n a_i^2 \epsilon_i = 0$ and $a_i \neq 0$, for at least one pair of indexes and the differentiable function f . Then, there is a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $A_{\bar{g}} = T$ if and only if $f_{ii} = a_i^2 f, \forall i = 1, \dots, n$, and φ is a solution of the equation*

$$\varphi'' - f\varphi = 0.$$

Next we present two examples for the Theorem (2.5), considering particular solutions for the equation $\varphi'' - f\varphi = 0$.

The study of oscillations is an important part of mechanics because of the frequency with which they occur. The simple swaying of leaves of a tree, radio waves, sound, and light are typical examples where oscillatory motion occurs. Below, we present an application of the physics of the above theorem, with the equation of free oscillations, obtained when $f = k$.

Example. Let (\mathbb{R}^n, g) , with $n \geq 3$, be a pseudo Euclidean space, with coordinates $x = (x_1, \dots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$. Consider $T = \sum_{i,j} f_{ij} dx_i \otimes dx_j$, where $f_{ii} = ka_i^2, \forall i = 1, \dots, n$, and $f_{ij} = ka_i a_j, \forall i \neq j = 1, \dots, n$, and $\sum_{i=1}^n a_i^2 \epsilon_i = 0$. Then, there exists a $\bar{g} = \frac{1}{\varphi^2}g$ such that $A_{\bar{g}} = T$, if and only if the function φ is satisfies:

$$\varphi(\xi) = \begin{cases} C_1 \sinh(\xi\sqrt{|k|}) + C_2 \cosh(\xi\sqrt{|k|}) & \text{if } k < 0 \\ C_1 + C_2 \xi & \text{if } k = 0 \\ C_1 \sin(\xi\sqrt{k}) + C_2 \cos(\xi\sqrt{k}) & \text{if } k > 0. \end{cases}$$

Example. Let (\mathbb{R}^n, g) , with $n \geq 3$, be a pseudo Euclidean space, with coordinates $x = (x_1, \dots, x_n)$ and metric $g_{ij} = \delta_{ij}\epsilon_i$. Consider $T = \sum_{i,j}^n f_{ij} dx_i \otimes dx_j$, where $f_{ii} = (h^2 + h')a_i^2 \forall i = 1, \dots, n$ and $f_{ij} = (h^2 + h')a_i a_j \forall i \neq j = 1, \dots, n$, and $\sum_{i=1}^n a_i^2 \epsilon_i = 0$, and a differentiable function h . Then, there exists a $\bar{g} = \frac{1}{\varphi^2}g$ such that $A_{\bar{g}} = T$, if and only if the function φ is given by

$$\varphi(\xi) = C_1\varphi_0 + c_2\varphi_0 \int \frac{d\xi}{\varphi_0^2},$$

where C_1 and C_2 are arbitrary constants and $\varphi_0 = e^{\int h d\xi}$, with $h = h(\xi)$.

Next, we present a generalization of the Theorem (2.3), for locally conformally flat manifolds.

Let us now consider a locally conformally flat Riemannian manifold (M^n, g) . We can consider the problem (1.4) for a neighborhood $V \subset M$ with coordinates $x = (x_1, \dots, x_n)$ such that $g_{ij} = \frac{\delta_{ij}}{F^2}$, where F is a non-null, differentiable function in V . Given a tensor $T = \sum_{i,j} f_{ij}(x) dx_i \otimes dx_j$ defined in V , we want to find a metric $\bar{g} = \frac{1}{\phi^2}g$ such that $A_{\bar{g}} = T$. Considering that g and \bar{g} are translation invariant, where $\xi = \sum a_i x_i$ is the basic invariant of the action, we have, in a way analogous to Lemma (2.2), that the components of the given tensor T , are necessarily given by $f_{ij} = a_i a_j f(\xi)$ and $f_{ii} = f_{ii}(\xi)$, where f is a differentiable function.

Theorem 2.6. *Let (M^n, g) , with $n \geq 3$, be a locally conformally flat Riemannian manifold. Let V be an open subset of M with coordinates $x = (x_1, x_2, \dots, x_n)$ with $g_{ij} = \frac{1}{F^2(\xi)}\delta_{ij}$. Consider a non-diagonal tensor $T = \sum_{i=1}^n f_{ij}(\xi) dx_i \otimes dx_j$. Then, there is a metric $\bar{g} = \frac{1}{\phi^2}g$, with $\phi = \phi(\xi)$, such that $A_{\bar{g}} = T$ if and only if the functions f_{ij} and φ are given in Theorem (2.3) and $\phi = \frac{\varphi}{F}$.*

Analogously, the Theorems (2.4) and (2.5) can be extended to locally conformally flat manifolds.

In [22], Vioclovisk extended the concept of scalar curvature using the Schouten tensor, by introducing the σ_k curvatures, which are obtained from the eigenvalues of the Schouten tensor. Recall that $\sigma_0(g)$ has been defined as 1 and $\sigma_1(g)$ is the scalar curvature less than constant. Recently, many works have considered the $\sigma_2(g)$ prescribed problem. For more details see [7, 11], and [22].

We know that, given a function $h : M^n \rightarrow \mathbb{R}$, finding a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $\sigma_2(\bar{g}) = h$ is equivalent to studying the following partial differential equation (ver [9]).

$$\left[(\Delta\varphi)^2 - |Hess_g\varphi|^2 \right] \varphi^2 - (n-1)\Delta\varphi|\nabla\varphi|^2\varphi + \frac{n(n-1)}{4}|\nabla\varphi|^4 = 2h. \tag{2.5}$$

Let us show that, for certain functions h , the above equation admits infinite solutions.

Corollary 2.7. *Let (\mathbb{R}^n, g) , with $n \geq 3$, be the Euclidean space, with coordinates $x = (x_1, \dots, x_n)$ and metric $g_{ij} = \delta_{ij}$. Given the function g , which can be any positive function, K is a positive constant and $a \in \mathbb{R}$*

$$h(\xi) = \frac{K(n-1)}{2} n^2 a^4 g e^{-4 \int \sqrt{g} d\xi} \left[\frac{ng}{4} - \left(g - \frac{g'}{2\sqrt{g}} \right) \right], \tag{2.6}$$

Then, the partial differential equation,

$$\left[(\Delta\varphi)^2 - |Hess_g \varphi|^2 \right] \varphi^2 - (n-1) \Delta\varphi |\nabla\varphi|^2 \varphi + \frac{n(n-1)}{4} |\nabla\varphi|^4 = 2h,$$

has infinite solutions, globally defined in \mathbb{R}^n , given by

$$\varphi(\xi) = K e^{-\int \sqrt{g} d\xi}.$$

The geometric interpretation of this result is presented below. We show an example of a metric \bar{g} , conformal to the Euclidean metric, complete in \mathbb{R}^n , with curvature $\sigma_2(\bar{g})$ prescribed for h .

Corollary 2.8. *Let (\mathbb{R}^n, g) , with $n \geq 3$, be the Euclidean space, with coordinates $x = (x_1, \dots, x_n)$ and metric $g_{ij} = \delta_{ij} \varepsilon_i$. Given a function h , in (2.6) there exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ with φ given by (2.3) such that $\sigma_2(\bar{g}) = h$.*

By choosing the function g so that φ is limited, the metrics obtained in the corollary (2.8) will be complete in R^n .

3. Proof of the Main Results

Before proving our results, it follows from [15], that if (\mathbb{R}^n, g) is a pseudo-Euclidean space and $\bar{g} = g/\varphi^2$ is a conformal metric, then the Ricci tensor \bar{g} is given by

$$\text{Ric } \bar{g} = \frac{1}{\varphi^2} \left\{ (n-2)\varphi Hess_g \varphi + (\varphi \Delta_g \varphi - (n-1)|\nabla_g \varphi|^2) g \right\} \tag{3.1}$$

and the scalar curvature of \bar{g} is given by

$$\bar{K} = (n-1) (2\varphi \Delta_g \varphi - n|\nabla_g \varphi|^2). \tag{3.2}$$

In the rest of this section, we demonstrate the main results of this article.

Proof [Theorem 2.1]. Using the expressions (3.1) and (3.2), we can write the Schouten tensor of the metric \bar{g} as:

$$A_{\bar{g}} = \frac{Hess_g \varphi}{\varphi} - \frac{\|\nabla_g \varphi\|^2}{2\varphi^2} g,$$

where ∇_g denotes the gradient of the pseudo-Euclidean metric g .

In this case, studying the problem (1.4) with $T = \sum_{ij}^n f_{ij}(x) dx_i \otimes dx_j$, is equivalent to studying the following system of equations:

$$\begin{cases} 2\varphi\varphi_{,x_i x_i} - \sum_{k=1}^n \epsilon_k (\varphi_{,x_k})^2 \epsilon_i - 2\varphi^2 f_{ii} = 0, & \forall i : 1, \dots, n, \\ \varphi_{,x_i x_j} - f_{ij}\varphi = 0, & \forall i \neq j, \end{cases} \tag{3.3}$$

□

Proof [Lemma 2.2]. Because $\varphi = \varphi(\xi)$, we have that:

$$\varphi_{x_i} = \varphi' a_i, \varphi_{x_i x_i} = \varphi'' a_i^2 \quad \forall i = 1, \dots, n, \varphi_{x_i x_j} = \varphi'' a_i a_j \quad \forall i \neq j, \text{ and } \|\nabla_g \varphi\|^2 = \sum_{i=k}^n \epsilon_k (\varphi_{x_k})^2 = (\varphi')^2 \epsilon.$$

Substituting these in the system (3.3), we obtain:

$$\begin{cases} 2\varphi\varphi'' a_i^2 - (\varphi')^2 \epsilon \epsilon_i = 2\varphi^2 f_{ii}, & \forall i : 1, \dots, n, \\ \varphi'' a_i a_j - f_{ij}\varphi = 0, & \forall i \neq j. \end{cases} \tag{3.4}$$

In this case, it follows directly from the second equation of (3.4) that $f_{ij} = f(\xi) a_i a_j$, for $i \neq j = 1, \dots, n$, where $\frac{\varphi''}{\varphi} = f(\xi)$. Similarly, it follows from the first equation of the system (3.4) that $f_{ii} = f_{ii}(\xi)$, because

$$f_{ii} = \frac{\varphi''}{\varphi} a_i^2 - \left(\frac{\varphi'}{\sqrt{2}\varphi}\right)^2 \epsilon \epsilon_i \quad \forall i = 1, \dots, n.$$

□

Proof [Theorem 2.3]. It follows by Lemma (2.2) that finding $\bar{g} = \frac{1}{\varphi^2} g$ such that $A_{\bar{g}} = T$, with $\varphi = \varphi(\xi)$, is equivalent to studying the following system of equations:

$$\begin{cases} 2\varphi\varphi'' a_i^2 - (\varphi')^2 \epsilon \epsilon_i = 2\varphi^2 f_{ii}, & \forall i : 1, \dots, n, \\ \varphi'' - f(\xi)\varphi = 0, & \forall i \neq j. \end{cases}$$

From the second equation we have $\varphi'' = f\varphi$; substituting this in first equation above, we obtain

$$2a_i^2 f - \left(\frac{\varphi'}{\varphi}\right)^2 \epsilon = 2f_{ii}, \quad \forall i : 1, \dots, n.$$

Because f_{ii} depends only on ξ , we also have:

$$\left(\frac{\varphi'}{\varphi}\right)^2 = 2(f a_i^2 - f_{ii}) \forall i : 1, \dots, n$$

which is equivalent to

$$\frac{\varphi'}{\varphi} = \pm \sqrt{2\epsilon(f a_i^2 - f_{ii})} \tag{3.5}$$

And it follows directly from Eq. (3.5) that

$$\begin{cases} 2\epsilon(f a_i^2 - f_{ii}) > 0, & \forall i : 1, \dots, n, \\ f a_i^2 - f_{ii} = f a_j^2 - f_{jj}, & \forall i \neq j. \end{cases}$$

Considering $\epsilon_0 = \pm 1$ and integrating Eq. (3.5), we obtain

$$\varphi(\xi) = K e^{\epsilon_0 \int \sqrt{2\epsilon(f a_i^2 - f_{ii})} d\xi},$$

where K is a positive constant. Because $\varphi'' = f\varphi$, we get:

$$2\epsilon_0^2\epsilon\epsilon_i(fa_i^2 - f_{ii}) + \frac{\epsilon\epsilon_i(f'a_i^2 - f'_{ii})}{\sqrt{2\epsilon\epsilon_i(fa_i^2 - f_{ii})}} = f \quad \forall i = 1, \dots, n$$

which concludes the proof. □

Proof [Theorem 2.4]. If $\varphi = \varphi(\xi)$ and $\xi = a_1x_1$, then $\varphi_{,x_i} = \varphi a_i$, $\varphi_{,x_i x_i} = \varphi'' a_i^2$, $\forall i = 1, \dots, n$, and $\varphi_{,x_i x_j} = 0$, $\forall i \neq j$. It follows from (3.3) that $f_{ij}\varphi = 0$, that is, $f_{ij} = 0 \quad \forall i \neq j$, this is the tensor diagonal. In this case, the system (3.3) is equivalent to

$$\begin{cases} 2\varphi\varphi'' a_1^2 - (\varphi')^2 \epsilon_i \epsilon - 2\varphi^2 f_{11} = 0, & \forall i : 1, \dots, n \\ (\varphi')^2 \epsilon_i \epsilon + 2\varphi^2 f_{ii} = 0, & \forall i \neq 1, i = 1, \dots, n. \end{cases} \tag{3.6}$$

From the second equation, we have

$$\left(\frac{\varphi'}{\varphi}\right)^2 = -2\epsilon\epsilon_i f_{ii}. \tag{3.7}$$

and follows from (3.7) that:

$$\begin{cases} -2\epsilon\epsilon_i f_{ii} > 0, & \forall i : 2, \dots, n, \\ \epsilon_i f_{ii} = \epsilon_j f_{jj}, & \forall i \neq j, \end{cases}$$

Considering $\epsilon_0 = \pm 1$ and integrating Eq. (3.7), we obtain

$$\varphi(\xi) = K e^{\epsilon_0 \int \sqrt{-2\epsilon\epsilon_i f_{ii}} d\xi}$$

where K is a positive constant. Substituting $\varphi(\xi)$ in the first equation in (3.6), we obtain

$$f_{11} = -2a_1^2 \epsilon_0^2 \epsilon \epsilon_i f_{11} - \frac{\epsilon \epsilon_0 \epsilon_i f' a_i^2}{\sqrt{-2\epsilon\epsilon_i f_{ii}}} + \epsilon_0^2 \epsilon^2 \epsilon_i f_{ii} \quad \forall i = 1, \dots, n,$$

which concludes the proof. □

Proof [Theorem 2.5]. Because $\sum_{k=1}^n \epsilon_k a_k^2 = \epsilon = 0$, it follows that the system (3.4) can be reduced to $\varphi'' - f\varphi = 0$ and $f_{ii} = a_i^2 f$, $\forall i = 1, \dots, n$. In this case, the tensor T is given by:

$$T = f \begin{pmatrix} a_1^2 & a_1 a_2 & \cdots & a_1 a_n \\ \vdots & a_2^2 & \ddots & \vdots \\ -a_n a_1 & \cdots & \cdots & a_n^2 \end{pmatrix}$$

□

Proof [Theorem 2.6]. Consider $\phi = \varphi F$ and apply Theorem (2.3). □

Proof [Corollary 2.7]. Note that the functions $\varphi(\xi)$ and $h(\xi)$ clearly satisfy the given partial differential equation. □

Proof [Corollary 2.8]. The proof is an immediate consequence of equation (2.5) and Corollary (2.7). □

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References

- [1] Barbosa, E., Pina, R., Tenenblat, K.: On gradient Ricci solitons conformal to a pseudo-Euclidean space. *Isr. J. Math.* **200**, 213–224 (2014)
- [2] Besse, A.L.: *Einstein Manifolds*. *Ergeb. Math. Grenzgeb.* (3) 10. Springer, Berlin (1987)
- [3] Brozos-Vázquez, M., García-Río, E., Gavino-Fernández, S.: Locally conformally flat Lorentzian gradient Ricci solitons. *J. Geom. Anal.* **23**, 1196–1212 (2013)
- [4] Chang, A., Gursky, M., Yang, P.: An equation of Monge–Ampere type in conformal geometry, and four manifolds of positive Ricci curvature. *Ann. Math.* **155**, 709–787 (2002)
- [5] DeTurck, D., Yang, D.: Local existence of smooth metrics with prescribed curvature. In: *Nonlinear Problems in Geometry, Contemporary Mathematics*, vol. 51. American Mathematical Society, Providence, RI (1986)
- [6] Fernández-López, M., García-Río, E.: On gradient Ricci solitons with constant scalar curvature. *Proc. Am. Math. Soc.* **144**, 369–378 (2016)
- [7] Ge, Y., Lin, C.-S., Wang, G.: On the σ_2 -scalar curvature. *J. Differ. Geom.* **84**, 45–86 (2010)
- [8] Guan, P., Lin, C.-S., Wang, G.: Schouten tensor and some topological properties. *Commun. Anal. Geom.* **13**, 887–902 (2005)
- [9] Gonzalez, M.M., Mazzieri, L.: Singularities for a fully non-linear elliptic equation conformal geometry. *Proc. Am. Math. Soc.* (Preprint)
- [10] He, C., Petersen, P., Wylie, W.: On the classification of warped product Einstein metrics. *Commun. Anal. Geom.* **20**, 271–311 (2012)
- [11] He, Y., Sheng, W.: Prescribing the symmetric function of the eigenvalues of the Schouten tensor. *Proc. AMS* **139**, 1127–1136 (2011)
- [12] Hu, Z., Li, H., Simon, U.: Schouten curvature functions on locally conformally flat Riemannian manifolds. *J. Math.* **88**, 75–100 (2007)
- [13] Hu, Z., Nishikawa, S., Simon, U.: Critical metrics of the Schouten functional. *J. Geom.* **98**, 91–113 (2010)
- [14] Petersen, P., Wylie, W.: Rigidity of gradient Ricci solitons. *Pac. J. Math.* **241**, 329–345 (2009)
- [15] Pieterzack, M., Pina, R.: Prescribed diagonal Schouten tensor in locally conformally flat manifolds. *J. Geom.* **104**, 341–355 (2013)
- [16] Pina, R., Tenenblat, K.: Conformal metrics and Ricci tensors in the pseudo-Euclidean space. *Proc. Am. Math. Soc.* **129**, 1149–1160 (2001)

- [17] Pina, R., Tenenblat, K.: On metrics satisfying equation $R_{ij} - Kg_{ij}/2 = T_{ij}$ for constant tensors T . *J. Geom. Phys.* **40**, 379–383 (2002)
- [18] Pina, R., Tenenblat, K.: Conformal metrics and Ricci tensors on the sphere. *Proc. Am. Math. Soc.* **132**, 3715–3724 (2004)
- [19] Pina, R., Tenenblat, K.: On solutions of the Ricci curvature equation and the Einstein equation. *Isr. J. Math.* **171**, 61–76 (2009)
- [20] Pina, R., Adriano, L., Pieterzack, M.: Prescribed diagonal Ricci tensor in locally conformally flat manifolds. *J. Math. Anal. Appl.* **421**, 893–904 (2015)
- [21] Pulemotov, A.: Metrics with prescribed Ricci curvature on homogeneous spaces. *J. Geom. Phys.* **106**, 275–283 (2016)
- [22] Viaclovsky, J.A.: Conformal geometry, contact geometry and the calculus of variation. *Duke Math. J.* **101**, 283–316 (2000)

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