



An Extension Theorem for Conditionally Additive Functions and Its Application for the Equality Problem of Quasi-Arithmetic Expressions

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Abstract. The focus of this paper is the equality problem of quasi-arithmetic expressions. This class is a far generalization of the well-known class of quasi-arithmetic means. One of the main tools in the proof is an extension theorem of real homomorphisms from a subset with a very weak structure (dyadically closed set).

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1. Introduction

A two place function M is called a quasi-arithmetic mean (see e.g. [1]) if it can be written in the form

$$M(x, y) = \varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right), \quad x, y \in I, \quad (1)$$

where I is a non-empty, open interval, $\varphi: I \rightarrow \mathbb{R}$ is a continuous, strictly monotonic function. This class is a close relative of quasi-sums (see e.g. [19]) and it is in the center of research of several authors (see e.g. [8] and the references therein). It contains classical well-known means like the arithmetic mean, geometric mean, harmonic mean and so on. Currently the operator version is also defined and investigated (see e.g. [10]).

If we consider only the expression (1) itself, then invertibility of φ with Jensen convex image seems to be more natural assumption than continuity with strict monotonicity.

The following questions arise: What will be the resulted new class? Is it larger than the class of quasi-arithmetic means? The members of this new class will be means again or not? What are the necessary and sufficient conditions of equality of two members of this class?

The main goal of this paper to answer these questions, especially the last one (equality problem in this class).

The remaining part is organized as follows. In Sect. 2 an extension theorem is proved which is one of the main tools in the proof of the solution of the equality problem. Extension theorems has a quite rich literature because conditional problems concerning functional equations have an important role in this area (see e.g. [3, 7, 12, 24, 25]).

In Sect. 3 properties of quasi-arithmetic expressions are investigated. It is proved that this class is much larger than the class of quasi-arithmetic means. Examples make the connection between these classes more plausible.

In Sect. 4 the solution of the equality problem is solved in the class of quasi-arithmetic expressions. Such type of questions are vividly examined by several authors (see e.g. [4–6, 13–18, 20–23]).

At last, in Sect. 5 two open problems are shown concerning the characterization problem of quasi-arithmetic expressions.

2. An Extension Theorem for Conditionally Additive Functions

Let $D \subset \mathbb{R}$ be a subset, and

$$D_\Delta := \{(x, y) \in \mathbb{R}^2 \mid x, y, x + y \in D\}.$$

A function $g: D \rightarrow \mathbb{R}$ is called conditionally additive on D (see [7]) if

$$g(x) + g(y) = g(x + y), \quad (x, y) \in D_\Delta.$$

A set $D \subset \mathbb{R}$ is dyadically closed if $\frac{x+y}{2} \in D$ for all $x, y \in D$.

The dyadic hull of D is

$$\text{diad}D := \left\{ x \in \mathbb{R} \mid x = \sum_{i=1}^n \delta_i d_i, \text{ for some } n \in \mathbb{N} \text{ and } d_i \in D \right\},$$

where $\delta_i = \frac{m_i}{2^{k_i}}$, $m_i, k_i \in \mathbb{Z}$.

Theorem 1. *Assume that a real valued function is defined on a nonempty, dyadically closed subset of the reals, which contains zero. If it is conditionally additive on this set, then it is (not necessarily uniquely) extendible onto the whole real line.*

Lemma 1. *If $D \subset \mathbb{R}$ is nonempty, dyadically closed set, and it contains zero, then $\text{diad}(D \cap [0, r]) = \text{diad}D$ for all positive $r \in D$.*

Proof. Let $d \in \text{diad}D$, then $x = \sum_{i=1}^n \delta_i d_i$ and for an arbitrary $d_i \in D$ there exists $j_i \in \mathbb{N}$, such that $\text{sign}(d_i) \frac{1}{2^{j_i}} d_i < r$. So

$$x = \sum_{i=1}^n \delta_i d_i = \sum_{i=1}^n \frac{m_i}{2^{k_i}} d_i = \sum_{i=1}^n \underbrace{\frac{\text{sign}(d_i) 2^{j_i} m_i}{2^{k_i}} \text{sign}(d_i) \frac{1}{2^{j_i}} d_i}_{\in \text{diad}(D \cap [0, r])}$$

So, $\text{diad}D \subset \text{diad}(D \cap [0, r])$. The other direction of the inclusion is trivial. □

Proof of Theorem 1. If D has a single element, the statement is trivial.

We can assume without losses that D has at least two elements, and at least one is positive, say $0 < r \in D$. Because of the previous lemma, it is enough to prove the statement for $D_r := D \cap [0, r]$.

For an arbitrary $x \in \text{diad}D_r$ there is a unique integer $n_x \in \mathbb{Z}$ such that

$$n_x \frac{r}{2} \leq x < (n_x + 1) \frac{r}{2}.$$

Let $x' := x - n_x \frac{r}{2}$. Because of its definition $x' \in D_r$. Assume that $g: D_r \rightarrow \mathbb{R}$ is conditionally additive, and let's define

$$a: \text{diad}D_r \rightarrow \mathbb{R}, \quad a(x) := n_x g\left(\frac{r}{2}\right) + g(x').$$

Firstly, we prove that a is an extension of g . For this we distinguish two cases:

(i) If $x \in \text{diad}D \cap [0, \frac{r}{2}[$. Then $x = 0 \cdot \frac{r}{2} + x$, so

$$a(x) = 0 \cdot g\left(\frac{r}{2}\right) + g(x) = g(x).$$

(ii) If $x \in \text{diad}D \cap [\frac{r}{2}[$. Then $x = 1 \cdot \frac{r}{2} + x'$, so

$$a(x) = 1 \cdot g\left(\frac{r}{2}\right) + g(x') = g\left(\frac{r}{2} + x'\right) = g(x).$$

This entails a is really an extension of g .

Secondly, we prove that a is additive on $\text{diad}D_r = \text{diad}D$. Let $x, y \in \text{diad}D_r$ be arbitrary elements, and

$$x = n_x \frac{r}{2} + x', \quad y = n_y \frac{r}{2} + y'.$$

We distinguish two cases again:

(i) If $x' + y' \in D_{\frac{r}{2}}$, then

$$a(x + y) = (n_x + m_y) g\left(\frac{r}{2}\right) + g(x' + y') = a(x) + a(y).$$

(ii) If $x' + y' \in [\frac{r}{2}, r[\cap D$, then $z' + \frac{r}{2} = x' + y'$

$$\begin{aligned} a(x + y) &= (n_x + m_y + 1) g\left(\frac{r}{2}\right) + g(z') \\ &= n_x g\left(\frac{r}{2}\right) + m_y g\left(\frac{r}{2}\right) + g(x') + g(y') = a(x) + a(y). \end{aligned}$$

That is to say, a is additive on the required set.

Thirdly, we extend a onto the whole real line. It is clear that $\text{diad}D$ with the usual addition is a subsemigroup of the additive group of real numbers. We can apply the theorem of Dhombres and Ger (see [9] or [11, Theorem 18.1.1.]). Which exactly says that there is a (not necessarily unique) homomorphism $A: \mathbb{R} \rightarrow \mathbb{R}$ such that $A|_{\text{diad}D} = a$. \square

3. Properties of Quasi-Arithmetic Expressions

Let $I \subset \mathbb{R}$ be a proper interval, $\varphi: I \rightarrow \mathbb{R}$ be an invertible function such that its inverse has a dyadically closed domain. Then the following two place function

$$A_\varphi(x, y) := \varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right), \quad x, y \in I,$$

is called a quasi-arithmetic expression generated by φ .

Besides of the required conditions, A_φ is well-defined.

If φ continuous and strictly monotone, then it fulfils the requirements above. The resulted set is the class of quasi-arithmetic means.

If φ is an invertible additive function, then the corresponding quasi-arithmetic expression is the arithmetic mean. If $\varphi = a \circ \log$, where a is an invertible additive function again, we get the geometric mean. These phenomenon are called absorbing irregularity. In other words, a very regular expression can be generated by a very irregular one.

The following two observations shows that irregularity of the generating function is not always absorbed. As a consequence we have that the class of quasi-arithmetic expressions is really larger than the class of quasi-arithmetic means.

Observation 1. *The class of quasi-arithmetic means are strictly contained by the class of quasi arithmetic expressions.*

Proof. Let $I = \mathbb{R}$ and $a: \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary non-continuous, additive, self-bijection of the reals, and $\varphi = \exp \circ a$, then the quasi arithmetic expression generated by φ is the following

$$A_{\exp \circ a}(x, y) = a^{-1} \circ \log \left(\frac{\exp(a(x)) + \exp(a(y))}{2} \right), \quad x, y \in I.$$

We prove that A_φ is non-continuous, so it cannot be a quasi-arithmetic mean.

Assume that A_φ is continuous, then the image of every connected set by A_φ is connected.

$$A_{\exp \circ a}(\mathbb{R}, 0) = a^{-1} \circ \log \left(\frac{\exp(a(\mathbb{R})) + 1}{2} \right) = a([- \log 2, \infty[)$$

If the right hand side is connected, then it is an interval. Because a is non-continuous it is unbounded (from below and from above) on every non-empty

interval (see e.g. [11]). So $a([-\log 2, \infty[)$ is an unbounded interval. The function a is bijective, so it is surjective too, we have $a([-\log 2, \infty[) \neq \mathbb{R}$. This entails the existence of $b \in \mathbb{R}$ such that

$$a([-\log 2, \infty[) = [b, \infty[, \text{ or }]b, \infty[, \text{ or }]-\infty, b], \text{ or }]-\infty, b[.$$

In all cases we get the boundedness of a either from below or from above. It follows that a is continuous, which is a contradiction. \square

The quasi-arithmetic means have the intern property, that is, their values are always between the minimum and the maximum of the variables. Intern property of the construction in the previous proof is an open problem, however, it is possible to construct a quasi-arithmetic expression which is non-continuous and it is not a mean.

Observation 2. *There is a non-continuous quasi-arithmetic expression which is not a mean.*

Proof. ¹ Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be the following function

$$\varphi(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ -x, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $\varphi = \varphi^{-1}$, and $A_\varphi(1, \sqrt{2}) = \frac{\sqrt{2}-1}{2} < 1 < \sqrt{2}$. So, A_φ is not a mean.

If $x_n \rightarrow \sqrt{2}$, and $x_n \in \mathbb{Q}$, we have

$$A_\varphi(x_n, 1) = \frac{x_n + 1}{2} \rightarrow \frac{\sqrt{2} + 1}{2} \neq \frac{\sqrt{2} - 1}{2} = A_\varphi(\sqrt{2}, 1)$$

So, A_φ is not continuous. \square

4. Equality Problem of Quasi-Arithmetic Expressions

Here we solve the following problem. Let $\varphi, \psi: I \rightarrow \mathbb{R}$ be invertible functions with inverses having dyadically closed domains. What is the sufficient and necessary condition of the equality of the generated quasi-arithmetic expressions? That is to say, solve the following functional equation!

$$\varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right) = \psi^{-1} \left(\frac{\psi(x) + \psi(y)}{2} \right), \quad x, y \in I. \tag{2}$$

Theorem 2. *Two quasi-arithmetic expressions generated by φ and ψ are equal on the interval I if and only if there exists an invertible additive function $a: \mathbb{R} \rightarrow \mathbb{R}$ (not necessarily unique), and a real constant b such that*

$$\varphi(x) = a \circ \psi(x) + b, \quad x \in I.$$

¹Gyula Maksa's example, *Oral communication.*

Proof. From the assumption $A_\varphi = A_\psi$ we have

$$\psi \circ \varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right) = \frac{\psi(x) + \psi(y)}{2}, \quad x, y \in I.$$

Using the substitutions $\psi \circ \varphi^{-1} =: f: \text{dom}\varphi^{-1} \rightarrow \mathbb{R}$, $\varphi(x) =: u$, $\varphi(y) =: v$, from the previous equation we get the following.

$$f \left(\frac{u + v}{2} \right) = \frac{f(u) + f(v)}{2}, \quad u, v \in D =: \text{dom}\varphi^{-1}. \tag{3}$$

We can assume without losses that $0 \in D$. Substituting $v = 0$ and $b := f(0)$ into the equation above we have

$$f \left(\frac{u}{2} \right) = \frac{f(u) + b}{2}, \quad u \in D. \tag{4}$$

Using (3) and (4) we get

$$\frac{f(u) + f(v)}{2} = f \left(\frac{u + v}{2} \right) = \frac{f(u + v) + b}{2}, \quad (u, v) \in D_\Delta.$$

Let $g: D \rightarrow \mathbb{R}$, $g(u) := f(u) - b$, then the previous equation entails the conditional additivity of g , that is to say,

$$g(u + v) = g(u) + g(v), \quad (u, v) \in D_\Delta.$$

Applying the extension theorem (Theorem 1.) for g , we have that there is and additive function $a: \mathbb{R} \rightarrow \mathbb{R}$ such that $a|_D = g$. Using this and the definitions of the functions f and g , we have

$$a(u) = f(u) - b = \psi \circ \varphi^{-1}(u) - b, \quad u \in D.$$

This exactly means that

$$a \circ \varphi(x) + b = \psi(x), \quad x \in I.$$

So, the proof is ready. □

5. Open Problems

Characterization of quasi-arithmetic means is well-known. The characterization of quasi-arithmetic expressions is open at this moment. For the formulation of the characterization theorem of quasi-arithmetic means we need the following conditions of a two place function $M: I \times I \rightarrow \mathbb{R}$:

- (i) Strict monotonicity: if $x < x'$ then $M(x, y) < M(x', y)$ and the same for the second variable;
- (ii) Continuity;
- (iii) Bisymmetry: $M((x, y), M(u, v)) = M(M(x, u), M(y, v))$;
- (iv) Reflexivity: $M(x, x) = x$;
- (v) Symmetry: $M(x, y) = M(y, x)$;

The next theorem due to Aczél can be found in [2].

Theorem 3 (Aczél, 1948). *Conditions (i)–(v) are necessary and sufficient for the existence of a strictly increasing and continuous function $\varphi: I \rightarrow \mathbb{R}$ by which M has the form*

$$M(x, y) = \varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right), \quad x, y \in I.$$

Conditions (iii)–(v) are fulfilled by quasi-arithmetic expressions too. The conditions (i) and (iv) implies intern property, that is, the value of the function is between the minimum and the maximum of the variables.

These ideas are motivate the following open problems:

Open problem 1. Is it possible to construct a non-continuous quasi-arithmetic expression which is a mean?

Open problem 2. Is it true that (iii)–(v) (maybe with additional conditions different from (i) and (ii)) characterize quasi-arithmetic expressions and (i) with (iii)–(v) characterize quasi-arithmetic expressions with intern property?

If the answer is affirmative for the first, so is for the second.

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