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Coefficient Problems in the Subclasses of Close-to-Star Functions

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Abstract. For two subclasses of close-to-star functions we estimate early logarithmic coefficients, coefficients of inverse functions, Hankel determinant $H_{2,2}$ and Zalcman functional $J_{2,3}$. All results are sharp.

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1. Introduction

Given $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C}\colon |z| < r\}$, and let $\mathbb{D} := \mathbb{D}_1$. Let $\overline{\mathbb{D}} := \{z \in \mathbb{C}\colon |z| \le 1\}$ and $\mathbb{T} := \partial\mathbb{D}$. Let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D}$ and $\mathcal{A}$ be its subclass of f normalized by $f(0) := 0$ and $f'(0) := 1$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \ z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of all univalent functions and $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ of all starlike functions, namely, $f \in \mathcal{S}^*$ if $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is called close-to-star if there exist $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$ such that

$$\operatorname{Re} \frac{e^{i\beta} f(z)}{g(z)} > 0, \quad z \in \mathbb{D}. \tag{1.2}$$

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Denote by $\mathcal{CST}$ the class of all close-to-star functions introduced by Reade [30]. Note that $f \in \mathcal{CST}$ if and only if a function

$$F(z) := \int_0^z \frac{f(t)}{t} dt, \quad z \in \mathbb{D}, \tag{1.3}$$

is close-to-convex [15], [12, Vol. II, p. 3]. The class of close-to-star functions and its subclasses were intensively studied by various authors (e.g., MacGregor [25], Sakaguchi [32], Causey and Merkes [4]; for further references, see [12, Vol. II, pp. 97–104]). Given $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$, let $\mathcal{CST}_\beta(g)$ be the subclass of $\mathcal{CST}$ of all f satisfying (1.2). The classes $\mathcal{CST}_0(g_i),\ i = 1, 2, 3$, where

$$g_1(z) := \frac{z}{1 - z^2}, \quad g_2(z) := \frac{z}{(1 - z)^2}, \quad g_3(z) := z, \quad z \in \mathbb{D},$$

are particularly interesting and were separately studied by authors. In this paper we deal with the classes $\mathcal{CST}_0(g_1) =: \mathcal{ST}(\mathrm{i})$ and $\mathcal{CST}_0(g_2) =: \mathcal{ST}(1)$ which elements f in view of (1.2) satisfy the condition

$$\operatorname{Re}\left\{(1 - z^2)\frac{f(z)}{z}\right\} > 0, \quad z \in \mathbb{D}, \tag{1.4}$$

and

$$\operatorname{Re}\left\{(1 - z)^2\frac{f(z)}{z}\right\} > 0, \quad z \in \mathbb{D}, \tag{1.5}$$

respectively. Let us add the inequality (1.4) defines the subclass of the class of functions starlike in the direction of the real axis introduced by Robertson [31]. Moreover, each function F given by (1.3) over the class $\mathcal{ST}(\mathrm{i})$ maps univalently $\mathbb{D}$ onto a domain $F(\mathbb{D})$ convex in the direction of the imaginary axis. The concept of convexity in one direction belongs to Roberston [31] (see e.g., [12, p. 199]). Each function F given by (1.3) over the class $\mathcal{ST}(1)$ maps univalently $\mathbb{D}$ onto a domain $F(\mathbb{D})$ called convex in the positive the direction of the real axis, i.e., $\{w + it\colon t \geq 0\} \subset f(\mathbb{D})$ for every $w \in f(\mathbb{D})$ [2,8,9,11,20,21]. Let us remark that the condition (1.4) was generalized by replacing the expression $1 - z^2$ by the expression $1 - \alpha^2 z^2$ with $\alpha \in [0, 1]$ in [13].

In this paper we find the sharp estimates of early logarithmic coefficients (Sect. 2), of the Hankel determinant $H_{2,2}$ and of Zalcman functional $J_{2,3}$ (Sect. 3) and of the early inverse coefficients (Sect. 4) of functions in the classes $\mathcal{ST}(\mathrm{i})$ and $\mathcal{ST}(1)$. Since both classes $\mathcal{ST}(\mathrm{i})$ and $\mathcal{ST}(1)$ have a representation using the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{1.6}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{ST}(\mathrm{i})$ and $\mathcal{ST}(1)$ have a suitable representation expressed by the coefficients of functions in $\mathcal{P}$. Therefore to get the upper bounds of considered functionals our computing is based on parametric formulas for the second and third coefficients in $\mathcal{P}$. However both classes are rotation non-invariant. Thus to solve discussed problems we will apply a general formula for c_3 recently found in [7]. The formula (1.7) was proved by Carathéodory [3] (see e.g., [10, p. 41]). The formula (1.8) can be found in [28, p. 166]. The formula (1.9) was shown in a recent paper [7], where the extremal functions (1.11) and (1.12) were computed also. For $c_1 \geq 0$ the formula (1.9) is due to by Libera and Zlotkiewicz [22,23].

Lemma 1.1. *If $p \in \mathcal{P}$ is of the form (1.6), then*

$$c_1 = 2\zeta_1, \tag{1.7}$$
$$c_2 = 2\zeta_1^2 + 2(1-|\zeta_1|^2)\zeta_2 \tag{1.8}$$

and

$$\begin{aligned} c_3 = {} & 2\zeta_1^3 + 4(1-|\zeta_1|^2)\zeta_1\zeta_2 \\ & - 2(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3 \end{aligned} \tag{1.9}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1,2,3\}$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (1.7), namely,

$$p(z) = \frac{1+\zeta_1 z}{1-\zeta_1 z}, \quad z \in \mathbb{D}. \tag{1.10}$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (1.7)–(1.8), namely,

$$p(z) = \frac{1 + \left(\overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \zeta_2 z^2}{1 + \left(\overline{\zeta_1}\zeta_2 - \zeta_1\right) z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{1.11}$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\zeta_3 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1, c_2 and c_3 as in (1.7)–(1.9), namely,

$$\begin{aligned} & p(z) \\ & = \frac{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_2\right) z^2 + \zeta_3 z^3}{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 - \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_2\right) z^2 - \zeta_3 z^3}, \quad z \in \mathbb{D}. \end{aligned} \tag{1.12}$$

2. Logarithmic Coefficients

Given $f \in \mathcal{S}$ let

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}\backslash\{0\},\ \log 1 := 0. \tag{2.1}$$

The numbers γ_n are called logarithmic coefficients of f. Differentiating (2.1) and using (1.1) we get

$$\begin{aligned}\gamma_1 &= \frac{1}{2}a_2, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right),\\ \gamma_3 &= \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right).\end{aligned} \tag{2.2}$$

As it well known, the logarithmic coefficients play a crucial role in Milin conjecture ([26], see also [10, p. 155]). It is surprising that for the class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients $\mathcal{S}$ are known only for γ_1 and γ_2, namely,

$$|\gamma_1| \leq 1, \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{\mathrm{e}} = 0.635\ldots$$

and are unknown for $n \geq 3$. Logarithmic coefficients is one of the topic recently being of interest by various authors (e.g., [1,18,33]).

Logarithmic coefficients can be considered for functions f from the class $\mathcal{A}$ however under the assumption that the branch of logarithm $\mathbb{D} \ni z \mapsto \log f(z)/z$ exists. From (1.4) and (1.5) it follows that $g(z) := f(z)/z \neq 0$ in $\mathbb{D}\backslash\{0\}$ for $f \in \mathcal{ST}(\mathrm{i})$ and $f \in \mathcal{ST}(1)$. However $g(\mathbb{D})$ needs not be necessarily a simply connected domain. Therefore, let $\mathcal{ST}_0(\mathrm{i})$ and $\mathcal{ST}_0(1)$ be the subclasses of $\mathcal{ST}(\mathrm{i})$ and $\mathcal{ST}(1)$ respectively, of all functions f for which the branch $\mathbb{D} \ni z \mapsto \log f(z)/z$ with $\log 1 := 0$ exists.

Theorem 2.1. *If $f \in \mathcal{ST}_0(\mathrm{i})$ is of the form (1.1), then*

$$|\gamma_1| \leq 1, \quad |\gamma_2| \leq \frac{3}{2}, \quad |\gamma_3| \leq 1.$$

All inequalities are sharp.

Proof. By (1.4) there exists $p \in \mathcal{P}$ of the form (1.6) such that

$$(1 - z^2)\frac{f(z)}{z} = p(z). \tag{2.3}$$

Substituting the series (1.1) and (1.6) into (2.3) by equating the coefficients we get

$$a_2 = c_1, \quad a_3 = c_2 + 1, \quad a_4 = c_1 + c_3. \tag{2.4}$$

The inequality $|\gamma_1| \leq 1$ follows directly from (2.2), (2.4) and (1.7) with sharpness for the function f given by (2.3), where p is as in (1.10).

Substituting (1.7) and (1.8) into (2.4) from (2.2) it follows that

$$\begin{aligned}|\gamma_2| &= \frac{1}{2}\left|a_3 - \frac{1}{2}a_2^2\right| = \frac{1}{2}\left|c_2 - \frac{1}{2}c_1^2 + 1\right|\\ &= \frac{1}{2}\left|2(1-|\zeta_1|^2)\zeta_2 + 1\right| \leq \frac{1}{2} + (1-|\zeta_1|^2)|\zeta_2| \leq \frac{3}{2}\end{aligned}$$

with sharpness for the function f given by (2.3), where p is as in (1.11) with $\zeta_1 = 0$ and any $\zeta_2 \in \mathbb{T}$.

By (2.2) and (2.4) we have

$$6\gamma_3 = c_1^3 - 3c_1c_2 + 3c_3.$$

Hence and by (1.7)–(1.9) we get

$$3\gamma_3 = 2\zeta_1^3 - 3(1-|\zeta_1|^2)\overline{\zeta}_1\zeta_2^2 + 3(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3,$$

where $\zeta_i \in \overline{\mathbb{D}}$, $i = 1, 2, 3$. Thus by setting $x := |\zeta_1| \in [0,1]$ and $y := |\zeta_2| \in [0,1]$ we obtain

$$\begin{aligned} 3|\gamma_3| &\le 2x^3 + 3(1-x^2)xy^2 + 3(1-x^2)(1-y^2) \\ &= 2x^3 - 3x^2 + 3 - 3(1-x^2)(1-x)y^2 \\ &\le 2x^3 - 3x^2 + 3 \le 3, \quad (x,y) \in [0,1]\times[0,1]. \end{aligned}$$

Thus $|\gamma_3| \le 1$ with sharpness for the function f given by (2.3), where p is as in (1.12) with $\zeta_1 = \zeta_2 = 0$ and any $\zeta_3 \in \mathbb{T}$. □

Theorem 2.2. *If $f \in \mathcal{ST}_0(1)$ is of the form (1.1), then*

$$|\gamma_1| \le 2, \quad |\gamma_2| \le \frac{3}{2}, \quad |\gamma_3| \le \frac{1}{3}(1+\sqrt{2}).$$

All inequalities are sharp.

Proof. By (1.5) there exists $p \in \mathcal{P}$ of the form (1.6) such that

$$(1-z)^2\frac{f(z)}{z} = p(z). \tag{2.5}$$

Substituting the series (1.1) and (1.6) into (2.5) by equating the coefficients we get

$$a_2 = c_1 + 2, \quad a_3 = 3 + 2c_1 + c_2, \quad a_4 = 4 + 3c_1 + 2c_2 + c_3. \tag{2.6}$$

The inequality $|\gamma_1| \le 2$ follows directly from (2.2), (2.6) and (1.7) with sharpness for the function f given by (2.5), where p is as in (1.10).

Substituting (1.7) and (1.8) into (2.6) from (2.2) it follows that

$$\begin{aligned} |\gamma_2| &= \frac{1}{2}\left|a_3 - \frac{1}{2}a_2^2\right| = \frac{1}{2}\left|c_2 - \frac{1}{2}c_1^2 + 1\right| \\ &= \frac{1}{2}\left|2(1-|\zeta_1|^2)\zeta_2 + 1\right| \le \frac{1}{2} + (1-|\zeta_1|^2)|\zeta_2| \le \frac{3}{2} \end{aligned}$$

with sharpness for the function f given by (2.3), where p is as in (1.11) with $\zeta_1 = 0$ and any $\zeta_2 \in \mathbb{T}$.

By (2.2) and (2.6) we have

$$6\gamma_3 = 2 + c_1^3 + 3c_3 - 3c_1c_2.$$

Hence and by (1.7)–(1.9) we get

$$3\gamma_3 = 1 + \zeta_1^3 - 3(1-|\zeta_1|^2)\overline{\zeta}_1\zeta_2^2 + (1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3,$$

where $\zeta_i \in \overline{\mathbb{D}}$, $i = 1, 2, 3$. Thus by setting $x := |\zeta_1| \in [0, 1]$ and $y := |\zeta_2| \in [0, 1]$ we obtain

$$\begin{aligned} 3|\gamma_3| &\le 1 + x^3 - 3(1 - x^2)xy^2 + (1 - x^2)(1 - y^2) \\ &= 2 - x^2 + x^3 + (1 - x^2)(3x - 1)y^2 =: F(x, y). \end{aligned} \tag{2.7}$$

We have $F(1/3, y) = 52/27$. Moreover for $x \in (1/3, 1]$ and $x \in [0, 1/3)$ we get

$$F(x, y) \le F(x, 1) \le 1 + 3x - 2x^3 \le 1 + \sqrt{2}, \quad y \in [0, 1],$$

and

$$F(x, y) \le F(x, 0) \le 2 - x^2 + x^3 \le 2, \quad y \in [0, 1],$$

respectively. Thus by (2.7), $|\gamma_3| \le (1 + \sqrt{2})/3$ with sharpness for the function f given by (2.3), where p is as in (1.12) with $\zeta_1 = 1/\sqrt{2}$, $\zeta_2 = \mathrm{i}$ and any $\zeta_3 \in \mathbb{T}$. □

3. Zalcman Functional and Hankel Determinant

Now we compute the sharp upper bound of the Zalcman functional $J_{2,3}(f) := a_2a_3 - a_4$ being a special case of the generalized Zalcman functional $J_{n,m}(f) := a_na_m - a_{n+m-1}$, $n, m \in \mathbb{N}\setminus\{1\}$, which was investigated by Ma [24] for $f \in \mathcal{S}$ (see also [29] for relevant results on this functional). We will find also the sharp bound of the second Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$. Both functionals $J_{2,3}$ and $H_{2,2}$ have been studied recently by various authors (see e.g., [5,6,14,16,17,19,27]).

Theorem 3.1. *If $f \in \mathcal{ST}(\mathrm{i})$ is of the form (1.1), then*

$$|a_2a_3 - a_4| \le 2.$$

The inequality is sharp with the extremal function

$$f(z) = \frac{z}{(1 - z)^2}, \quad z \in \mathbb{D}. \tag{3.1}$$

Proof. From (2.4) by using (1.7)–(1.9) it follows that

$$\begin{aligned} |a_2a_3 - a_4| &= |c_1c_2 - c_3| \\ &= 2\left|\zeta_1^3 + 2(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 - (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3\right| \\ &\le 2\left[|\zeta_1|^3 + 2(1 - |\zeta_1|^2)|\zeta_1||\zeta_2|^2 - (1 - |\zeta_1|^2)(1 - |\zeta_2|^2\right] \\ &= 2\left[1 - |\zeta_1|^2 + |\zeta_1|^3 - 2(1 - |\zeta_1|^2)(1 - |\zeta_1|)|\zeta_2|^2\right] \\ &\le 2\left(1 - |\zeta_1|^2 + |\zeta_1|^3\right) \le 2, \end{aligned} \tag{3.2}$$

with sharpness for the function (3.1).

To find sharp estimate for $H_{2,2}$ over $\mathcal{ST}(\mathrm{i})$ we use the following lemma. □

Proposition 3.2.

$$|4z^2-4z-1| \le \begin{cases} 1+4|z|-4|z|^2, & |z| \le (-1+\sqrt{2})/2, \\ \sqrt{2}(1+4|z|^2), & (-1+\sqrt{2})/2 \le |z| \le 1. \end{cases} \tag{3.3}$$

Proof. Since the inequality (3.3) clearly holds for $z=0$, assume that $z=re^{i\theta}$ with $0<r\le 1$ and $0\le\theta<2\pi$. A simple computation gives

$$|4z^2-4z-1|^2 = \varphi(\cos\theta), \tag{3.4}$$

where $\varphi\colon[-1,1]\to\mathbb{R}$ is a function defined by

$$\varphi(x) := -16r^2x^2 - 8r(4r^2-1)x + 16r^4 + 24r^2 + 1.$$

Note that $\varphi'(x)=0$ occurs only when $x=(1-4r^2)/(4r) =: x_0$.

When $r\le(-1+\sqrt{2})/2$, we have $x_0>1$ or $1-4r-4r^2>0$. Therefore

$$\varphi'(x) \ge 8r(1-4r-4r^2) > 0, \quad x\in[-1,1].$$

Hence we get

$$\varphi(x) \le \varphi(1) = (1+4r-4r^2)^2. \tag{3.5}$$

Thus from (3.4) and (3.5) it follows that the inequality (3.3) holds for $|z|\le(-1+\sqrt{2})/2$.

When $(-1+\sqrt{2})/2\le r\le 1$, we have $x_0\in[-1,1]$. Then

$$\varphi(x) \le \varphi(x_0) = 2(1+4r^2)^2, \quad x\in[-1,1]. \tag{3.6}$$

Combining (3.4) and (3.6) we see that the inequality (3.3) holds for $(-1+\sqrt{2})/2\le|z|\le 1$. □

Theorem 3.3. *If $f\in\mathcal{ST}(\mathrm{i})$ is of the form (1.1), then*

$$|a_2a_4-a_3^2| \le \frac{28}{3}. \tag{3.7}$$

The inequality is sharp with the extremal function

$$f(z) = \frac{z(3+z+3z^2)}{3(1-z^2)^2}, \quad z\in\mathbb{D}. \tag{3.8}$$

Proof. From (2.4) by using (1.7)–(1.9)we have

$$\begin{aligned} a_2a_4-a_3^2 &= c_1^2+c_1c_3-c_2^2-2c_2-1 \\ &= 4\zeta_1^2-4\zeta_1-1-4(1-|\zeta_1|^2)\zeta_2-4(1-|\zeta_1|^2)\zeta_2^2 \\ &\quad + 4\zeta_1(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3, \end{aligned} \tag{3.9}$$

where $\zeta_i\in\overline{\mathbb{D}},\ i=1,2,3$. Let $x:=|\zeta_1|\in[0,1]$ and $y=|\zeta_2|\in[0,1]$.

Assume first that $x\in[0,x_0]$, where $x_0:=(-1+\sqrt{2})/2$. Then by (3.9) and Proposition 3.2 for $y\in[0,1]$ we get

$$\begin{aligned} |a_2a_4-a_3^2| &\le 1+8x-4x^2-4x^3+4(1-x^2)y \\ &\quad + 4x(1-x^2)y^2 =: F(x,y). \end{aligned}$$

Clearly, for each $x \in [0, x_0]$, the function $[0,1] \ni y \mapsto F(\cdot, y)$ is increasing and therefore for $y \in [0,1]$,

$$F(x,y) \le F(x,1) = 9 + 4x - 12x^2 \le \frac{28}{3} = 9.333\ldots. \tag{3.10}$$

Assume now that $x \in [x_0, 1]$. Then by (3.9) and Proposition 3.2 for $y \in [0,1]$ we get

$$\begin{aligned}|a_2a_4 - a_3^2| \le \sqrt{2} + 4x + 4\sqrt{2}x^2 - 4x^3 + 4(1-x^2)y \\ + 4(1-x^2)(1-x)y^2 =: G(x,y).\end{aligned}$$

Note first that

$$G(1,y) = 5\sqrt{2} = 7.071\ldots, \quad y \in [0,1]. \tag{3.11}$$

Clearly, for each $x \in [x_0, 1]$, the function $[0,1] \ni y \mapsto G(\cdot, y)$ is increasing and therefore for $y \in [0,1]$,

$$\begin{aligned}G(x,y) &\le G(x,1) = 8 + \sqrt{2} - 4(2-\sqrt{2})x^2 \\ &\le -2 + 8\sqrt{2} = 9.133\ldots.\end{aligned}$$

Hence, from (3.10) and (3.11) it follows that the inequality (3.7) is true. Equality in (3.7) holds for the function f given by (2.3), where p is given by (1.12) with $\zeta_1 := 1/6$ and $\zeta_2 = \zeta_3 := 1$, i.e., for the function (3.8). □

Theorem 3.4. *If $f \in \mathcal{ST}(1)$ is of the form (1.1), then*

$$|a_2a_3 - a_4| \le 20.$$

The inequality is sharp with the extremal function

$$f(z) = \frac{z(1+z)}{(1-z)^3}, \quad z \in \mathbb{D}. \tag{3.12}$$

Proof. From (2.6), by using (1.7) and the inequality $|c_1c_2 - c_3| \le 2$ which was proved in (3.2), we obtain

$$\begin{aligned}|a_2a_3 - a_4| &= |2 + 4c_1 + 2c_1^2 + c_1c_2 - c_3| \\ &\le 2 + 4|c_1| + 2|c_1|^2 + |c_1c_2 - c_3| \le 20.\end{aligned}$$

with sharpness for the function (3.12). □

Theorem 3.5. *If $f \in \mathcal{ST}(1)$ is of the form (1.1), then*

$$|a_2a_4 - a_3^2| \le 17. \tag{3.13}$$

The inequality is sharp with the extremal function (3.12).

Proof. From (2.6) by using (1.7)–(1.9)we have

$$\begin{aligned} a_2a_4 - a_3^2 &= -1 - 2c_1 - c_1^2 - 2c_1c_2 - 2c_2 - c_2^2 + c_1c_3 + 2c_3 \\ &= -1 - 4\zeta_1 - 8\zeta_1^2 - 4\zeta_1^3 - 4(1-|\zeta_1|^2)\zeta_2 \\ &\quad - 4(1+\zeta_1)(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 \\ &\quad + 4(1+\zeta_1)(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3, \end{aligned} \tag{3.14}$$

where $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1,2,3\}$. Set $x := |\zeta_1| \in [0,1]$ and $y =: |\zeta_2| \in [0,1]$. By (3.14) we have

$$\begin{aligned} |a_2a_4 - a_3^2| \le 5 + 8x + 4x^2 + 4(1-x^2)y \\ - 4(1-x^2)^2y^2 =: F(x,y), \quad x, y \in [0,1]. \end{aligned}$$

Note first that

$$F(1,y) = 17, \quad y \in [0,1]. \tag{3.15}$$

Let now $x \in [0,1)$. Then for $y \in [0,1]$ we have

$$\frac{\partial F}{\partial y} = 4(1-x^2)[1-2(1-x^2)y] = 0$$

iff $y = 1/2(1-x^2) =: y_0$. Since $y_0 \ge 1$ for each $x \in [1/\sqrt{2}, 1)$, so then the function $[0,1] \ni y \mapsto F(\cdot, y)$ is increasing and therefore

$$F(x,y) \le F(x,1) = 5 + 8x + 8x^2 - 4x^4 \le 17, \quad y \in [0,1]. \tag{3.16}$$

For $x \in [0, 1/\sqrt{2})$ we have

$$\begin{aligned} F(x,y) \le F(x,y_0) = F\left(x, \frac{1}{2(1-x^2)}\right) \\ = 6 + 8x + 4x^2 \le 8 + 4\sqrt{2} = 13.656\ldots, \quad y \in [0,1]. \end{aligned}$$

Hence by (3.15) and (3.16) it follows that the inequality (3.13) is true. Equality in (3.13) holds for the function f defined by (3.12). □

4. Inverse Coefficients

Since $\mathcal{ST}(\mathrm{i})$ is a compact class and $f'(0) = 1$ for every $f \in \mathcal{ST}(\mathrm{i})$, there exists $r_0 \in (0,1)$ such that every $f \in \mathcal{ST}(\mathrm{i})$ is invertible in the disk $\mathbb{D}_{r_0}$. Thus there exists $\delta > 0$ such that the inverse function $\hat{f}$ of $f_{|\mathbb{D}_{r_0}}$ has a series expansion in the disk $\mathbb{D}_\delta$ of the form

$$\hat{f}(w) = w + \sum_{n=2}^{\infty} \beta_n w^n, \quad w \in \mathbb{D}_\delta. \tag{4.1}$$

Thus for $f \in \mathcal{ST}(\mathrm{i})$ of the form (1.1) the following relations hold (see e.g., [12, Vol. I, p. 57])

$$\beta_2 = -a_2, \quad \beta_3 = 2a_2^2 - a_3, \quad \beta_4 = -5a_2^3 + 5a_2a_3 - a_4. \tag{4.2}$$

Similar situation holds for the class $\mathcal{ST}(1)$.

Theorem 4.1. *If $\hat{f}$ is the inverse function of $f \in \mathcal{ST}(\mathrm{i})$ of the form (4.1), then*
(i) $|\beta_2| \le 2$;
(ii) $|\beta_3| \le 7$;
(iii) $|\beta_4| \le 30$.
All inequalities are sharp with the extremal function

$$f(z) = \frac{z(1+\mathrm{i}z)}{(1-z^2)(1-\mathrm{i}z)}, \quad z \in \mathbb{D}. \tag{4.3}$$

Proof. Substituting (2.4) into (4.2) we get

$$\beta_2 = -c_1, \quad \beta_3 = 2c_1^2 - c_2 - 1 \tag{4.4}$$

and

$$\beta_4 = -5c_1^3 + 5c_1c_2 + 4c_1 - c_3. \tag{4.5}$$

By (4.4) and (1.7) the inequality (i) follows immediately. From (4.4) with (1.7) and (1.8) we have

$$\begin{aligned}|\beta_3| &= |2c_1^2 - c_2 - 1| = \left|6\zeta_1^2 - 2(1-|\zeta_1|^2)\zeta_2 - 1\right| \\ &\le 6|\zeta_1|^2 + 2(1-|\zeta_1|^2)||\zeta_2| + 1 \le 4|\zeta_1|^2 + 3 \le 7.\end{aligned}$$

Now we prove (iii). By (4.5) and (1.7)–(1.9) we have

$$\begin{aligned}|\beta_4| &= \big|-22\zeta_1^3 + 8\zeta_1 + 16(1-|\zeta_1|^2)\zeta_1\zeta_2 \\ &\quad + 2(1-|\zeta_1|^2)\overline{\zeta}_1\zeta_2^2 - 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3\big| \\ &\quad \le 2 + 8x - 2x^2 + 22x^3 + 16x(1-x^2)y - 2(1-x)^2(1+x)y^2 \\ &=: F(x,y),\end{aligned}$$

where $\zeta_i \in \overline{\mathbb{D}}$, $i = 1,2,3$, $x := |\zeta_1| \in [0,1]$ and $y := |\zeta_2| \in [0,1]$.

Note first that

$$F(1,y) = 30, \quad y \in [0,1]. \tag{4.6}$$

Let now $x \in [0,1)$. Then for $y \in [0,1]$ we have

$$\frac{\partial F}{\partial y} = 4(1-x^2)[4x - (1-x)y] = 0$$

iff $y = 4x/(1-x) =: y_0$. Since $y_0 \ge 1$ for each $x \in [1/5,1)$, so then the function $[0,1] \ni y \mapsto F(\cdot,y)$ is increasing and therefore

$$F(x,y) \le F(x,1) = 26x + 4x^3 \le 30, \quad y \in [0,1]. \tag{4.7}$$

For $x \in [0,1/5)$ we have

$$\begin{aligned}F(x,y) &\le F(x,y_0) = F\left(x, \frac{4x}{1-x}\right) \\ &= 2 + 72x + 30x^2 - 10x^3 \le \frac{438}{25} = 15.52, \quad y \in [0,1].\end{aligned} \tag{4.8}$$

Hence by (4.6)–(4.8) it follows that the inequality in (iii) is true.

All inequalities are sharp with the extremal function (4.3). □

Theorem 4.2. *If $\hat{f}$ is the inverse function of $f \in \mathcal{ST}(1)$ of the form* (4.1), *then*

(i) $|\beta_2| \le 4$;
(ii) $|\beta_3| \le 23$;
(iii) $|\beta_4| \le 156$.

All inequalities are sharp with the extremal function

$$f(z) = \frac{z(1+z)}{(1-z)^3}, \quad z \in \mathbb{D}. \tag{4.9}$$

Proof. Substituting (2.6) into (4.2) we get

$$\beta_2 = -c_1 - 2, \quad \beta_3 = 2c_1^2 + 6c_1 - c_2 + 5 \tag{4.10}$$

and

$$\beta_4 = -5c_1^3 - 20c_1^2 + 2c_1 + 5c_1c_2 + 8c_2 - c_3 - 14. \tag{4.11}$$

By (4.10) and (1.7) the inequality (i) follows immediately. From (4.10) with (1.7) and (1.8) we have

$$\begin{aligned} |\beta_3| &= |2c_1^2 + 6c_1 - c_2 + 5| = \left|6\zeta_1^2 + 12\zeta_1 - 2(1-|\zeta_1|^2)\zeta_2 + 5\right| \\ &\le 6|\zeta_1|^2 + 12|\zeta_1| + 2(1-|\zeta_1|^2) + 5 = 4|\zeta|^2 + 12|\zeta| + 7 \le 23. \end{aligned}$$

Now we prove (iii). By (4.11) and (1.7)–(1.9) we have

$$\begin{aligned} |\beta_4| &= \big|-22\zeta_1^3 - 64\zeta_1^2 - 56\zeta_1 - 14 + 16(1-|\zeta_1|^2)\zeta_2 \\ &\quad + 16(1-|\zeta_1|^2)\zeta_1\zeta_2 + 2(1-|\zeta_1|^2)\overline{\zeta}_1\zeta_2^2 \\ &\quad - 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3\big| \\ &\le 22x^3 + 62x^2 + 56x + 16 + 16(1-x^2)(1+x)y \\ &\quad - 2(1-x^2)(1-x)y^2 =: F(x,y), \end{aligned}$$

where $\zeta_i \in \overline{\mathbb{D}}$, $i = 1,2,3$, $x := |\zeta_1| \in [0,1]$ and $y := |\zeta_2| \in [0,1]$.

Note first that

$$F(1,y) = 156, \quad y \in [0,1]. \tag{4.12}$$

Let now $x \in [0,1)$. Then for $y \in [0,1]$ we have

$$\frac{\partial F}{\partial y} = 4(1-x^2)[4(1+x) - (1-x)y = 0$$

iff $y = 4(1+x)/(1-x) =: y_0$. Since $y_0 \ge 1$ for each $x \in (0,1)$, so the function $[0,1] \ni y \mapsto F(\cdot, y)$ is increasing and therefore

$$F(x,y) \le F(x,1) = 4x^3 + 48x^2 + 74x + 30 \le 156, \quad y \in [0,1].$$

Hence and from (4.12) it follows that the inequality in (iii) is true.

All inequalities are sharp with the extremal function (4.9). □

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