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Results in Mathematics

Multiple Solutions for $(p, 2)$-Equations with Resonance and Concave Terms

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Abstract. We consider parametric Dirichlet problems driven by the sum of a p-Laplacian $(p > 2)$ and a Laplacian ($(p, 2)$-equation) and with a reaction term which exhibits competing nonlinearities. We prove two multiplicity theorems. In the first the competing terms are not decoupled, the dependence on the parameter is not necessarily linear and the reaction term has a general polynomial growth, possibly supercritical. We produce three nontrivial solutions for small values of the parameter. We provide sign information for all solutions (two of constant sign and the third nodal). Then we decouple the competing nonlinearities and allow for resonance to occur at $\pm\infty$. We produce six nontrivial smooth solutions for small values of the parameter. We provide sign information for five of these solutions.

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1. Introduction

In a recent paper Papageorgiou–Winkert [32] examined the following nonhomogeneous parametric Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = f(z, u(z)) - \lambda |u(z)|^{q-2} u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{I_λ}$$

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Problem (I_λ) is defined on a bounded domain $\Omega \subseteq \mathbb{R}^N$ with smooth boundary $\partial\Omega$ and Δ_p (with $2 < p < +\infty$) is the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right) \quad \forall u \in W_0^{1,p}(\Omega).$$

In the reaction term (right hand side) $1 < q < 2$, $\lambda > 0$ is a parameter and the perturbation $f(z,\zeta)$ is a Carathéodory function (that is, for all $\zeta \in \mathbb{R}$, $\zeta \longmapsto f(z,\zeta)$ is measurable and for almost all $z \in \Omega$, $\zeta \longmapsto f(z,\zeta)$ is continuous), which near $\pm\infty$ is $(p-1)$-linear and resonance can occur with respect to the principal (first) eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ from the left. This makes the energy (Euler) functional of the problem coercive. Hence problem (I_λ) is an equation with competing nonlinearities, namely a concave term and a $(p-1)$-linear perturbation. Note that in (I_λ) the concave nonlinearity $\zeta \longmapsto \lambda|\zeta|^{q-2}\zeta$ enters in the reaction with a negative sign. In [32] the authors provide a multiplicity theorem for all small values of the parameter $\lambda > 0$.

Our aim in the present paper, is to study the complementary situation. Namely, again we deal with a problem with competing nonlinearities. However, now the concave term enters with a positive sign and asymptotically at $\pm\infty$, the reaction function is $(p-1)$-linear and resonant with respect to any nonprincipal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. In fact in the first part of the paper, the competing nonlinearities in the reaction are not decoupled and the dependence on the parameter $\lambda > 0$ need not be linear.

So, now the problem under consideration is the following:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = f_\lambda(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{P_λ}$$

with $2 < p < +\infty$. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^3-boundary $\partial\Omega$ and for every $\lambda > 0$, f_λ is a Carathéodory function on $\Omega \times \mathbb{R}$, exhibiting a concave nonlinearity near zero and is $(p-1)$-linear near $\pm\infty$. We prove two multiplicity theorems for small values of the parameter $\lambda > 0$. In the first multiplicity theorem (Theorem 3.6), we assume that f_λ admits arbitrary polynomial growth not necessarily subcritical. We prove the existence of at least three nontrivial smooth solutions, all with sign information (one positive, one negative and the third nodal (sign changing)). Then we decouple the competing terms and improve the regularity and growth at the reaction term $f_\lambda(z,\cdot)$ (more precisely, we assume that $f_\lambda(z,\zeta) = \lambda|\zeta|^{q_\lambda-2}\zeta + f_0(z,\zeta)$ for almost all $z \in \Omega$, $f_0(z,\cdot) \in C^1(\mathbb{R})$ and it is $(p-1)$-linear near $\pm\infty$ interacting with a nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ (resonant equation)). We prove the existence of at least six nontrivial smooth solution. We provide sign information for five of them (two positive, two negative and the fifth nodal).

Equations driven by the sum of a p-Laplacian and a Laplacian, known in the literature as $(p,2)$-equations, arise in problems of mathematical physics; see Benci–D'Avenia–Fortunato–Pisani [4] (quantum physics) and Cherfils–Il$'$yasov [6] (plasma physics). Recently there have been some existence and multiplicity results for such equations. We mention the works of Aizicovici–Papageorgiou–Staicu [2], Cingolani–Degiovanni [7], Gasiński–Klimczak–Papageorgiou [11], Gasiński–Papageorgiou [15,17–20], Mugnai–Papageorgiou [27], Papageorgiou–Rădulescu [28–30], Papageorgiou–Smyrlis [31], Papageorgiou–Winkert [32], Sun [34], Sun–Zhang–Su [35], Yang–Bai [37]. Only [32] deals with equations exhibiting competing nonlinearities.

2. Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Suppose that $\varphi \in C^1(X;\mathbb{R})$. We say that φ satisfies the *Cerami condition*, if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence."

This is a compactness-type condition on the functional φ which compensates for the fact that the ambient space X is not necessarily locally compact (X is in general infinite dimensional). It leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. A basic result in this theory, is the so called *mountain pass theorem* due to Ambrosetti-Rabinowitz [3]. Here we state the result in a slightly more general form (see Gasiński–Papageorgiou [12, p. 648]).

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X;\mathbb{R})$ satisfies the Cerami condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \varrho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \ \|u - u_0\| = \varrho\} = m_\varrho,$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)),$$

with

$$\Gamma = \big\{\gamma \in C\big([0, 1]; X\big) : \gamma(0) = u_0,\ \gamma(1) = u_1\big\},$$

then $c \geqslant m_\varrho$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi'(u) = 0$, $\varphi(u) = c$).

In the study of problem (P_λ), we will make use of the Sobolev space $W^{1,p}_0(\Omega)$ and of the Banach space

$$C^1_0(\overline{\Omega}) = \big\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\big\}.$$

On account of the Poincaré inequality, on $W^{1,p}_0(\Omega)$ we can use the norm

$$\|u\| = \|Du\|_p \quad \forall u \in W^{1,p}_0(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \Omega, \frac{\partial u}{\partial n}\big|_{\partial\Omega} < 0\}.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative of u defined by $(Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Recall that $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$.

Let $\zeta \in \mathbb{R}$. We set $\zeta^\pm = \max\{\pm\zeta, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^- \quad \text{and} \quad |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and if $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we set

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \forall u \in W_0^{1,p}(\Omega)$$

(the Nemytskii or superposition map corresponding to the function $g(z,\zeta)$).

Let $f_0\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,\zeta)| \leqslant a_0(z)(1 + |\zeta|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{all } \zeta \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$, $2 < p < r < p^*$ where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geqslant N. \end{cases}$$

We set

$$F_0(z,\zeta) = \int_0^\zeta f_0(z,s)\, ds$$

and consider the C^1-functional $\varphi_0\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z,u)\, dz \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.1}$$

The next proposition is a particular case of a more general result of Gasiński–Papageorgiou [14].

Proposition 2.2. *If φ_0 is defined by (2.1) and $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $\varrho_0 > 0$, such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \forall h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \leqslant \varrho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $\varrho_1 > 0$, such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \forall h \in W_0^{1,p}(\Omega), \|h\| \leqslant \varrho_1.$$

Since we will be dealing with resonant equations, we need to know the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in}\, \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.2}$$

with $1 < p < +\infty$. We say that $\widehat{\lambda} \in \mathbb{R}$ is an *eigenvalue* of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (2.2) admits a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$ known as an *eigenfunction* corresponding to the eigenvalue $\widehat{\lambda}$. The nonlinear regularity theory (see, for example, Gasiński–Papageorgiou [12, pp. 737–738]), implies that $\widehat{u} \in C_0^1(\overline{\Omega})$. We know that $(-\Delta_p, W_0^{1,p}(\Omega))$ admits a smallest eigenvalue $\widehat{\lambda}_1(p)$ which has the following properties:

- $\widehat{\lambda}_1(p) > 0$ and it is isolated in the spectrum $\sigma_0(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \varepsilon) \cap \sigma_0(p) = \emptyset$).
- $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}, \widehat{v} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1(p)$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R}\backslash\{0\}$).
- we have

$$\widehat{\lambda}_1(p) = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right\}. \tag{2.3}$$

From the second property (simplicity of $\widehat{\lambda}_1(p) > 0$), we infer that the eigenfunctions corresponding to $\widehat{\lambda}_1(p)$ do not change sign. By $\widehat{u}_1(p)$ we denote the positive L^p-normalized (that is, $\|\widehat{u}_1(p)\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(p)$. We already mentioned that $\widehat{u}_1(p) \in C_+\backslash\{0\}$. In fact the nonlinear maximum principle (see Gasiński–Papageorgiou [12, p. 738]) implies that $\widehat{u}_1(p) \in \operatorname{int} C_+$. In (2.3) the infimum is realized on the one dimensional eigenspace corresponding to $\widehat{\lambda}_1(p) > 0$. Since $\sigma_0(p) \subseteq (0, +\infty)$ is closed and $\widehat{\lambda}_1(p) > 0$ is isolated, then the second eigenvalue is well defined by

$$\widehat{\lambda}_2(p) = \min\left\{\widehat{\lambda} \in \sigma_0(p) :\ \widehat{\lambda} > \widehat{\lambda}_1(p)\right\}.$$

To produce additional eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$, we employ the Ljusternik-Schnirelmann minimax scheme, which generates a whole sequence $\{\widehat{\lambda}_k(p)\}_{k\geqslant 1}$ of strictly increasing eigenvalues such that $\widehat{\lambda}_k(p) \longrightarrow +\infty$ as $k \to +\infty$. These eigenvalues are known as *variational eigenvalues* and we can have at least three such sequences of variational eigenvalues depending on the index used in the Ljusternik-Schnirelmann minimax scheme (see Cingolani–Degiovanni [7, p. 1198]). The three sequences coincide in the first two elements. Here, we use the sequence constructed using the Fadell–Rabinowitz [9] cohomological index. So, we have

$$\widehat{\lambda}_n(p) = \left\{\sup_{u\in E} \|Du\|_p^p : E \subseteq M,\ E\, \text{is symmetric}, \text{index}\, (E) \geqslant n\right\},$$

with $M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}$ (here $\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$ and index $(\cdot)$ is the Fadell–Rabinowitz [9] cohomological index). We do not know if these variational eigenvalues exhaust $\sigma_0(p)$. This is the case if $p = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential equations), see Gasiński–Papageorgiou [12]. We know that every eigenfunction $\widehat{u}$ corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(p)$ is nodal (sign changing).

Now suppose that X is a Banach space and $\varphi \in C^1(X;\mathbb{R})$, $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\},$$
$$\varphi^c = \{u \in X : \varphi(u) \leqslant c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Recall that $H_k(Y_1, Y_2) = 0$ for all $k \in -\mathbb{N}$. If K_φ^c is isolated, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \forall k \in \mathbb{N}_0,$$

with U being a neighbourhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition of critical groups is independent of the choice of the neighbourhood of U.

Suppose that $\varphi \in C^1(X;\mathbb{R})$ satisfies the Cerami condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \in \mathbb{N}_0.$$

The second deformation theorem (see e.g., Gasiński–Papageorgiou [12, p. 628]), implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Finally, for $1 < p < +\infty$, we define the map $A_p\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) by setting

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p}(\Omega). \tag{2.4}$$

From Gasiński–Papageorgiou [12, p. 746], we have the following result.

Proposition 2.3. *The map $A_p\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ defined by (2.4) is continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$.

If $p = 2$, then we write $A_2 = A$ and we have $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

3. Three Nontrivial Smooth Solutions

In this section, we consider problem (P_λ) with a reaction $f_\lambda(z,\zeta)$ of arbitrary polynomial growth in $\zeta \in \mathbb{R}$, not necessarily subcritical. We prove a multiplicity theorem producing three nontrivial smooth solutions all with sign information.

The conditions on the reaction term f_λ are the following.

$\underline{H_1}$: For every $\lambda > 0$, $f_\lambda \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f_\lambda(z,0) = 0$ for almost all $z \in \Omega$ and

(i) $|f_\lambda(z,\zeta)| \leqslant a_\lambda(z) + c|\zeta|^{r-1}$ for almost all $z \in \Omega$, all $\zeta \in \mathbb{R}$, all $\lambda \in (0,\lambda_0]$ with $a_\lambda \in L^\infty(\Omega)$, $c > 0$, $2 < r < +\infty$ and $\|a_\lambda\|_\infty \longrightarrow 0$ as $\lambda \searrow 0$;

(ii) for every $\lambda \in (0,\lambda_0]$, there exist $q_\lambda \in (1,2)$, $\delta_0^\lambda > 0$ and $\eta_\lambda > 0$ such that
$$\eta_\lambda |\zeta|^{q_\lambda} \leqslant\ f_\lambda(z,\zeta)\zeta \leqslant\ q_\lambda F_\lambda(z,\zeta) \quad \text{for a.a. } z \in \Omega,\ \text{all } |\zeta| \leqslant \delta_0^\lambda,$$
where $F_\lambda(z,\zeta) = \int_0^\zeta f_\lambda(z,s)\,ds$.

Remark 3.1. Hypothesis $H_1(i)$ permits supercritical polynomial growth for $f_\lambda(z,\cdot)$. Hypothesis $H_1(ii)$ implies the presence of a concave term near zero.

Example 3.2. The following functions satisfy hypotheses H_1. For the sake of simplicity we drop the z-dependence.
$$f^1_\lambda(\zeta) = \lambda|\zeta|^{q_\lambda-2}\zeta + |\zeta|^{r-2}\zeta \quad \text{with } 1 < q_\lambda < 2 < r < +\infty,$$
$$f^2_\lambda(\zeta) = \lambda(|\zeta|^{q_\lambda-2}\zeta - |\zeta|^{r-2}\zeta) \quad \text{with } 1 < q_\lambda < 2 < r < +\infty,$$
$$f^3_\lambda(\zeta) = \lambda|\zeta|^{q_\lambda-2}\zeta + |\zeta|^{p-2}\zeta\ln(1+|\zeta|) \quad \text{with}\, 1 < q_\lambda < 2 < p < +\infty.$$
Functions f^1_λ and f^3_λ correspond to "concave-convex" reactions but without the subcritical growth. Also in f^2_λ the "convex" term fails to satisfy the Ambrosetti-Rabinowitz condition (see Ambrosetti-Rabinowitz [3]). The function f^2_λ corresponds to the usual superdiffusive logistic reaction.

First we produce two nontrivial constant sign smooth solutions.

Proposition 3.3. *If hypotheses H_1 hold, then there exists $\lambda^* \in (0,\lambda_0]$ such that for every $\lambda \in (0,\lambda^*)$ problem (P_λ) admits at least two nontrivial constant sign smooth solutions*
$$u_\lambda \in \operatorname{int} C_+ \quad \textit{and} \quad v_\lambda \in\ -\operatorname{int} C_+.$$

Proof. First we produce the positive solution. We consider the following auxiliary Dirichlet problem
$$\begin{cases} -\Delta e(z) = 1 & \text{in}\, \Omega, \\ e|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$
This problem has a unique solution $e \in \operatorname{int} C_+$. In fact since we assumed that $\partial\Omega$ is a C^3-manifold, standard regularity theory (see Troianiello [36, Theorem 3.23, page 189]) implies that $e \in C^2(\overline{\Omega})$.

Claim. There exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$, we can find $\xi_0^\lambda > 0$ for which we have

$$\|a_\lambda\|_\infty + c(\xi_0^\lambda \|e\|_\infty)^{r-1} < \xi_0^\lambda - (\xi_0^\lambda)^{p-1}\|\Delta_p e\|_\infty$$

(recall that $e \in C^2(\overline{\Omega})$, hence $\Delta_p e \in C(\overline{\Omega})$).

Suppose that the Claim is not true. Then we can find a sequence $\{\lambda_n\}_{n\geqslant 1} \subseteq (0,1)$ such that $\lambda_n \searrow 0$ and

$$\|a_{\lambda_n}\|_\infty + c(\xi\|e\|_\infty)^{r-1} \geqslant \xi - \xi^{p-1}\|\Delta_p e\|_\infty \quad \forall n \geqslant 1, \xi > 0.$$

We let $n \to +\infty$. Using hypothesis $H_1(i)$ we obtain

$$c(\xi\|e\|_\infty)^{r-1} \geqslant \xi(1 - \xi^{p-2}\|\Delta_p e\|_\infty),$$

so

$$c\xi^{r-2}\|e\|_\infty^{r-1} \geqslant 1 - \xi^{p-2}\|\Delta_p e\|_\infty.$$

But recall that $2 < p, r$ and $\xi > 0$ is arbitrary. So, we let $\xi \searrow 0$ and we reach a contradiction. This proves the Claim.

Let $\overline{u}_\lambda = \xi_0^\lambda e \in \operatorname{int} C_+ \cap C^2(\overline{\Omega})$. For $\lambda \in (0, \lambda^*)$, we have

$$\begin{aligned} -\Delta_p \overline{u}_\lambda(z) - \Delta \overline{u}_\lambda(z) &= (\xi_0^\lambda)^{p-1}(-\Delta_p e(z)) + \xi_0^\lambda \\ &\geqslant f_\lambda(z, \overline{u}_\lambda(z)) \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{3.2}$$

(see hypothesis $H_1(i)$ and the Claim).

For $\lambda \in (0, \lambda^*)$ we introduce the following truncation of the reaction $f_\lambda(z, \cdot)$:

$$\widehat{f}_\lambda^+(z, \zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ f_\lambda(z, \zeta) & \text{if } 0 \leqslant \zeta \leqslant \overline{u}_\lambda(z), \\ f_\lambda(z, \overline{u}_\lambda(z)) & \text{if } \overline{u}_\lambda(z) < \zeta. \end{cases} \tag{3.3}$$

This is a Carathéodory function. We set

$$\widehat{F}_\lambda^+(z, \zeta) = \int_0^\zeta \widehat{f}_\lambda^+(z, s)\, ds$$

and consider the C^1-functional $\widehat{\varphi}_\lambda^+ : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{F}_\lambda^+(z, u)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently $\widehat{\varphi}_\lambda^+$ is coercive (see (3.3)) and by the Sobolev embedding theorem, we see that it is also sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda^+(u_\lambda) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\varphi}_\lambda^+(u). \tag{3.4}$$

We choose $t \in (0,1)$ small such that

$$t\widehat{u}_1(2)(z) \in [0, \delta_0^\lambda] \quad \forall z \in \overline{\Omega}, \ t\widehat{u}_1(2) \leqslant \overline{u}_\lambda \tag{3.5}$$

(recall that $\widehat{u}_1(2), \overline{u}_\lambda \in \operatorname{int} C_+$ and use Lemma 3.6 of Filippakis-Papageorgiou [10]). From (3.3) and (3.5) and hypothesis $H_1(ii)$, we have

$$\widehat{\varphi}_\lambda^+(t\widehat{u}_1(2)) \leqslant \frac{t^p}{p}\|D\widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\widehat{\lambda}_1(2) - \frac{t^{q_\lambda}}{q_\lambda}\int_\Omega \eta_\lambda \widehat{u}_1(2)^{q_\lambda}\,dz$$

(recall that $\|\widehat{u}_1(2)\|_2 = 1$). Since $1 < q_\lambda < 2 < p$ (see hypothesis $H_1(ii)$), by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\widehat{\varphi}_\lambda^+(t\widehat{u}_1(2)) < 0,$$

so

$$\widehat{\varphi}_\lambda^+(u_\lambda) < 0 = \widehat{\varphi}_\lambda^+(0)$$

(see (3.4)), hence $u_\lambda \neq 0$. From (3.4), we have

$$(\widehat{\varphi}_\lambda^+)'(u_\lambda) = 0,$$

so

$$A_p(u_\lambda) + A(u_\lambda) = N_{\widehat{f}_\lambda^+}(u_\lambda) \,\text{in}\, W^{-1,p'}(\Omega). \tag{3.6}$$

On (3.6) first we act with $-u_\lambda^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_\lambda^-\|_p^p + \|Du_\lambda^-\|_2^2 = 0$$

(see (3.3)), so $u_\lambda \geqslant 0$, $u_\lambda \neq 0$.

Also, on (3.6) we act with $(u_\lambda - \overline{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A_p(u_\lambda), (u_\lambda - \overline{u}_\lambda)^+\rangle + \langle A(u_\lambda), (u_\lambda - \overline{u}_\lambda)^+\rangle \\
&= \int_\Omega f(z, \overline{u}_\lambda)(u_\lambda - \overline{u}_\lambda)^+\,dz \\
&\leqslant \langle A_p(\overline{u}_\lambda), (u_\lambda - \overline{u}_\lambda)^+\rangle + \langle A(\overline{u}_\lambda), (u_\lambda - \overline{u}_\lambda)^+\rangle
\end{aligned}$$

(see (3.3) and (3.2)), thus

$$\langle A_p(u_\lambda) - A_p(\overline{u}_\lambda), (u_\lambda - \overline{u}_\lambda)^+\rangle + \|D(u_\lambda - \overline{u}_\lambda)^+\|_2^2 \leqslant 0$$

and hence $u_\lambda \leqslant \overline{u}_\lambda$.

So, we have proved that

$$u_\lambda \in [0, \overline{u}_\lambda], \tag{3.7}$$

where $[0, \overline{u}_\lambda] = \{u \in W_0^{1,p}(\Omega) : 0 \leqslant u(z) \leqslant \overline{u}_\lambda(z) \,\text{for a.a.}\, z \in \Omega\}$. On account of (3.7), equation (3.6) becomes

$$A_p(u_\lambda) + A(u_\lambda) = N_{f_\lambda}(u_\lambda)$$

(see (3.3)), so

$$\begin{cases} -\Delta_p u_\lambda(z) - \Delta u_\lambda(z) = f_\lambda(z, u_\lambda(z)) & \text{in}\, \Omega, \\ u_\lambda|_{\partial\Omega} = 0. \end{cases} \tag{3.8}$$

From Ladyzhenskaya-Uraltseva [23, p. 286], we have

$$u_\lambda \in L^\infty(\Omega).$$

So, we can apply Theorem 1 of Lieberman [24] and infer that

$$u_\lambda \in C_+\backslash\{0\}.$$

Hypotheses H_1 imply that we can find $\widehat{\xi}_\lambda > 0$ such that

$$f_\lambda(z,\zeta) + \widehat{\xi}_\lambda \zeta^{p-1} \geqslant 0 \quad \text{for a.a. } z \in \Omega, \text{all } 0 \leqslant \zeta \leqslant \|\overline{u}_\lambda\|_\infty.$$

Then from (3.7) and (3.8), it follows that

$$\Delta_p u_\lambda(z) + \Delta u_\lambda(z) \leqslant \widehat{\xi}_\lambda u_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \tag{3.9}$$

Let $a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$. Since $2 < p$, we have

$$a \in C^1(\mathbb{R}^N;\mathbb{R}^N),$$

and

$$\nabla a(y) = |y|^{p-2}\left(I + (p-2)\frac{y \otimes y}{|y|^2}\right) + I.$$

Thus

$$(\nabla a(y)\xi,\xi)_{\mathbb{R}^N} \geqslant |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^N.$$

Then from (3.9) and the tangency principle of Pucci–Serrin [33, p. 35], we have

$$u_\lambda(z) > 0 \quad \forall z \in \Omega.$$

Invoking the boundary point lemma of Pucci–Serrin [33, p. 120], we conclude that

$$u_\lambda \in \operatorname{int} C_+.$$

In a similar fashion, using this time $\overline{v}_\lambda = -\overline{u}_\lambda \in (-\operatorname{int} C_+) \cap C^2(\overline{\Omega})$, we produce a negative solution $v_\lambda \in -\operatorname{int} C_+$. □

Let S_λ^+ (respectively S_λ^-) be the set of positive (respectively negative) solutions of problem (P_λ). From Proposition 3.3 and its proof, we have that

$$\emptyset \neq S_\lambda^+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_\lambda^- \subseteq -\operatorname{int} C_+ \quad \forall \lambda \in (0,\lambda^*).$$

Moreover, as in Filippakis–Papageorgiou [10, Lemmata 4.1 and 4.2], we show that

- S_λ^+ is downward directed (that is, if $u_1, u_2 \in S_\lambda^+$, then we can find $u \in S_\lambda^+$ such that $u \leqslant u_1$, $u \leqslant u_2$).
- S_λ^- is upward directed (that is, if $v_1, v_2 \in S_\lambda^-$, then we can find $v \in S_\lambda^-$ such that $v_1 \leqslant v$, $v_2 \leqslant v$).

Next we show that the set S_λ^+ admits a minimal element (that is, we can find a smallest positive solution for problem (P_λ), $\lambda \in (0,\lambda^*)$) and the set S_λ^- admits a maximal element (that is, we can find a biggest negative solution for problem (P_λ), $\lambda \in (0,\lambda^*)$). These solutions are know as "extremal" constant sign solutions.

Proposition 3.4. *If hypotheses* H_1 *hold and* $\lambda \in (0, \lambda^*)$, *then problem* (P_λ) *has a smallest positive solution* $u_\lambda^* \in \operatorname{int} C_+$ *and a biggest negative solution* $v_\lambda^* \in -\operatorname{int} C_+$.

Proof. Since S_λ^+ is downward directed, we can restrict ourselves to the set

$$\widehat{S}_\lambda^+ = S_\lambda^+ \cap [0, \overline{u}_\lambda] \neq \emptyset \tag{3.10}$$

(see the proof of Proposition 3.3). Hypotheses H_1 imply that

$$f_\lambda(z,\zeta)\zeta \geqslant \eta_\lambda |\zeta|^{q_\lambda} - c_1|\zeta|^r \quad \text{for a.a.}\, z \in \Omega, \text{all}\, \zeta \in \mathbb{R}, \tag{3.11}$$

with $c_1 = c_1(\lambda) > 0$. We introduce the following Carathéodory function

$$k_\lambda^+(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ \eta_\lambda \zeta^{q_\lambda - 1} - c_1 \zeta^{r-1} & \text{if } 0 \leqslant \zeta \leqslant \overline{u}_\lambda(z), \\ \eta_\lambda \overline{u}_\lambda^{q_\lambda - 1}(z) - c_1 \overline{u}_\lambda(z)^{r-1} & \text{if } \overline{u}_\lambda(z) < \zeta \end{cases} \tag{3.12}$$

and consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = k_\lambda^+(z, u(z)) & \text{in}\, \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{Au_λ}$$

We solve problem (Au_λ). To this end, let $\psi_\lambda^+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\psi_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega K_\lambda^+(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega),$$

where

$$K_\lambda^+(z,\zeta) = \int_0^\zeta k_\lambda^+(z,s)\,ds.$$

From (3.12) it is clear that ψ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda^+(\widetilde{u}_\lambda) = \inf_{u \in W_0^{1,p}(\Omega)} \psi_\lambda^+(u). \tag{3.13}$$

As in the proof of Proposition 3.3, since $1 < q_\lambda < 2 < p, r$ for $t \in (0,1]$ small (at least such that $t\widehat{u}_1(2)(z) \in [0, \delta_0^\lambda]$ for all $z \in \overline{\Omega}$), we have

$$\psi_\lambda^+(t\widehat{u}_1(2)) < 0,$$

so

$$\psi_\lambda^+(\widetilde{u}_\lambda) < 0 = \psi_\lambda^+(0)$$

(see (3.13)), hence $\widetilde{u}_\lambda \neq 0$.

From (3.13), we have

$$(\psi_\lambda^+)'(\widetilde{u}_\lambda) = 0,$$

so

$$A_p(\widetilde{u}_\lambda) + A(\widetilde{u}_\lambda) = N_{k_\lambda^+}(\widetilde{u}_\lambda). \tag{3.14}$$

On (3.14) we act with $-\widetilde{u}_\lambda^- \in W_0^{1,p}(\Omega)$ and using (3.12), we obtain that $\widetilde{u}_\lambda \geqslant 0$, $\widetilde{u}_\lambda \neq 0$. Also, on (3.14) we act with $(\widetilde{u}_\lambda - \overline{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A_p(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - \overline{u}_\lambda)^+\rangle + \langle A(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - \overline{u}_\lambda)^+\rangle \\
&\quad = \int_\Omega (\eta_\lambda \overline{u}_\lambda^{q_\lambda - 1} - c_1 \overline{u}_\lambda^{r-1})(\widetilde{u}_\lambda - \overline{u}_\lambda)^+ \, dz \\
&\quad \leqslant \int_\Omega f(z, \overline{u}_\lambda)(\widetilde{u}_\lambda - \overline{u}_\lambda)^+ \, dz \\
&\quad \leqslant \langle A_p(\overline{u}_\lambda), (\widetilde{u}_\lambda - \overline{u}_\lambda)^+\rangle + \langle A(\overline{u}_\lambda), (\widetilde{u}_\lambda - \overline{u}_\lambda)^+\rangle
\end{aligned}$$

(see (3.12), (3.11) and (3.2) in the proof of Proposition 3.3), so

$$\langle A_p(\widetilde{u}_\lambda) - A_p(\overline{u}_\lambda), (\widetilde{u}_\lambda - \overline{u}_\lambda)^+\rangle + \|D(\widetilde{u}_\lambda - \overline{u}_\lambda)^+\|_2^2 \leqslant 0,$$

thus $\widetilde{u}_\lambda \leqslant \overline{u}_\lambda$. So, we have proved that

$$\widetilde{u}_\lambda \text{ solves } (Au_\lambda) \quad \text{and} \quad \widetilde{u}_\lambda \in [0, \overline{u}_\lambda] \setminus \{0\}. \tag{3.15}$$

As before (see the proof of Proposition 3.3), using the nonlinear regularity theory (see Lieberman [24, Theorem 1]) and the nonlinear maximum principle (see Pucci–Serrin [33, pp. 35, 120]), we have

$$\widetilde{u}_\lambda \in \operatorname{int} C_+.$$

Next we show that this is the unique positive solution of (Au_λ). To this end, we consider the integral functional $j\colon L^1(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|Du^{\frac{1}{2}}\|_p^p + \frac{1}{2}\|Du^{\frac{1}{2}}\|_2^2 & \text{if } u \geqslant 0, u^{\frac{1}{2}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We set

$$G_0(t) = \frac{1}{p}t^p + \frac{1}{2}t^2 \quad \forall t \geqslant 0.$$

Then the map $t \longmapsto G_0(t^{\frac{1}{2}})$ is convex on $\mathbb{R}_+ = [0, +\infty)$. If

$$G(y) = G_0(|y|) \quad \forall y \in \mathbb{R}^N,$$

then for all $u \in \operatorname{dom} j = \{y \in L^1(\Omega) : j(y) < +\infty\}$ (the effective domain of j), we have $j(u) = G(\nabla u^{\frac{1}{2}})$. Using Lemma 1 of Diaz–Saa [8], we see that j is convex (see also Gasiński–Papageorgiou [18, proof of Proposition 7]).

Suppose that $\widehat{u}_\lambda \in W_0^{1,p}(\Omega)$ is another positive solution of (Au_λ). Again we have

$$\widehat{u}_\lambda \in [0, \overline{u}_\lambda] \cap \operatorname{int} C_+. \tag{3.16}$$

Given $h \in C_0^1(\overline{\Omega})$, for $|t| < 1$ small, we have

$$\widetilde{u}_\lambda^2 + th \in \operatorname{dom} j \quad \text{and} \quad \widehat{u}_\lambda^2 + th \in \operatorname{dom} j.$$

It is easy to see that j is Gâteaux differentiable at $\widetilde{u}_\lambda^2$ and at $\widehat{u}_\lambda^2$ in the direction of h and using the chain rule, we have

$$j'(\widetilde{u}_\lambda^2)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \widetilde{u}_\lambda - \Delta \widetilde{u}_\lambda}{\widetilde{u}_\lambda} h\, dz \quad \forall h \in C_0^1(\overline{\Omega}),$$
$$j'(\widehat{u}_\lambda^2)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \widehat{u}_\lambda - \Delta \widehat{u}_\lambda}{\widehat{u}_\lambda} h\, dz \quad \forall h \in C_0^1(\overline{\Omega}).$$

The convexity of j implies the monotonicity of j'. Hence

$$\begin{aligned} 0 &\leqslant \int_\Omega \left(\frac{1}{2}\int_\Omega \frac{-\Delta_p \widetilde{u}_\lambda - \Delta \widetilde{u}_\lambda}{\widetilde{u}_\lambda} - \frac{1}{2}\int_\Omega \frac{-\Delta_p \widehat{u}_\lambda - \Delta \widehat{u}_\lambda}{\widehat{u}_\lambda}\right)(\widetilde{u}_\lambda^2 - \widehat{u}_\lambda^2)\, dz \\ &= \int_\Omega \left(\eta_\lambda\left(\frac{1}{\widetilde{u}_\lambda^{2-q_\lambda}} - \frac{1}{\widehat{u}_\lambda^{2-q_\lambda}}\right) - c_1(\widetilde{u}_\lambda^{r-2} - \widehat{u}_\lambda^{r-2})\right)(\widetilde{u}_\lambda^2 - \widehat{u}_\lambda^2)\, dz \end{aligned}$$

(see (3.14), (3.12) and (3.16)), so $\widetilde{u}_\lambda = \widehat{u}_\lambda$ (recall that $1 < q_\lambda < 2 < r$).

Therefore $\widetilde{u}_\lambda \in [0, \overline{u}_\lambda] \cap \operatorname{int} C_+$ is the unique solution of (Au_λ).

Claim. $\widetilde{u}_\lambda \leqslant u$ for all $u \in \widehat{S}_\lambda^+$ (see (3.10)).

Let $u \in \widehat{S}_\lambda^+$ and consider the Carathéodory function $g_\lambda^+ \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g_\lambda^+(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ k_\lambda^+(z,\zeta) & \text{if } 0 \leqslant \zeta \leqslant u(z), \\ k_\lambda^+(z,u(z)) & \text{if } u(z) < \zeta. \end{cases} \tag{3.17}$$

We set

$$G_\lambda^+(z,\zeta) = \int_0^\zeta g_\lambda^+(z,s)\, ds$$

and consider the C^1-functional $\mu_\lambda^+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\mu_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_\lambda^+(z,u)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

The functional μ_λ^+ is coercive (see (3.17)) and sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda^* \in W_0^{1,p}(\Omega)$ such that

$$\mu_\lambda^+(\widetilde{u}_\lambda^*) = \inf_{u \in W_0^{1,p}(\Omega)} \mu_\lambda^+(u) < 0 = \mu_\lambda^+(0) \tag{3.18}$$

(see the proof of Proposition 3.3), so

$$\widetilde{u}_\lambda^* \neq 0.$$

From (3.18), we have

$$(\mu_\lambda^+)'(\widetilde{u}_\lambda^*) = 0,$$

so

$$A_p(\widetilde{u}_\lambda^*) + A(\widetilde{u}_\lambda^*) = N_{g_\lambda^+}(\widetilde{u}_\lambda^*). \tag{3.19}$$

On (3.19) we act with $-(\widetilde{u}_\lambda^*)^- \in W_0^{1,p}(\Omega)$ and obtain $\widetilde{u}_\lambda^* \geqslant 0$, $\widetilde{u}_\lambda^* \neq 0$. Then we act with $(\widetilde{u}_\lambda^* - u)^+ \in W_0^{1,p}(\Omega)$ and have

$$\begin{aligned}
&\langle A_p(\widetilde{u}_\lambda^*) + A(\widetilde{u}_\lambda^*), (\widetilde{u}_\lambda^* - u)^+ \rangle \\
&= \int_\Omega k_\lambda^+(z,u)(\widetilde{u}_\lambda^* - u)^+ \, dz \\
&= \int_\Omega (\eta_\lambda u^{q_\lambda - 1} - c_1 u^{r-1})(\widetilde{u}_\lambda^* - u)^+ \, dz \\
&\leqslant \int_\Omega f_\lambda(z,u)(\widetilde{u}_\lambda^* - u)^+ \, dz \\
&= \langle A_p(u) + A(u), (\widetilde{u}_\lambda^* - u)^+ \rangle
\end{aligned}$$

(see (3.17), (3.11) and use $u \in \widehat{S}_\lambda^+$), so

$$\langle A_p(\widetilde{u}_\lambda^*) - A_p(u), (\widetilde{u}_\lambda^* - u)^+ \rangle + \|D(\widetilde{u}_\lambda^* - u)^+\|_2^2 \leqslant 0,$$

thus

$$\widetilde{u}_\lambda^* \leqslant u.$$

Therefore, we have proved that

$$\widetilde{u}_\lambda^* \in [0, u] \quad \text{and} \quad \widetilde{u}_\lambda^* \neq 0,$$

so

$$\widetilde{u}_\lambda^* \text{ is a positive solution of } (Au_\lambda)$$

(see (3.12) and (3.17)), thus

$$\widetilde{u}_\lambda^* = \widetilde{u}_\lambda$$

(problem (Au_λ) has a unique positive solution), hence

$$\widetilde{u}_\lambda \leqslant u \quad \forall u \in \widehat{S}_\lambda^+.$$

This proves the Claim.

Invoking Lemma 3.10 of Hu–Papageorgiou [22, p. 178], we can find a decreasing sequence $\{u_n\}_{n \geqslant 1} \subseteq \widehat{S}_\lambda^+$ such that

$$\inf \widehat{S}_\lambda^+ = \inf_{n \geqslant 1} u_n.$$

Evidently the sequence $\{u_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u_\lambda^* \text{ in } L^p(\Omega). \tag{3.20}$$

We have

$$A_p(u_n) + A(u_n) = N_{f_\lambda}(u_n) \quad \forall n \geqslant 1. \tag{3.21}$$

On (3.21) we act with $u_n - u_\lambda^* \in W_0^{1,p}(\Omega)$.

Note that the sequence $\{N_{f_\lambda}(u_n)\}_{n\geqslant 1} \subseteq L^{p'}(\Omega)$ is bounded (see hypothesis $H_1(i)$ and recall that $u_n \leqslant \overline{u}_\lambda$ for all $n \geqslant 1$). So, we have

$$\lim_{n\to+\infty} \big(\langle A_p(u_n), u_n - u_\lambda^*\rangle + \langle A(u_n), u_n - u_\lambda^*\rangle\big) = 0,$$

so

$$\limsup_{n\to+\infty} \big(\langle A_p(u_n), u_n - u_\lambda^*\rangle + \langle A(u_\lambda^*), u_n - u_\lambda^*\rangle\big) \leqslant 0$$

(from the monotonicity of A), thus

$$\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u_\lambda^*\rangle \leqslant 0$$

(see (3.20)) and hence

$$u_n \longrightarrow u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \tag{3.22}$$

(see Proposition 2.3).

So, if in (3.21) we pass to the limit as $n \to +\infty$ and use (3.22), then

$$A_p(u_\lambda^*) + A(u_\lambda^*) = N_{f_\lambda}(u_\lambda^*). \tag{3.23}$$

From the Claim we know that

$$\widetilde{u}_\lambda \leqslant u_n \quad \forall n \geqslant 1,$$

so

$$\widetilde{u}_\lambda \leqslant u_\lambda^*$$

(see (3.22)), thus

$$u_\lambda^* \in \widehat{S}_\lambda^+ \quad \text{and} \quad u_\lambda^* = \inf \widehat{S}_\lambda^+$$

(see (3.23)). Similarly, working on the negative semiaxis with $\widetilde{v}_\lambda = -\widetilde{u}_\lambda \in -\operatorname{int} C_+$ and recalling that S_λ^- is upward directed (this allows us to focus on $\widehat{S}_\lambda^- = S_\lambda^- \cap [\overline{v}_\lambda, 0]$), we produce

$$v_\lambda^* \in S_\lambda^- \quad \text{with } v_\lambda^* = \sup S_\lambda^-.$$

$\square$

Using these extremal constant sign solutions, we can produce a nodal (that is, sign changing) solution. Indeed, we consider the order interval

$$[v_\lambda^*, u_\lambda^*] = \{y \in W_0^{1,p}(\Omega) : \ v_\lambda^*(z) \leqslant y(z) \leqslant u_\lambda^* \quad \text{for a.a. } z \in \Omega\}$$

and look for a nontrivial solution $u \in [v_\lambda^*, u_\lambda^*]$, $u \neq v_\lambda^*$, $u \neq u_\lambda^*$. The extremality of u_λ^* and v_λ^* implies that u must be nodal.

To implement this strategy, we introduce the Carathéodory function $d_\lambda \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$d_\lambda(z,\zeta) = \begin{cases} f_\lambda(z, v_\lambda^*(z)) & \text{if } \zeta < v_\lambda^*(z), \\ f_\lambda(z,\zeta) & \text{if } v_\lambda^*(z) \leqslant \zeta \leqslant u_\lambda^*(z), \\ f_\lambda(z, u_\lambda^*(z)) & \text{if } u_\lambda^* < \zeta. \end{cases} \tag{3.24}$$

We set

$$D_\lambda(z,\zeta) = \int_0^\zeta d_\lambda(z,s)\,ds$$

and consider the functional $\gamma_\lambda\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\gamma_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega D_\lambda(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently $\gamma_\lambda \in C^1(W_0^{1,p}(\Omega))$. We also consider the positive and negative truncations of $d_\lambda(z,\cdot)$, that is the Carathéodory functions

$$d_\lambda^\pm(z,\zeta) = d_\lambda(z,\pm\zeta^\pm) \quad \forall (z,\zeta) \in \Omega \times \mathbb{R}.$$

We set

$$D_\lambda^\pm(z,\zeta) = \int_0^\zeta d_\lambda^\pm(z,s)\,ds$$

and consider the C^1-functionals $\gamma_\lambda^\pm\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\gamma_\lambda^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega D_\lambda^\pm(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Proposition 3.5. *If hypotheses H_1 hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) admits a nodal solution $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$.*

Proof. As before (see the proof of Proposition 3.4), using (3.24), we show that

$$\begin{aligned} K_{\gamma_\lambda} &\subseteq [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}),\\ K_{\gamma_\lambda^+} &\subseteq [0, u_\lambda^*] \cap C_+,\\ K_{\gamma_\lambda^-} &\subseteq [v_\lambda^*, 0] \cap (-C_+). \end{aligned}$$

The extremality of $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ (see Proposition 3.4), imply that

$$K_{\gamma_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}), \quad K_{\gamma_\lambda^+} = \{0, u_\lambda^*\}, \quad K_{\gamma_\lambda^-} = \{0, v_\lambda^*\}. \tag{3.25}$$

Claim. $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ are local minimizers of γ_λ.

From (3.24) we see that γ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\lambda^* \in W_0^{1,p}(\Omega)$ such that

$$\gamma_\lambda^+(\widehat{u}_\lambda^*) = \inf_{u \in W_0^{1,p}(\Omega)} \gamma_\lambda^+(u) < 0 = \gamma_\lambda^+(0),$$

so

$$\widehat{u}_\lambda^* \neq 0 \quad \text{and} \quad \widehat{u}_\lambda^* \in K_{\gamma_\lambda^+},$$

thus

$$\widehat{u}_\lambda^* = u_\lambda^*$$

(see (3.25)). We know that $u_\lambda^* \in \operatorname{int} C_+$ (see Proposition 3.4) and $\gamma_\lambda|_{C_+} = \gamma_\lambda^+|_{C_+}$. It follows that

$$u_\lambda^* \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \gamma_\lambda,$$

thus

$$u_\lambda^* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \gamma_\lambda$$

(see Proposition 2.2).

Similarly for $v_\lambda^* \in -\operatorname{int} C_+$, using this time the functional γ_λ^-. This proves the Claim.

We may assume that $\gamma_\lambda(v_\lambda^*) \leqslant \gamma_\lambda(u_\lambda^*)$. The reasoning is similar if the opposite inequality holds. Also, we assume that K_{γ_λ} is finite. Otherwise, by (3.25), we see that already we have an infinity of distinct nodal solutions and by the nonlinear regularity theory (see Lieberman [24, Theorem 1]) they belong in $C_0^1(\overline{\Omega})$. So, we are done. Then on account of the Claim, we can find $\varrho \in (0,1)$ small such that

$$\begin{cases} \gamma_\lambda(v_\lambda^*) \leqslant \gamma_\lambda(u_\lambda^*) < \inf\{\gamma_\lambda(u) : \|u - u_\lambda^*\| = \varrho\} = m_\lambda^\varrho, \\ \|v_\lambda^* - u_\lambda^*\| > \varrho \end{cases} \tag{3.26}$$

(see Gasiński–Papageorgiou [13, proof of Theorem 2.12]). Also, γ_λ is coercive (see (3.24)). It follows that

$$\gamma_\lambda \text{ satisfies the Cerami condition} \tag{3.27}$$

(see Papageorgiou–Winkert [32]). Then (3.26) and (3.27) permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $y_\lambda \in W_0^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\gamma_\lambda} \quad \text{and} \quad m_\lambda^\varrho \leqslant \gamma_\lambda(y_\lambda). \tag{3.28}$$

From Motreanu–Motreanu–Papageorgiou [26, p. 168], we have

$$C_1(\gamma_\lambda, y_\lambda) \neq 0, \tag{3.29}$$

while from (3.26) and (3.28) we infer that

$$y_\lambda \notin \{v_\lambda^*, u_\lambda^*\}. \tag{3.30}$$

Hypothesis $H_1(ii)$ and Proposition 4.1 of Gasiński–Papageorgiou [18], imply that

$$C_k(\gamma_\lambda, 0) = 0 \quad \forall k \geqslant 0. \tag{3.31}$$

From (3.29) and (3.31) it follows that $y_\lambda \neq 0$. This fact together with (3.25) and (3.30) lead to the conclusion that

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}) \text{ is a nodal solution of } (P_\lambda).$$

$\square$

Now we can state our first multiplicity theorem. We stress that in this result the reaction term $f_\lambda(z,\cdot)$ can have arbitrary polynomial growth, in particular it can be supercritical. Our multiplicity theorem provides sign information for all solutions produced.

Theorem 3.6. *If hypotheses H_1 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0,\lambda^*)$ problem (P_λ) admits at least three nontrivial smooth solutions*

$$u_\lambda \in \operatorname{int} C_+, \quad v_\lambda \in -\operatorname{int} C_+ \quad \textit{and} \quad y_\lambda \in [v_\lambda, u_\lambda] \cap C_0^1(\overline{\Omega}) \textit{nodal.}$$

4. Six Nontrivial Smooth Solutions

In this section we study problem (P_λ) when the reaction term $f_\lambda(z,\cdot)$ is $(p-1)$-linear at $\pm\infty$ and resonant with respect to a nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. This is another distinguishing feature from the work of Papageorgiou–Winkert [32]. There asymptotically at $\pm\infty$ we have resonance with respect to $\widehat{\lambda}_1(p) > 0$ from the left and so the corresponding energy (Euler) functional is coercive. In contrast here the energy functional is indefinite and this is the source of additional difficulties, which lead to different techniques.

Now the reaction f_λ has the form

$$f_\lambda(z,\zeta) = \lambda|\zeta|^{q_\lambda-2}\zeta + f_0(z,\zeta) \quad \forall (z,\zeta) \in \Omega \times \mathbb{R}, \tag{4.1}$$

with $1 < q_\lambda < 2$.

On the perturbation f_0 we impose the following conditions.

$\underline{H_2}$: $f_0\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function, such that for almost all $z \in \Omega$, $f_0(z,0) = 0$ and $f_0(z,\cdot) \in C^1(\mathbb{R})$, $f_0(z,\cdot)$ is nondecreasing on $\mathbb{R}$ and

(i) $|(f_0)'_\zeta(z,\zeta)| \leqslant a(z)(1+|\zeta|^{r-2})$ for almost all $z \in \Omega$, all $\zeta \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $2 < r < p^*$;

(ii) there exists integer $m \geqslant 2$ such that

$$\lim_{\zeta\to\pm\infty} \frac{f_0(z,\zeta)}{|\zeta|^{p-2}\zeta} = \widehat{\lambda}_m(p) \text{ uniformly for a.a. } z \in \Omega;$$

(iii) if $F_0(z,\zeta) = \int_0^\zeta f_0(z,s)\,ds$, then there exists $\tau_\lambda > q_\lambda$ such that

$$\lim_{\zeta\to\pm\infty} \frac{f_0(z,\zeta)\zeta - pF_0(z,\zeta)}{|\zeta|^{\tau_\lambda}} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iv) we have

$$\lim_{\zeta\to 0} \frac{f_0(z,\zeta)}{|\zeta|^{p-2}\zeta} = 0 \text{ uniformly for a.a. } z \in \Omega.$$

Remark 4.1. From (4.1) and hypothesis $H_2(i)$, we have that

$$|f_\lambda(z,\zeta)| \leqslant \widehat{a}_\lambda(z) + \widehat{c}|\zeta|^{r-1} \quad \text{for a.a. } z \in \Omega, \text{all } \zeta \in \mathbb{R}, \text{all } \lambda \in (0,\lambda_0),$$

with $\widehat{a}_\lambda \in L^\infty(\Omega)$, $\widehat{c} > 0$ and $\|\widehat{a}_\lambda\|_\infty \longrightarrow 0$ as $\lambda \searrow 0$. So, we preserve the framework of Sect. 3. However, we have added two new conditions concerning the behaviour of $f_\lambda(z, \cdot)$ near $\pm\infty$ (hypotheses $H_2(ii), (iii)$). Hypothesis $H_2(ii)$ makes the problem resonant. The resonance is with respect to any nonprincipal variational eigenvalue $\widehat{\lambda}_m(p) > 0$ (recall $m \geqslant 2$).

Example 4.2. The following functions are of the form (4.1) and satisfy hypotheses H_2. For the sake of simplicity we drop the z-dependence.

$$f^1_\lambda(\zeta) = \lambda|\zeta|^{q_\lambda-2}\zeta + \widehat{\lambda}_m(p)|\zeta|^{p-2}\zeta - c_1|\zeta|^{\tau_\lambda-2}\zeta,$$

$$f^2_\lambda(\zeta) = \lambda|\zeta|^{q_\lambda-2}\zeta + \begin{cases} c_2\zeta & \text{if } |\zeta| \leqslant 1, \\ \widehat{\lambda}_m(p)|\zeta|^{p-2}\zeta - \eta|\zeta|^{\tau_\lambda-2}\zeta & \text{if } |\zeta| > 1, \end{cases}$$

with $1 < q_\lambda < 2 < \tau_\lambda < p$, $c_1 < \frac{\lambda(q_\lambda-1)}{\lambda_0}$, $\eta = \frac{\widehat{\lambda}_m(p)(p-2)}{\tau_\lambda-2}$, $c_2 = \widehat{\lambda}_m(p) - \eta$.

First we produce two additional constant sign smooth solutions.

According to Theorem 3.6, for all $\lambda \in (0, \lambda^*)$, we already have three nontrivial smooth solutions

$$u_\lambda \in \operatorname{int} C_+, \quad v_\lambda \in -\operatorname{int} C_+ \quad \text{and} \quad y_\lambda \in [v_\lambda, u_\lambda] \cap C^1_0(\overline{\Omega}) \text{ nodal}.$$

On account of Proposition 3.4, without any loss of generality we may assume that u_λ and v_λ are extremal (that is, $u_\lambda = u^*_\lambda \in \operatorname{int} C_+$ and $v_\lambda = v^*_\lambda \in -\operatorname{int} C_+$).

Proposition 4.3. *If hypotheses H_2 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has two more constant sign smooth solutions*

$$\widehat{u}_\lambda \in \operatorname{int} C_+ \quad \textit{with } \widehat{u}_\lambda - u_\lambda \in \operatorname{int} C_+,$$
$$\widehat{v}_\lambda \in -\operatorname{int} C_+ \quad \textit{with } v_\lambda - \widehat{v}_\lambda \in \operatorname{int} C_+$$

Proof. Using the solution $u_\lambda \in \operatorname{int} C_+$, we introduce the following truncation of $f_\lambda(z, \cdot)$

$$g^+_\lambda(z, \zeta) = \begin{cases} f_\lambda(z, u_\lambda(z)) & \text{if } \zeta \leqslant u_\lambda(z), \\ f_\lambda(z, \zeta) & \text{if } u_\lambda(z) < \zeta. \end{cases} \tag{4.2}$$

This is a Carathéodory function. We set

$$G^+_\lambda(z, \zeta) = \int_0^\zeta g^+_\lambda(z, s)\, ds$$

and consider the C^1-functional $\chi^+_\lambda : W^{1,p}_0(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\chi^+_\lambda(u) = \frac{1}{p}\|Du\|^p_p + \frac{1}{2}\|Du\|^2_2 - \int_\Omega G^+_\lambda(z, u)\, dz \quad \forall u \in W^{1,p}_0(\Omega).$$

Claim 1. $K_{\chi^+_\lambda} \subseteq [u_\lambda) \cap C^1_0(\overline{\Omega})$, where

$$[u_\lambda) = \{u \in W^{1,p}_0(\Omega) : u_\lambda(z) \leqslant u(z) \quad \text{for a.a. } z \in \Omega\}.$$

Let $u \in K_{\chi^+_\lambda}$. Then

$$(\chi^+_\lambda)'(u) = 0,$$

so

$$A_p(u) + A(u) = N_{g_\lambda^+}(u). \tag{4.3}$$

On (4.3) we act with $(u_\lambda - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A_p(u), (u_\lambda - u)^+\rangle + \langle A(u), (u_\lambda - u)^+\rangle \\
&= \int_\Omega f_\lambda(z, u_\lambda)(u_\lambda - u)^+ \, dz \\
&= \langle A_p(u_\lambda), (u_\lambda - u)^+\rangle + \langle A(u_\lambda), (u_\lambda - u)^+\rangle
\end{aligned}$$

(see (4.2) and since $u_\lambda \in S_\lambda^+$, so

$$\langle A_p(u) - A_p(u_\lambda), (u_\lambda - u)^+\rangle + \|D(u_\lambda - u)^+\|_2^2 = 0,$$

thus

$$u_\lambda \leqslant u.$$

Also, the nonlinear regularity theory implies that $u \in C_0^1(\overline{\Omega})$. This proves Claim 1.

Recall that $u_\lambda \leqslant \overline{u}_\lambda$ (see the proof of Proposition 3.3). We may assume that

$$K_{\chi_\lambda^+} \cap [0, \overline{u}_\lambda] = \{u_\lambda\}. \tag{4.4}$$

Otherwise, on account of Claim 1, we already have a second positive solution $\widehat{u}_\lambda \in C_0^1(\overline{\Omega})$ with $u_\lambda \leqslant \widehat{u}_\lambda$. Moreover, as we will show below we have $\widehat{u}_\lambda - u_\lambda \in \operatorname{int} C_+$. Therefore we are done.

Claim 2. $u_\lambda \in \operatorname{int} C_+$ is a local minimizer of χ_λ^+.

We consider the following truncation of $g_\lambda^+(z, \cdot)$:

$$\widehat{g}_\lambda^+(z, \zeta) = \begin{cases} g_\lambda^+(z, \zeta) & \text{if } \zeta \leqslant \overline{u}_\lambda(z), \\ g_\lambda^+(z, \overline{u}_\lambda(z)) & \text{if } \overline{u}_\lambda(z) < \zeta. \end{cases} \tag{4.5}$$

This is a Carathéodory function. We set

$$\widehat{G}_\lambda^+(z, \zeta) = \int_0^\zeta \widehat{g}_\lambda^+(z, s) \, ds$$

and consider the C^1-functional $\widehat{\chi}_\lambda^+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\chi}_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{G}_\lambda^+(z, u) \, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

This functional is coercive (see (4.5)) and sequentially weakly lower semicontinuous. So, there is $w_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\chi}_\lambda^+(w_\lambda) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\chi}_\lambda^+(u), \tag{4.6}$$

thus

$$w_\lambda \in K_{\widehat{\chi}_\lambda^+}. \tag{4.7}$$

As above (see the proof of Claim 1), we can show that

$$K_{\widehat{\chi}_\lambda^+} \subseteq [u_\lambda, \overline{u}_\lambda] \cap C_0^1(\overline{\Omega}). \tag{4.8}$$

From (4.5) it follows that

$$K_{\chi_\lambda^+} \cap [u_\lambda, \overline{u}_\lambda] = K_{\widehat{\chi}_\lambda^+}. \tag{4.9}$$

Then (4.4), (4.7), (4.8), (4.9) imply that

$$w_\lambda = u_\lambda. \tag{4.10}$$

Let

$$B(y) = \frac{1}{p}|y|^p + \frac{1}{2}|y|^2 \quad \forall y \in \mathbb{R}^N.$$

Then $B \in C^2(\mathbb{R}^N; \mathbb{R})$ and

$$a(y) = \nabla B(y) = |y|^{p-2}y + y \quad \forall y \in \mathbb{R}^N,$$

so

$$\nabla a(y) = |y|^{p-2}\left(I + (p-2)\frac{y \otimes y}{|y|^2}\right) + I \quad \forall y \in \mathbb{R}^N.$$

Evidently

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \forall u \in W_0^{1,p}(\Omega)$$

and

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^N.$$

Since $u_\lambda \neq \overline{u}_\lambda$, the tangency principle of Pucci–Serrin [33, p. 35] implies that

$$(\overline{u}_\lambda - u_\lambda)(z) > 0 \quad \forall z \in \Omega. \tag{4.11}$$

We have

$$\begin{aligned} -\Delta_p \overline{u}_\lambda - \Delta \overline{u}_\lambda &\geqslant \lambda \overline{u}_\lambda^{q_\lambda - 1} + f_0(z, \overline{u}_\lambda) \\ &\geqslant \lambda u_\lambda^{q_\lambda - 1} + f_0(z, u_\lambda) = -\Delta_p u_\lambda - \Delta u_\lambda \quad \text{for a.a. } z \in \Omega \end{aligned}$$

(see (3.2), (4.1) and use the facts that $\overline{u}_\lambda \geqslant u_\lambda$, $f_0(z, \cdot)$ is nondecreasing and $u_\lambda \in S_\lambda^+$).

Let $\widehat{e}_\lambda = \overline{u}_\lambda - u_\lambda \in C_+ \setminus \{0\}$. From Guedda–Véron [21] (see also Gasiński–Papageorgiou [16, Lemma 2.9]), we know that $\widehat{e}_\lambda$ satisfies

$$L\widehat{e}_\lambda(z) \geqslant 0 \quad \text{for a.a. } z \in \Omega, \quad \widehat{e}_\lambda|_{\partial\Omega} = 0, \tag{4.12}$$

with $L(y) = -\operatorname{div}(H(z)Dy)$ for all $y \in W_0^{1,p}(\Omega)$, where the coefficient matrix $H(z) = (h_{ij}(z))_{i,j=1}^N$ is defined by

$$h_{ij}(z) = |D\overline{u}_\lambda(z)|^{p-2}\big(\delta_{ij}|D\overline{u}_\lambda(z)|^{p-2} + (p-2)D_i\overline{u}_\lambda(z)D_j\overline{u}_\lambda(z)\big) + 1.$$

Recalling that $\overline{u}_\lambda \in \operatorname{int} C_+$, we see that the linear differential operator L is strictly elliptic. So, from (4.11), (4.12) and the strong maximum principle (see, for example Gasiński–Papageorgiou [12, p. 738]), we have

$$\frac{\partial \widehat{e}_\lambda}{\partial n}\Big|_{\partial\Omega} < 0,$$

so

$$\widehat{e}_\lambda = \overline{u}_\lambda - u_\lambda \in \ \operatorname{int} C_+ \tag{4.13}$$

(see (4.11)). From (4.13), (4.6), (4.10) and since $\chi_\lambda^+|_{[0,\overline{u}_\lambda]} = \widehat{\chi}_\lambda^+|_{[0,\overline{u}_\lambda]}$ (see (4.5) and (4.6)), it follows that

$$u_\lambda \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizers of } \chi_\lambda^+,$$

so

$$u_\lambda \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizers of } \chi_\lambda^+$$

(see Proposition 2.2). This proves Claim 2.

Due to Claim 1, we may assume that $K_{\chi_\lambda^+}$ is finite. Then Claim 2 implies that we can find $\varrho \in (0,1)$ small such that

$$\chi_\lambda^+(u_\lambda) < \inf\left\{\chi_\lambda^+(u): \ \|u-u_\lambda\| = \varrho\right\} = m_\lambda^+. \tag{4.14}$$

Hypothesis $H_2(ii)$ and the fact that $m \geqslant 2$, imply that

$$\chi_\lambda^+(t\widehat{u}_1(p)) \longrightarrow -\infty \text{ as } t \to +\infty. \tag{4.15}$$

Claim 3. The functional χ_λ^+ satisfies the Cerami condition.

Consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that $\{\chi_\lambda^+(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1+\|u_n\|)(\chi_\lambda^+)'(u_n) \longrightarrow 0 \text{ in } W^{-1,p'}(\Omega). \tag{4.16}$$

From (4.16) we have

$$\begin{aligned} &\Big|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega g_\lambda^+(z,u_n)h\,dz\Big| \\ &\quad \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega), \end{aligned} \tag{4.17}$$

with $\varepsilon_n \searrow 0$.

In (4.17) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. We obtain

$$\|Du_n^-\|_p^p + \|Du_n^-\|_2^2 \leqslant M_1 \quad \forall n \geqslant 1,$$

for some $M_1 > 0$, so

$$\text{the sequence } \{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{4.18}$$

From (4.17) and (4.18), we have

$$\begin{aligned} &\Big|\langle A_p(u_n^+), h\rangle + \langle A(u_n^+), h\rangle - \int_\Omega g_\lambda^+(z,u_n^+)h\,dz\Big| \\ &\quad \leqslant M_2 \quad \forall h \in W_0^{1,p}(\Omega), n \geqslant 1, \end{aligned} \tag{4.19}$$

for some $M_2 > 0$. We show that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. We argue by contradiction. So, suppose that at least for a subsequence, we have $\|u_n^+\| \longrightarrow +\infty$. We set $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \geqslant 1$. Then $\|y_n\| = 1$ and $y_n \geqslant 0$ for all $n \geqslant 1$. Passing to a subsequence is necessary, we may assume that

$$y_n \xrightarrow{w} y \,\text{in}\, W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \,\text{in}\, L^p(\Omega), \tag{4.20}$$

with $y \geqslant 0$. From (4.19), we obtain

$$\begin{aligned} &\Big|\langle A_p(y_n), h\rangle + \frac{1}{\|u_n^+\|^{p-2}}\langle A(y_n), h\rangle - \int_\Omega \frac{N_{g_\lambda^+}(u_n^+)}{\|u_n^+\|^{p-1}} h\,dz\Big| \\ &\quad \leqslant \frac{M_2}{\|u_n^+\|^{p-1}} \quad \forall h \in W_0^{1,p}(\Omega), n \geqslant 1, \end{aligned} \tag{4.21}$$

In (4.21) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$. Since the sequence $\{\frac{N_{g_\lambda^+}(u_n^+)}{\|u_n^+\|^{p-1}}\}_{n\geqslant 1}$ is bounded (see (4.1), (4.2) and hypothesis $H_1(i)$) and recalling that $p > 2$, by passing to the limit as $n \to +\infty$ in (4.21), we obtain

$$\lim_{n\to+\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$

so

$$y_n \longrightarrow y \,\text{in}\, W_0^{1,p}(\Omega)$$

(see (4.20) and Proposition 2.3) hence $\|y\| = 1$ and $y \geqslant 1$.

Also, hypothesis $H_2(ii)$ implies that

$$\frac{N_{g_\lambda^+}(u_n^+)}{\|u_n^+\|^{p-1}} \longrightarrow \widehat{\lambda}_m(p) y^{p-1} \,\text{in}\, L^{p'}(\Omega) \tag{4.22}$$

(see Aizicovici–Papageorgiou–Staicu [1, Proposition 14]). Then, if in (4.21) we pass to the limit as $n \to +\infty$ and use (4.22) and the facts that $\|y\| = 1$ and $p > 2$ (recall that $\|u_n^+\| \longrightarrow +\infty$), we get

$$\langle A_p(y), h\rangle = \widehat{\lambda}_m(p) \int_\Omega y^{p-1} h\,dz \quad \forall h \in W_0^{1,p}(\Omega),$$

so

$$\begin{cases} -\Delta_p y(z) = \widehat{\lambda}_m(p) y(z)^{p-1} & \text{for a.a. } z \in \Omega, \\ y|_{\partial\Omega} = 0, \end{cases}$$

thus $y \equiv 0$ or y is nodal (since $m \geqslant 2$), a contradiction to the fact that $y \geqslant 0$.

This proves that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded, thus the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded too (see (4.18)).

Hence, passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \,\text{in}\, W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u \,\text{in}\, L^p(\Omega). \tag{4.23}$$

In (4.17) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (4.23). Then

$$\lim_{n \to +\infty} \big(\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle\big) = 0,$$

so

$$\limsup_{n \to +\infty} \big(\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle\big) \leqslant 0$$

(since A is monotone), hence

$$\limsup_{n \to +\infty} \langle A_p(u_n), u_n - u\rangle \leqslant 0$$

and thus

$$u_n \longrightarrow u \ \text{ in } W_0^{1,p}(\Omega)$$

(see (4.18) and Proposition 2.3). This proves Claim 3.

Combining (4.14), (4.15) and Claim 3, we see that we can use the mountain pass theorem (see Theorem 2.1) and produce $\widehat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u}_\lambda \in K_{\chi_\lambda^+} \quad \text{and} \quad m_\lambda^+ \leqslant \chi_\lambda^+(\widehat{u}_\lambda). \tag{4.24}$$

From (4.24), (4.14) and Claim 1, we have

$$\widehat{u}_\lambda \in C_0^1(\overline{\Omega}) \quad \text{and} \quad \widehat{u}_\lambda - u_\lambda \in C_+ \setminus \{0\}.$$

As we did earlier (see the proof of Claim 2) for the pair $\{u_\lambda, \overline{u}_\lambda\}$, exploiting the monotonicity of $f_\lambda(z, \cdot)$, we obtain

$$\widehat{u}_\lambda - u_\lambda \in \operatorname{int} C_+.$$

Similarly, working on the negative semiaxis with $v_\lambda \in -\operatorname{int} C_+$ and $\overline{v}_\lambda = -\overline{u}_\lambda \in -\operatorname{int} C_+$, we produce a second nontrivial solution $\widehat{v}_\lambda \in -\operatorname{int} C_+$ of (P_λ) which satisfies

$$v_\lambda - \widehat{v}_\lambda \in \operatorname{int} C_+.$$

$\square$

So far, we have produced five nontrivial smooth solutions for (P_λ) ($\lambda \in (0, \lambda^*)$), all with sign information. We have

$$u_\lambda, \widehat{u}_\lambda \in \operatorname{int} C_+ \text{ with } \widehat{u}_\lambda - u_\lambda \in \operatorname{int} C_+ \text{ (two positive solutions)},$$
$$v_\lambda, \widehat{v}_\lambda \in -\operatorname{int} C_+ \text{ with } v_\lambda - \widehat{v}_\lambda \in \operatorname{int} C_+ \text{ (two negative solutions)},$$
$$y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda, u_\lambda] \text{ nodal}.$$

Next using critical groups (Morse theory), we will produce a sixth nontrivial smooth solution. However, we cannot provide any sign information for this sixth solution.

So, let $\lambda \in (0, \lambda^*)$ and let $\varphi_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy (Euler) functional for problem (P_λ) when the reaction term has the form (4.1). Therefore

$$\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{q_\lambda}\|u\|_{q_\lambda}^{q_\lambda} - \int_\Omega F_0(z, u)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega)) \cap C^2(W_0^{1,p}(\Omega)\backslash\{0\})$.

Proposition 4.4. *If hypotheses $H_2(i), (ii)$ and (iii) hold, then for every $\lambda \in (0, \lambda_0)$ the functional φ_λ satisfies the Cerami condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence, such that

$$|\varphi_\lambda(u_n)| \leqslant M_3 \quad \forall n \geqslant 1, \tag{4.25}$$

with $M_3 > 0$ and

$$(1 + \|u_n\|)\varphi'_\lambda(u_n) \longrightarrow 0 \text{ in } W^{-1,p'}(\Omega). \tag{4.26}$$

From (4.26), we have

$$\begin{aligned} &\Big|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \lambda \int_\Omega |u_n|^{q_\lambda - 2} u_n h\, dz - \int_\Omega f_0(z, u_n) h\, dz\Big| \\ &\quad\leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega), \end{aligned} \tag{4.27}$$

with $\varepsilon_n \searrow 0$. Choosing $h = u_n \in W_0^{1,p}(\Omega)$ in (4.27), we obtain

$$-\|Du_n\|_p^p - \|Du_n\|_2^2 + \lambda\|u_n\|_{q_\lambda}^{q_\lambda} + \int_\Omega f_0(z, u_n) u_n\, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{4.28}$$

On the other hand from (4.25), we have

$$\|Du_n\|_p^p + \frac{p}{2}\|Du_n\|_2^2 - \frac{\lambda p}{q_\lambda}\|u_n\|_{q_\lambda}^{q_\lambda} - \int_\Omega p F_0(z, u_n)\, dz \leqslant p M_3 \quad \forall n \geqslant 1. \tag{4.29}$$

We add (4.28) and (4.29). Then

$$\begin{aligned} &\big(\frac{p}{2} - 1\big)\|Du_n\|_2^2 + \int_\Omega \big(f_0(z, u_n) u_n - p F_0(z, u_n)\big)\, dz \\ &\quad\leqslant M_4 + \lambda\big(\frac{p}{q_\lambda} - 1\big)\|u_n\|_{q_\lambda}^{q_\lambda} \quad \forall n \geqslant 1, \end{aligned}$$

for some $M_4 > 0$, so

$$\int_\Omega \big(f_0(z, u_n) u_n - p F_0(z, u_n)\big)\, dz \leqslant M_4 + \lambda\big(\frac{p}{q_\lambda} - 1\big)\|u_n\|_{q_\lambda}^{q_\lambda} \quad \forall n \geqslant 1$$

(since $2 < p$), thus

$$\begin{aligned} &\frac{1}{\|u_n\|^{\tau_\lambda}} \int_\Omega \big(f_0(z, u_n) u_n - p F_0(z, u_n)\big)\, dz \\ &\quad\leqslant M_5 + \big(\frac{1}{\|u_n\|^{\tau_\lambda}} + \lambda \frac{1}{\|u_n\|^{\tau_\lambda - q_\lambda}}\big) \quad \forall n \geqslant 1 \end{aligned} \tag{4.30}$$

for some $M_5 > 0$ (recall that $q_\lambda < p$).

Claim. The sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

We argue indirectly. So, suppose that the Claim is not true. Hence at least for a subsequence, we have

$$\|u_n\| \longrightarrow +\infty. \tag{4.31}$$

Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so passing to a subsequence if necessary, we may assume that

$$y_n \longrightarrow y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \text{ in } L^p(\Omega). \tag{4.32}$$

From (4.27), we have

$$\begin{aligned} \Big|\langle A_p(y_n), h\rangle &+ \frac{1}{\|u_n\|^{p-2}}\langle A(y_n), h\rangle \\ &- \frac{\lambda}{\|u_n\|^{p-q_\lambda}} \int_\Omega |y_n|^{q_\lambda - 2} y_n \, dz - \int_\Omega \frac{N_{f_0}(u_n)}{\|u_n\|^{p-1}} h \, dz\Big| \\ \leqslant &\frac{\varepsilon_n}{(1+\|u_n\|)\|u_n\|^{p-1}} \quad \forall n \geqslant 1. \end{aligned} \tag{4.33}$$

Hypotheses $H_2(i), (ii)$ imply that the sequence $\left\{\frac{N_{f_0}(u_n)}{\|u_n\|^{p-1}}\right\}_{n \geqslant 1} \subseteq L^{p'}(\Omega)$ is bounded and so for at least a subsequence, we have

$$\frac{N_{f_0}(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p)|y|^{p-2}y \text{ in } L^{p'}(\Omega) \tag{4.34}$$

(see hypothesis $H_2(ii)$ and Gasiński–Papageorgiou [13]). In (4.33) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to +\infty$. Using (4.31), the fact that $p > 2$ and (4.34), we obtain

$$\lim_{n \to +\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$

so

$$y_n \longrightarrow y \text{ in } W_0^{1,p}(\Omega) \tag{4.35}$$

(see (4.32) and Proposition 2.3), hence $\|y\| = 1$. Let

$$C = \{z \in \Omega : y(z) \neq 0\}.$$

Then from (4.35) we see that $|C|_N > 0$ and

$$|u_n(z)| \longrightarrow +\infty \quad \text{for a.a. } z \in C,$$

thus

$$\frac{f_0(z, u_n(z))u_n(z) - pF_0(z, u_n(z))}{|u_n(z)|^{\tau_\lambda}} \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega. \tag{4.36}$$

From (4.36) and Fatou's lemma, we have

$$\lim_{n \to +\infty} \int_C \frac{f_0(z, u_n)u_n - pF_0(z, u_n)}{|u_n|^{\tau_\lambda}} \, dz = +\infty. \tag{4.37}$$

Hypothesis $H_2(iii)$ implies that we can find $M_6 > 0$ such that

$$\frac{f_0(z, \zeta)\zeta - pF_0(z, \zeta)}{|\zeta|^{\tau_\lambda}} \geqslant 0 \quad \text{for a.a. } z \in \Omega, \text{all } |\zeta| \geqslant M_6. \tag{4.38}$$

Then assuming without any loss of generality that $\|u_n\| \geqslant 1$ for all $n \geqslant 1$ (see (4.31)), we have

$$\begin{aligned}
&\frac{1}{\|u_n\|^{\tau_\lambda}} \int_\Omega (f_0(z,u_n)u_n - pF_0(z,u_n))\,dz \\
&= \int_\Omega \frac{f_0(z,u_n)u_n - pF_0(z,u_n)}{|u_n|^{\tau_\lambda}} |y_n|^{\tau_\lambda}\,dz \\
&= \int_{\{|u_n|\geqslant M_6\}} \frac{f_0(z,u_n)u_n - pF_0(z,u_n)}{|u_n|^{\tau_\lambda}} |y_n|^{\tau_\lambda}\,dz \\
&\quad + \frac{1}{\|u_n\|^{\tau_\lambda}} \int_{\{|u_n|<M_6\}} (f_0(z,u_n)u_n - pF_0(z,u_n))\,dz \\
&\geqslant \int_{\{|u_n|\geqslant M_6\}\cap C} \frac{f_0(z,u_n)u_n - pF_0(z,u_n)}{|u_n|^{\tau_\lambda}} |y_n|^{\tau_\lambda}\,dz - M_7 \\
&\geqslant \int_C \frac{f_0(z,u_n)u_n - pF_0(z,u_n)}{|u_n|^{\tau_\lambda}} |y_n|^{\tau_\lambda}\,dz - M_8 \quad \forall n \geqslant 1,
\end{aligned}$$

for some $M_7, M_8 > 0$ (see hypothesis $H_2(i)$), so

$$\lim_{n\to+\infty} \frac{1}{\|u_n\|^{\tau_\lambda}} \int_\Omega (f_0(z,u_n)u_n - pF_0(z,u_n))\,dz = +\infty \tag{4.39}$$

(see (4.37)). Comparing (4.39) and (4.30) and recalling that $\tau_\lambda > q_\lambda$ and that $\|u_n\| \longrightarrow +\infty$ (see (4.31)), we have a contradiction. This proves the Claim.

On account of the Claim, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u \text{ in } L^p(\Omega). \tag{4.40}$$

In (4.27) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (4.40). Then

$$\lim_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle = 0,$$

so

$$u_n \longrightarrow u \text{ in } W_0^{1,p}(\Omega)$$

(see (4.40) and Proposition 2.3), hence φ_λ satisfies the Cerami condition. □

Proposition 4.5. *If hypotheses $H_2(i), (ii)$ and (iii) hold, then for every $\lambda \in (0, \lambda_0)$, we have*

$$C_m(\varphi_\lambda, \infty) \neq 0.$$

Proof. Let

$$\vartheta \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p)) \backslash \sigma_0(p)$$

and consider the C^1-functional $\gamma\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\gamma(u) = \frac{1}{p}\|Du\|_p^p - \frac{\vartheta}{p}\|u\|_p^p \quad \forall u \in W_0^{1,p}(\Omega).$$

We consider the homotopy $h_\lambda(t,u)$ defined by

$$h_\lambda(t,u) = (1-t)\varphi_\lambda(u) + t\gamma(u) \quad \forall (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Claim. There exist $\xi_\lambda \in \mathbb{R}$ and $\widehat{\delta}_\lambda > 0$ such that

$$\text{if}\, h_\lambda(t,u) \leqslant \xi_\lambda, \text{then } (1+\|u\|)\|(h_\lambda)'_u(t,u)\|_* \geqslant \widehat{\delta}_\lambda \quad \forall t \in [0,1].$$

We argue by contradiction. So, suppose that the Claim is not true. Note that h_λ maps bounded sets into bounded sets. Hence, we can find two sequences $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$\begin{cases} t_n \longrightarrow t, \quad \|u_n\| \longrightarrow +\infty, \quad h_\lambda(t_n,u_n) \longrightarrow -\infty \\ (1+\|u_n\|)(h_\lambda)'_u(t_n,u_n) \longrightarrow 0 \end{cases} \tag{4.41}$$

From the last convergence in (4.37), we have

$$\begin{aligned} &\Big|\langle A_p(u_n),h\rangle + (1-t_n)\langle A(u_n),h\rangle - (1-t_n)\int_\Omega |u_n|^{q_\lambda-2}u_n h\,dz \\ &\quad -(1-t_n)\int_\Omega f_0(z,u_n)h\,dz - t_n\vartheta\int_\Omega |u_n|^{p-2}u_n h\,dz\Big| \\ &\leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega), \end{aligned} \tag{4.42}$$

with $\varepsilon_n \searrow 0$.

Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \,\text{in}\, W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \,\text{in}\, L^p(\Omega). \tag{4.43}$$

From (4.42), we have

$$\begin{aligned} &\Big|\langle A_p(y_n),h\rangle + \frac{1-t_n}{\|u_n\|^{p-2}}\langle A(y_n),h\rangle - \frac{(1-t_n)\lambda}{\|u_n\|^{p-q_\lambda}}\int_\Omega |y_n|^{q_\lambda-2}y_n h\,dz \\ &\quad -(1-t_n)\int_\Omega \frac{N_{f_0}(u_n)}{\|u_n\|^{p-1}}h\,dz - t_n\vartheta\int_\Omega |y_n|^{p-2}y_n h\,dz\Big| \\ &\leqslant \frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \quad \forall n \geqslant 1. \end{aligned} \tag{4.44}$$

As before, choosing $h = y_n - y \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to +\infty$ and recalling that $q_\lambda < 2 < p$ (see also (4.41)), via Proposition 2.3 (see also (4.43)), we obtain

$$y_n \longrightarrow y \,\text{in}\, W_0^{1,p}(\Omega), \tag{4.45}$$

hence $\|y\| = 1$. Passing the limit as $n \to +\infty$ in (4.44) and using (4.41) and (4.45), we obtain

$$\langle A_p(y),h\rangle = \vartheta_t\int_\Omega |y|^{p-2}yh\,dz \quad \forall h \in W_0^{1,p}(\Omega),$$

with $\vartheta_t = (1-t)\widehat{\lambda}_m(p) + t\vartheta$, so

$$\begin{cases} -\Delta_p y(z) = \vartheta_t |y(z)|^{p-2} y(z) & \text{for a.a. } z \in \Omega, \\ y|_{\partial\Omega} = 0. \end{cases} \tag{4.46}$$

If $\vartheta_t \notin \sigma_0(p)$, then (4.46) implies that $y = 0$, which contradicts (4.45).

If $\vartheta_t \in \sigma_0(p)$ and $C = \{z \in \Omega : y(z) \neq 0\}$, then $|C|_N > 0$ and as in the proof of Proposition 4.4, we have

$$\lim_{n\to+\infty} \frac{1}{\|u_n\|^{\tau_\lambda}} \int_\Omega (f_0(z,u_n)u_n - pF_0(z,u_n)\, dz = +\infty. \tag{4.47}$$

From the third convergence in (4.41), we see that we can find $n_0 \geqslant 1$ such that

$$\begin{aligned} &\|Du_n\|_p^p + \frac{(1-t_n)p}{2}\|Du_n\|_2^2 - \frac{(1-t_n)p}{q_\lambda}\|u_n\|_{q_\lambda}^{q_\lambda} \\ &\quad -(1-t_n)\int_\Omega pF_0(z,u_n)\, dz - t_n\vartheta\|u_n\|_p^p \leqslant -1 \quad \forall n \geqslant n_0. \end{aligned} \tag{4.48}$$

In (4.42) we choose $h = u_n \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &-\|Du_n\|_p^p - (1-t_n)\|Du_n\|_2^2 + (1-t_n)\|u_n\|_{q_\lambda}^{q_\lambda} \\ &\quad +(1-t_n)\int_\Omega f_0(z,u_n)u_n\, dz + t_n\vartheta\|u_n\|_p^p \leqslant \varepsilon_n \quad \forall n \geqslant 1. \end{aligned} \tag{4.49}$$

We add (4.48), (4.49) and use the fact that $p > 2$, to obtain

$$\begin{aligned} &(1-t_n)\int_\Omega (f_0(z,u_n)u_n - pF_0(z,u_n))\, dz \\ &\quad \leqslant (1-t_n)\lambda(\frac{p}{q_\lambda} - 1)\|u_n\|_{q_\lambda}^{q_\lambda} \quad \forall n \geqslant n_1 \geqslant n_0, \end{aligned}$$

so

$$\frac{1-t_n}{\|u_n\|^{\tau_\lambda}} \int_\Omega (f_0(z,u_n)u_n - pF_0(z,u_n))\, dz \leqslant \frac{(1-t_n)M_9}{\|u_n\|^{\tau_\lambda - q_\lambda}} \quad \forall n \geqslant n_1,$$

for some $M_9 > 0$. We can always assume that $t_n \neq 1$ for all $n \geqslant 1$ or otherwise $t = 1$ and since $\vartheta \notin \sigma_0(p)$, we infer that $y = 0$, a contradiction to (4.45). Hence

$$\frac{1}{\|u_n\|^{\tau_\lambda}} \int_\Omega (f_0(z,u_n)u_n - pF_0(z,u_0))\, dz \leqslant \frac{M_9}{\|u_n\|^{\tau_\lambda - q_\lambda}} \quad \forall n \geqslant 1. \tag{4.50}$$

Comparing (4.50) and (4.47), we have a contradiction (recall that $q_\lambda < \tau_\lambda$). This prove the Claim.

This above argument with minor changes also shows that for all $t \in [0,1]$, h_λ satisfies the Cerami condition (see the proof of Proposition 4.4). So, from Chang [5, p. 334] (see also Liang–Su [25, Proposition 3.2]), we have

$$C_k(h_\lambda(0,\cdot),\infty) = C_k(h_\lambda(1,\cdot),\infty) \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi_\lambda, \infty) = C_k(\gamma, \infty) \quad \forall k \geqslant 0. \tag{4.51}$$

Since $\vartheta \notin \sigma_0(p)$, we have $K_\gamma = \{0\}$. Therefore

$$C_k(\gamma, \infty) = C_k(\gamma, 0) \quad \forall k \geqslant 0,$$

so

$$C_m(\gamma, \infty) \neq 0$$

(see Cingolani–Degiovanni [7]), thus

$$C_m(\varphi_\lambda, \infty) \neq 0$$

(see (4.51)). □

Now we are ready for the second multiplicity theorem concerning problem (P_λ).

Theorem 4.6. *If (4.1) and hypotheses H_2 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) admits at least six nontrivial smooth solutions*

$$\begin{aligned} &u_\lambda, \widehat{u}_\lambda \in \operatorname{int} C_+, \quad \textit{with } \widehat{u}_\lambda - u_\lambda \in \operatorname{int} C_+, \\ &v_\lambda, \widehat{v}_\lambda \in -\operatorname{int} C_+, \quad \textit{with } v_\lambda - \widehat{v}_\lambda \in \operatorname{int} C_+, \\ &y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda, u_\lambda] \textit{ nodal and } \widehat{y}_\lambda \in C_0^1(\overline{\Omega}) \setminus \{0\}. \end{aligned}$$

Proof. From Theorem 3.6 and Proposition 4.3, we know that we can find $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least five smooth solutions all with sign information

$$\begin{aligned} &u_\lambda, \widehat{u}_\lambda \in \operatorname{int} C_+, \quad \text{with } \widehat{u}_\lambda - u_\lambda \in \operatorname{int} C_+, \\ &v_\lambda, \widehat{v}_\lambda \in -\operatorname{int} C_+, \quad \text{with } v_\lambda - \widehat{v}_\lambda \in \operatorname{int} C_+ \end{aligned}$$

and

$$y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda, u_\lambda] \text{ nodal}.$$

We can always assume that u_λ and v_λ are extremal (that is $u_\lambda = u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda = v_\lambda^* \in -\operatorname{int} C_+$; see Proposition 3.4). From the proof of Proposition 3.3 we know that

$$u_\lambda \in \operatorname{int} C_+ \text{ is a minimizer of } \widehat{\varphi}_\lambda^+ \text{ and } \overline{u}_\lambda - u_\lambda \in \operatorname{int} C_+, \tag{4.52}$$

$$v_\lambda \in -\operatorname{int} C_+ \text{ is a minimizer of } \widehat{\varphi}_\lambda^- \text{ and } v_\lambda - \overline{v}_\lambda \in \operatorname{int} C_+. \tag{4.53}$$

We have

$$\varphi_\lambda\big|_{[0,\overline{u}_\lambda]} = \widehat{\varphi}_\lambda^+\big|_{[0,\overline{u}_\lambda]} \quad \text{and} \quad \varphi_\lambda\big|_{[\overline{v}_\lambda,0]} = \widehat{\varphi}_\lambda^-\big|_{[\overline{v}_\lambda,0]}.$$

So, from (4.52) and (4.53) it follows that

$$C_k(\varphi_\lambda, u_\lambda) = C_k(\varphi_\lambda, v_\lambda) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0 \tag{4.54}$$

(see Proposition 2.2). From the proof of Proposition 4.3, we know that

$$\widehat{u}_\lambda \text{ is a critical point of mountain pass type for } \chi_\lambda^+, \tag{4.55}$$

$$\widehat{v}_\lambda \text{ is a critical point of mountain pass type for } \chi_\lambda^-. \tag{4.56}$$

Note that

$$\varphi_\lambda\big|_{[u_\lambda)} = \chi_\lambda^+\big|_{[u_\lambda)} + \mu_\lambda^+ \quad \text{and} \quad \varphi_\lambda\big|_{(v_\lambda]} = \chi_\lambda^-\big|_{(v_\lambda]} + \mu_\lambda^-$$

with $\mu_\lambda^+, \mu_\lambda^- \in \mathbb{R}$ (see (4.2)), where

$$(v_\lambda] = \{v \in W_0^{1,p}(\Omega) : v(z) \leqslant v_\lambda(z) \text{ for a.a. } z \in \Omega\}.$$

Since

$$\widehat{u}_\lambda - u_\lambda \in \operatorname{int} C_+ \quad \text{and} \quad v_\lambda - \widehat{v}_\lambda \in \operatorname{int} C_+,$$

from (4.55) and (4.56) it follows that

$$C_1(\varphi_\lambda, \widehat{u}_\lambda) \neq 0, \quad C_1(\varphi_\lambda, \widehat{v}_\lambda) \neq 0,$$

so

$$C_k(\varphi_\lambda, \widehat{u}_\lambda) = C_k(\varphi_\lambda, \widehat{v}_\lambda) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0 \tag{4.57}$$

(see Papageorgiou–Rădulescu [28] and Papageorgiou–Smyrlis [31]). Also recall that

$$y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda, u_\lambda]$$

and y_λ is a critical point of mountain pass type for the functional γ_λ (see the proof of Proposition 3.5). Hence

$$C_1(\gamma_\lambda, y_\lambda) \neq 0$$

(see Motreanu–Motreanu–Papageorgiou [26, p. 177]). Also note that

$$\gamma_\lambda\big|_{[v_\lambda, u_\lambda]} = \varphi_\lambda\big|_{[v_\lambda, u_\lambda]}$$

(see (3.24)). So, it follows that

$$C_1(\varphi_\lambda, y_\lambda) \neq 0,$$

thus

$$C_k(\varphi_\lambda, y_\lambda) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0 \tag{4.58}$$

(as before see [28] and [31]). The presence of the concave term $\lambda|\zeta|^{q_\lambda - 2}\zeta$ (see (4.1)) and hypothesis $H_2(iv)$ imply that

$$C_k(\varphi_\lambda, 0) = 0 \quad \forall k \geqslant 0 \tag{4.59}$$

(see Gasiński–Papageorgiou [18, Proposition 4.1]). From Proposition 4.5, we know that

$$C_m(\varphi_\lambda, \infty) \neq 0.$$

Therefore there exists $\widehat{y}_\lambda \in K_{\varphi_\lambda}$ such that

$$C_m(\varphi_\lambda, \widehat{y}_\lambda) \neq 0 \tag{4.60}$$

with $m \geqslant 2$. Comparing (4.60) with (4.54), (4.57), (4.58), (4.59) we infer that

$$\widehat{y}_\lambda \notin \{u_\lambda, v_\lambda, \widehat{u}_\lambda, \widehat{v}_\lambda, y_\lambda, 0\},$$

so $\widehat{y}_\lambda \in C_0^1(\overline{\Omega})\backslash\{0\}$ (nonlinear regularity) is the sixth nontrivial solution. $\square$

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