L^∞ -Norm Estimates of Weak Solutions via Their Morse Indices for the m -Laplacian Problems

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Recommended by Dong Ye.

Abstract. This work is devoted to obtain the L^p and the L^∞ -estimates of solutions via their Morse indices to the following m -Laplacian problems

$$\begin{cases} -\Delta_m u = f(x, u) & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary, $N > m > 2$ and $f \in C(\bar{\Omega} \times \mathbb{R})$ which will be specified later. As far as we know, it seems to be the first time that such explicit estimates are obtained for a nonlinear degenerate problems. So, our main results extend and complement previously L^∞ -estimates results in the literature.

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1. Introduction

A priori L^∞ -estimate (also called a priori L^∞ -bound) is an estimate for the size of a solution. “A priori” is a Latin expression which means “from before” and refers to the fact that the estimate for the solution is derived before the solution is known to exist. A priori estimates play important role in the proof of existence of the solution. Indeed, when variational methods are not applicable, the existence of a solution can be proved using other methods as far as we can derive a priori L^∞ -bounds for all possible solutions (see [7, 17]).

Ladyzhenskaya and Ural'tseva [14] proved that every weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of (1) belongs to $C^\beta(\bar{\Omega})$ for some $\beta \in (0, 1)$, and a more minute estimate such that $u \in C^{1+\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ can be derived from the result of Tolksdorf [19]. Furthermore, DiBenedetto [3] and Lewis [13] assures that every bounded weak solution of (1) enjoys $C_{loc}^{1+\alpha}$ regularity.

There exist various methods which provide a priori L^∞ -estimates of solutions of elliptic problems: In [16], Ôtani obtained an L^∞ -estimate of weak solutions which the proof of this regularity result relies on Moser's iteration scheme and a certain "Pohozaev-type inequality" which is valid for all weak solutions in P , where $P := \{u \in W_0^{1,p}(\Omega) \cap L^q(\Omega); x_i |u|^{q-2} u \in L^{\frac{p}{p-1}}(\Omega), i = 1, 2, \dots, N\} \subset W_0^{1,p(\Omega)} \cap L^\infty(\Omega)$.

Another method is the Morse index which was first explored (in harmonic equations and the subcritical case) by Bahri and Lions [2] and extended by Farina [8] to $1 < p < p_c(N)$, where $p_c(N)$ is the so-called Joseph-Lundgren exponent. The basic argument used in the previous works is blow-up technique which was first introduced by Gidas and Spruck [9]. Motivated by [2], based on some local interior estimates and careful boundary estimates, Yang obtained in a famous paper [20] the first explicite L^∞ -bounds via the Morse index and he controlled the L^p and L^∞ norm of solution by polynomial growth in Morse index. Note that the explicite L^∞ -estimate was improved in [10] under a large assumptions and extended in [15] to the bi-harmonic and tri-harmonic semilinear problems and for the polyharmonic [11] for any classical solution u . Inspired by the previously mentioned works, we are concerned to establish some L^p and L^∞ estimates for weak solution of (1) via their Morse indices. However, in this work, we use a different approach to those explored in [15, 20] because here we are working with the m -Laplacian operator which is a nonlinear degenerate operator and a standard boot-strap iteration do not yield any extra regularity of the solution which only gives $C_{loc}^{1,\alpha}(\Omega)$ regularity, for some $\alpha \in (0, 1)$ (we refer the interested readers to [4–6, 12, 16, 18]). Therefore, Eq. (1) is to be understood with its weak formulation.

Definition 1.1. *We say that $u \in C_{loc}^{1,\alpha}(\Omega)$ is a weak solution of (1) if for every $\varphi \in C_c^1(\bar{\Omega})$, one has*

$$\int_{\Omega} |\nabla u|^{m-2} (\nabla u, \nabla \varphi) dx = \int_{\Omega} f(x, u) \varphi dx. \quad (2)$$

Formally, The linearized operator of (2) at u is given by

$$\begin{aligned} \Lambda_u(v, \varphi) = & \int_{\Omega} |\nabla u|^{m-2} (\nabla v, \nabla \varphi) dx + (m-2) \int_{\Omega} |\nabla u|^{m-4} (\nabla u, \nabla v) (\nabla u, \nabla \varphi) dx \\ & - \int_{\Omega} f'(x, u) v \varphi dx, \end{aligned} \quad (3)$$

for all $v, \varphi \in C_c^1(\Omega)$. We observe that if we consider

$$\rho = |\nabla u|^{m-2} \quad (4)$$

then $\rho \in L^1_{loc}(\Omega)$ by the $C^{1,\alpha}$ regularity of u .

Observe that, this weight is 1 when $m = 2$, so we will be mostly concerned with the case $m > 2$, in the case $1 < m < 2$ the weight $|\nabla u|^{m-2}$ might not belong to the space $L^1_{loc}(\Omega)$. This prevents the use of classical estimates in weighted Sobolev spaces and in this case it is not even clear which definition of Morse index would be the natural one.

Definition 1.2. We say that a weak solution $C^{1,\alpha}(\Omega)$

- is stable if $\Lambda_u(\varphi, \varphi) \geq 0$, for all $\varphi \in C^1_c(\Omega)$,
- has Morse index of a solution u (denoted $i(u)$) equal to $K \geq 1$ if K is the maximal dimension of all subspace $X \subset C^1_c(\Omega)$, such that $\Lambda_u(\varphi, \varphi) < 0$ in $X \setminus \{0\}$,
- is stable outside a compact set $K \subset \Omega$ if $\Lambda_u(\varphi, \varphi) \geq 0$ for any $\varphi \in C^1_c(\Omega \setminus K)$.

Remark 1.1. It is well known that, if u has finite Morse index, then u is stable outside a compact set. Also, for future use, we point out that, if u has finite Morse index, for any $\varphi \in C^1_c(\Omega)$, for any compact set $K \subset \Omega$, we have

$$(m-1) \int_{\Omega} |\nabla u|^{m-2} |\nabla(u\varphi^m)|^2 - \int_{\Omega} f'(x, u) u^2 \varphi^{2m} \geq 0. \quad (5)$$

Before stating our results, we present first our assumptions on the non-linearity f .

Let $f \in C(\overline{\Omega} \times \mathbb{R})$ and $f'(x, s) = \frac{\partial f}{\partial s}(x, s)$ exists for all $(x, s) \in \overline{\Omega} \times \mathbb{R}$ such that:

(h_0) (Super-linearity) there exists $\mu > 0$ such that for $|s| > s_0$ and $x \in \Omega$

$$f'(x, s)s^2 \geq (m-1+\mu)f(x, s)s > 0;$$

(h_1) (Subcritical growth) there exists $0 < \theta < 1$ such that for $|s| > s_0$ and $x \in \Omega$

$$\left(\frac{mN}{N-m} - \theta \right) F(x, s) \geq f(x, s)s;$$

(h_2) there is a constant $C \geq 0$ such that for $x \in \Omega$

$$|\nabla_x F(x, s)| \leq C(F(x, s) + 1).$$

From (h_0) and (h_1), we may easily see that there exists two positive constants C_1 and C_2 such that:

$$\frac{N-m}{mN-\theta(N-m)} f(x, s)s - C_1 \leq F(x, s) \leq (m+\mu)^{-1} f(x, s)s + C_1, \quad (6)$$

$$f(x, s)s \geq C_2 (|s|^{m+\mu} - 1), \quad (7)$$

$$|f(x, s)| \leq C_1 \left(|s|^{\frac{mN}{N-m}-\theta-1} + 1 \right). \quad (8)$$

Our first result is to claim an explicite estimate of $|\nabla u|_{L^m(\Omega)}$ via the Morse index:

Theorem 1.1. *Assume that f satisfies $(h_0) - (h_2)$ with $m > 2$, then there exists a positive constant $C = C(\Omega, f)$ such that for any weak solution $u \in C_{loc}^{1,\alpha}(\Omega)$ of (1) with finite Morse index $i(u)$, we have*

$$\int_{\Omega} |\nabla u|^m \leq C(i(u) + 1)^\alpha, \text{ where } \alpha = \frac{m(m + \mu)}{\mu} + 1.$$

The proof of Theorem 1.1 is based on local estimate from a combination of the quadratic form and (1) and a variant of the Pohozaev identity (see [4, 16]) which is key point to finish the proof. Precisely, we will test the Eq. (1) against $\psi \nabla u \cdot x$, where ψ is an appropriate cut-off function allowing to avoid the spherical integrals raised in Yang's proof [20] which are very difficult to control. We will adapt the Moser iteration technique [1, 16] to prove that the L^∞ norm evolves less rapidly than polynomial growth of the Morse indices of solutions.

Theorem 1.2. *Let $u \in C_{loc}^{1,\alpha}(\Omega)$ be a weak solution of (1) and f satisfies $(h_0) - (h_2)$ with $m > 2$, then there exists a positive constant $C = C(\Omega, f)$ such that*

$$\|u\|_{L^\infty} \leq C(i(u) + 1)^{\beta'},$$

where $\beta' = \left(\frac{3m}{4\theta(N-m)^2}\right) \left(\frac{m(m+\mu)}{\mu} + 1\right) \left(\frac{mN}{(N-m)} - 1 + q\right)$.

Remark 1.2. *Giving attention to [4, 15, 20], we think that some similar L^∞ estimates via Morse index for solutions of the m -polyharmonic Kirchhoff equation can be proved (work in preparation).*

2. Notations and Preliminaries

Consider a cut-off function $\psi \in C_c^1(B_R(y))$ with $0 \leq \psi \leq 1$ where $y \in \mathbb{R}^N$ and $B_R(y) = B(y, \frac{3R}{7})$ denotes the open ball of center y and **radius** $\frac{3R}{7}$. We denote $\partial\Omega_R(y) := \partial\Omega \cap B_R(y)$, v the unit normal vector on $\partial\Omega$ and for $x \in \Omega \cap B_R(y)$, $n := x - y$. Set now for $R > 0$

$$\begin{aligned} \Omega_1(R) &:= \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{R}{2} \right\} \quad \text{and} \\ \Omega_2(R) &:= \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) \leq \frac{R}{3} \right\}. \end{aligned}$$

To derive a global estimate from local controls on $\Omega \cap B_R(y)$, we will cover $\Omega \setminus \Omega_2(R)$ with finite number of balls $B_R(y)$, $y \in \Omega_1(R)$ which yields in particular that $\partial\Omega_R(y) = \emptyset$. However we will cover $\Omega_2(R)$ by $B_R(y)$ with

$$y \in \Gamma(R) := \left\{ x \in \mathbb{R}^N \setminus \Omega : \text{dist}(x, \partial\Omega) = \frac{R}{21} \right\}, \quad (9)$$

and clearly $\partial\Omega_R(y) \neq \emptyset$. Precisely, since $\bar{\Omega}$ is compact, then there exist $\{y_1, \dots, y_{k_0}\} \subset \Omega_1$ and $\{y_{k_0+1}, \dots, y_{l_0}\} \subset \Gamma$ such that

$$\bar{\Omega} \subset \bigcup_{i=1}^{l_0} B_R(y_i). \quad (10)$$

So we need first to control the following boundary term raised in the next variant of Pohozaev identity.

Lemma 2.1. *There exists $0 < R_1(\Omega) < 1$ such that for any $0 < R < R_1(\Omega)$ and $y \in \Gamma(R)$, there holds*

$$\int_{\partial\Omega_R(y)} \left(|\nabla u|^{m-2} (\nabla u \cdot n) (\nabla u \cdot v) - \frac{1}{m} v \cdot n |\nabla u|^m \right) \psi \leq 0.$$

Proof. First we may exhibit $R_1 = R_1(\Omega) > 0$ such that if $0 < R \leq R_1$ we have $v \cdot n \leq 0$ for all $x \in \partial\Omega_R(y)$ (see the proof of lemma 2.2 in [20]).

According to $u = 0$ on $\partial\Omega$, so $v = \pm 1 \frac{\nabla u}{|\nabla u|}$ for all $x \in \partial\Omega_R(y)$ with $\nabla u \neq 0$. Therefore, there holds

$$\begin{aligned} & \int_{\partial\Omega_R(y)} \left(|\nabla u|^{m-2} (\nabla u \cdot n) (\nabla u \cdot v) - \frac{1}{m} v \cdot n |\nabla u|^m \right) \psi \\ &= \int_{\partial\Omega_R(y)} \left(|\nabla u|^{m-2} \left(\epsilon v \cdot n |\nabla u| \right) \left(\nabla u \cdot \epsilon \frac{\nabla u}{|\nabla u|} \right) - \frac{1}{m} v \cdot n |\nabla u|^m \right) \psi \\ &= \left(1 - \frac{1}{m} \right) \int_{\partial\Omega_R(y)} v \cdot n |\nabla u|^m \psi \leq 0. \end{aligned}$$

□

According to Lemma 2.1, we may establish

Lemma 2.2. *Let $u \in C_{loc}^{1,\alpha}(\Omega)$ be a weak solution of (1). Assume that f satisfies (h_0) – (h_2) , then there are $0 < R_0 \leq R_1(\Omega)$ and $C > 0$ verifying $0 < R < R_0(\Omega)$ and $y \in \Gamma(R) \cup \Omega_1(R)$, there holds*

$$\begin{aligned} & \int_{B_R(y) \cap \Omega} (f(x, u)u + |\nabla u|^m) \psi \\ & \leq C \left(1 + \int_{B_R(y) \cap \Omega} (|\nabla u|^m + |u|^m + f(x, u)u) |\nabla \psi| \right). \end{aligned} \quad (11)$$

Proof. Multiplying (1) by $\nabla u \cdot n \psi$, and integrating by part, we obtain

$$\begin{aligned} & \int_{B_R(y) \cap \Omega} |\nabla u|^{m-2} \nabla u \cdot \nabla (\nabla u \cdot n \psi) - \int_{\partial(B_R(y) \cap \Omega)} |\nabla u|^{m-2} (\nabla u \cdot v) \nabla u \cdot n \psi \\ &= \int_{B_R(y) \cap \Omega} f(x, u) \nabla u \cdot n \psi. \end{aligned} \quad (12)$$

For the left hand side of (12), by direct calculations, we get

$$\begin{aligned}
 I_1 &:= \int_{B_R(y) \cap \Omega} |\nabla u|^{m-2} \nabla u \cdot \nabla (\nabla u \cdot n \psi) \\
 &\quad - \int_{\partial(\Omega \cap B_R(y))} |\nabla u|^{m-2} (\nabla u \cdot v) \nabla u \cdot n \psi \\
 &= \frac{1}{m} \int_{B_R(y) \cap \Omega} \nabla (|\nabla u|^m) \cdot n \psi + \int_{B_R(y) \cap \Omega} |\nabla u|^m \psi \\
 &\quad + \int_{B_R(y) \cap \Omega} |\nabla u|^{m-2} (\nabla u \cdot \nabla \psi) \nabla u \cdot n \\
 &\quad - \int_{\partial(\Omega \cap B_R(y))} |\nabla u|^{m-2} (\nabla u \cdot v) (\nabla u \cdot n) \psi.
 \end{aligned}$$

Integrate again by parts, we derive

$$I_1 = A_m + B_m - C_m, \quad (13)$$

where

$$\begin{aligned}
 A_m &= \frac{m-N}{m} \int_{B_R(y) \cap \Omega} |\nabla u|^m \psi - \frac{1}{m} \int_{B_R(y) \cap \Omega} |\nabla u|^m n \cdot \nabla \psi, \\
 B_m &= \int_{B_R(y) \cap \Omega} |\nabla u|^{m-2} (\nabla u \cdot n) (\nabla u \cdot \nabla \psi), \\
 C_m &= \int_{\partial \Omega_R(y)} \left(|\nabla u|^{m-2} (\nabla u \cdot v) (\nabla u \cdot n) - \frac{1}{m} v \cdot n |\nabla u|^m \right) \psi.
 \end{aligned}$$

Now integrate by parts the right hand side of (12) and taking into account that $F(x, u) = 0$ on $\partial \Omega_R(y)$, we obtain

$$\begin{aligned}
 I_2 &:= \int_{B_R(y) \cap \Omega} f(x, u) (\nabla u \cdot n \psi) \\
 &= \int_{B_R(y) \cap \Omega} \nabla (F(x, u)) \cdot n \psi - \int_{B_R(y) \cap \Omega} \nabla_x F(x, u) \cdot n \psi \\
 &= -N \int_{B_R(y) \cap \Omega} F(x, u) \psi - \int_{B_R(y) \cap \Omega} (F(x, u)) \nabla \psi \cdot n \\
 &\quad - \int_{B_R(y) \cap \Omega} \nabla_x F(x, u) \cdot n \psi.
 \end{aligned} \quad (14)$$

Hence, for $0 < R < R_1(\Omega)$, combining (12), (13), (14), Lemma 2.1 and (h_2) , yields

$$\begin{aligned}
& \frac{mN}{N-m} \int_{B_R(y) \cap \Omega} F(x, u) \psi - \int_{B_R(y) \cap \Omega} |\nabla u|^m \psi \\
& \leq C \left(1 + R \int_{B_R(y) \cap \Omega} F(x, u) \psi + \int_{B_R(y) \cap \Omega} (|\nabla u|^m + f(x, u)) u |\nabla \psi| \right). \quad (15)
\end{aligned}$$

In view of (h_1) , there are $0 < R_0 < R_1(\Omega)$ and $C > 0$ such that for $0 < R \leq R_0$, we have

$$\begin{aligned}
& \frac{\theta}{2} \int_{B_R(y) \cap \Omega} F(x, u) \psi + \int_{B_R(y) \cap \Omega} f(x, u) u \psi - \int_{B_R(y) \cap \Omega} |\nabla u|^m \psi \\
& \leq C \left(1 + \int_{B_R(y) \cap \Omega} (|\nabla u|^m + f(x, u)) u |\nabla \psi| \right). \quad (16)
\end{aligned}$$

Multiplying now (1) by $u\psi$ and integrating by parts over $B_R(y) \cap \Omega$ and using Young's inequality, we can conclude that

$$\begin{aligned}
\int_{B_R(y) \cap \Omega} |\nabla u|^m \psi - \int_{B_R(y) \cap \Omega} f(x, u) u \psi & \leq \int_{B_R(y) \cap \Omega} |\nabla u|^{m-1} |u| |\nabla \psi| \\
& \leq \int_{B_R(y) \cap \Omega} (|\nabla u|^m + |u|^m) |\nabla \psi|_{j_0}.
\end{aligned}$$

Combining the two last inequalities and (6), the estimate (11) of Lemma 2.2 is proved. $\square$

2.1. Proof of Theorem 1.1

Proof. First we establish the following L^p -estimate via the Morse index which will be combined with Lemma 2.2 to prove Theorem 1.1. Let $u \in C_{loc}^{1,\alpha}(\Omega)$ be a weak solution of Eq. (1), $R_0 > 0$ obtained in Lemma 2.2 and $y \in \Gamma(R_0) \cup \Omega_1(R_0)$. We denote by $A_a^b = \{x \in \mathbb{R}^N; a < |x - y| < b\}$. For $j = 1, 2, \dots, i(u) + 1$, we designate by

$$A_j := A_{a_j}^{b_j} \quad \text{with} \quad a_j = R_0 \frac{2(j + i(u))}{4(i(u) + 1)}, \quad b_j = R_0 \frac{2(j + i(u)) + 1}{4(i(u) + 1)}. \quad (17)$$

Let φ_j be a family of C^1 functions satisfying the following conditions:

- $\varphi_j = 1$ for $2(j + i(u)) < |x| < 2(j + i(u)) + 1$;
- $\text{supp}(\varphi_j) \subset \{2(j + i(u)) - \frac{1}{2} \leq |x| \leq 2(j + i(u)) + \frac{3}{2}\}$;
- $\|\nabla \varphi_j\|_\infty \leq C$.

We will use the test function $\phi_j^k \in C_c^1(B_{R_0}(y))$, defined by $\phi_j^k(x) := \varphi_j \left(\frac{4(i(u) + 1)(x - y)}{R_0} \right)$, then $\phi_j^k(x) = 1$ in A_j and

$$\|\nabla \phi_j^k\|_\infty \leq \frac{C}{R_0} (1 + i(u)). \quad (18)$$

Lemma 2.3. *Let $m > 2$ and $u \in C_{loc}^{1,\alpha}(\Omega)$ be a weak solution to (1), with Morse index $i(u)$. If f satisfies (h_0) , then there exists a positive constant $C = C(\Omega, f)$ and $j_0 \in \{1, 2, \dots, 1 + i(u)\}$ such that*

$$\int_{\Omega \cap A_{j_0}} |\nabla u|^m + \int_{\Omega \cap A_{j_0}} f(x, u)u \leq C(1 + i(u))^\delta, \text{ where } \delta = \frac{m(m + \mu)}{\mu}. \quad (19)$$

Proof. Let $k > m$ will be defined later. It is easy to see that $\text{supp}(u\phi_j^m) \cap \text{supp}(u\phi_l^m) = \emptyset$, $\forall 1 \leq l \neq j \leq 1 + i(u)$, so the family $u\phi_j^m$, $1 \leq j \leq 1 + i(u)$ are mutually orthogonal for the quadratic form Λ_u (see (3)). Hence, by the definition of $i(u)$ defined by (5), there exists $j_0 \in \{1, 2, \dots, 1 + i(u)\}$ such that

$$(m - 1) \int_{\Omega} |\nabla u|^{m-2} |\nabla(u\phi_{j_0}^m)|^2 - \int_{\Omega} f'(x, u)u^2 \phi_{j_0}^{2m} \geq 0,$$

hence

$$\begin{aligned} \int_{\Omega} f'(x, u)u^2 \phi_{j_0}^{2m} - (m - 1) \int_{\Omega} |\nabla u|^{m-2} [|\nabla u|^2 \phi_{j_0}^{2m} \\ + u^2 |\nabla \phi_{j_0}^m|^2 + 2|u| |\nabla u| \phi_{j_0}^m |\nabla \phi_{j_0}^m|] \leq 0. \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega} f'(x, u)u^2 \phi_{j_0}^{2m} - (m - 1) \int_{\Omega} |\nabla u|^m \phi_{j_0}^{2m} \\ \leq m^2(m - 1) \int_{\Omega} u^2 |\nabla u|^{m-2} \phi_{j_0}^{2(m-2)+2} |\nabla \phi_{j_0}|^2 \\ + 2m(m - 1) \int_{\Omega} |u| |\nabla u|^{m-1} \phi_{j_0}^{2(m-1)+1} |\nabla \phi_{j_0}|. \end{aligned}$$

Applying Young's inequality and using (h_0) we derive that for any $\epsilon > 0$ there is $C_{\epsilon, m} > 0$ such that

$$\begin{aligned} (m - 1 + \mu) \int_{\Omega} f(x, u)u\phi_{j_0}^{2m} - (m - 1 + \epsilon) \int_{\Omega} |\nabla u|^m \phi_{j_0}^{2m} \\ \leq C_{\epsilon, m} \int_{\Omega} |u|^m \phi_{j_0}^m |\nabla \phi_{j_0}|^m. \end{aligned} \quad (20)$$

Multiplying now (1) by $u\phi_{j_0}^{2m}$ and integrating by parts, we have

$$\int_{\Omega} |\nabla u|^m \phi_{j_0}^{2m} - \int_{\Omega} f(x, u)u\phi_{j_0}^{2m} \leq 2m \int_{\Omega} |\nabla u|^{m-1} \phi_{j_0}^{2(m-1)+1} |u| |\nabla \phi_{j_0}|.$$

As above using Young's inequality, there holds

$$(1 - \epsilon) \int_{\Omega} |\nabla u|^m \phi_{j_0}^{2km} - \int_{\Omega} f(x, u)u\phi_{j_0}^{2km} \leq C_{\epsilon, m} \int_{\Omega} |u|^m \phi_{j_0}^m |\nabla \phi_{j_0}|^m. \quad (21)$$

Multiplying (21) by $(m - 1 + \mu')$ with $0 \leq \mu' \leq \mu$ and combining with (20), we obtain

$$\begin{aligned} & ((1 - \epsilon)(m - 1 + \mu') - (m - 1 + \epsilon)) \int_{\Omega} |\nabla u|^m \phi_{j_0}^{2m} + (\mu - \mu') \int_{\Omega} f(x, u) u \phi_{j_0}^{2m} \\ & \leq C_{\epsilon, m} \int_{\Omega} |u|^m \phi_{j_0}^m |\nabla \phi_{j_0}|^m. \end{aligned}$$

Choose $0 < \epsilon < \frac{\mu'}{m + \mu'}$ so that $(1 - \epsilon)(m - 1 + \mu') - (m - 1 + \epsilon) > 0$, it follows

$$\int_{\Omega} |\nabla u|^m \phi_{j_0}^{2m} + \int_{\Omega} f(x, u) u \phi_{j_0}^{2m} \leq C_{\epsilon, m} \int_{\Omega} |u|^m \phi_{j_0}^m |\nabla \phi_{j_0}|^m.$$

Now substituting in the last inequality, ϕ_{j_0} by $\phi_{j_0}^k$ (where $k > m$ will be defined later), then (18) implies

$$\begin{aligned} & (1 - \epsilon) \int_{\Omega} |\nabla u|^m \phi_{j_0}^{2km} - \int_{\Omega} f(x, u) u \phi_{j_0}^{2km} \\ & \leq C_{\epsilon, m, k} \left(\frac{1 + i(u)}{R_0} \right)^m \int_{\Omega} |u|^m \phi_{j_0}^{m(2k-1)}. \end{aligned}$$

Applying Young's inequality, we derive that for any $\epsilon > 0$ there is $C_{\epsilon, m, k} > 0$ such that

$$\begin{aligned} & \int_{\Omega} |\nabla u|^m \phi_{j_0}^{2km} + \int_{\Omega} f(x, u) u \phi_{j_0}^{2km} \leq C_{\epsilon, m, k} \left(\frac{1 + i(u)}{R_0} \right)^{m \cdot \frac{m+\mu}{\mu}} \\ & + \epsilon \int_{\Omega} |u|^{m+\mu} \phi_{j_0}^{(m+\mu)(2k-1)}. \end{aligned}$$

Choose $k = \frac{m+\mu}{2\mu}$ so that $2mk = (m + \mu)(2k - 1)$ and using (7), we deduce that

$$\int_{\Omega} |\nabla u|^m \phi_{j_0}^{2km} + \int_{\Omega} f(x, u) u \phi_{j_0}^{2km} \leq C_{\epsilon, m} \left(\frac{1 + i(u)}{R_0} \right)^{m \cdot \frac{m+\mu}{\mu}} + \epsilon \int_{\Omega} f(x, u) u \phi_{j_0}^{2km}.$$

For $\epsilon > 0$ sufficiently small, we obtain

$$\int_{\Omega} |\nabla u|^m \phi_{j_0}^{2km} + \int_{\Omega} f(x, u) u \phi_{j_0}^{2km} \leq C \left(\frac{1 + i(u)}{R_0} \right)^{\frac{m(m+\mu)}{\mu}}.$$

□

Now we turn to the proof of Theorem 1.1. Set $\rho := \frac{R_0}{40(i(u)+1)}$, so that $0 < \rho \leq \min\{1, \frac{b_{j_0} - a_{j_0}}{20}\}$, we denote by $A'_{j_0} := A_{a_{j_0} + \rho}^{b_{j_0} - \rho} \subset A_{j_0}$. We shall use Lemma 2.2 with an appropriate choice of the cut-off function $\psi = \psi_{j_0} \in C_c^1(B_{R_0}(y))$ satisfying

- $\psi_{j_0} \equiv 1$ if $|x - y| \leq a_{j_0} + \rho$, $\psi_{j_0}(x) \equiv 0$ if $|x - y| \geq b_{j_0} - \rho$;
- $\|\nabla \psi_{j_0}\|_{\infty} \leq \frac{C}{R_0}(1 + i(u))$,

where a_{j_0} and b_{j_0} are defined in (17). So applying Lemmas 2.2 with $\psi = \psi_{j_0}$ and $R = R_0$, we derive that

$$\int_{B_{R_0}(y) \cap \Omega} (f(x, u)u + |\nabla u|^m) \psi_{j_0} \leq C \left(1 + \int_{B_{R_0}(y) \cap \Omega} (|\nabla u|^m + |u|^m + f(x, u)u) |\nabla \psi_{j_0}| \right).$$

In view of Young's inequality, (h_0) and (7), the above inequality implies

$$\int_{B(y, a_{j_0}) \cap \Omega} (f(x, u)u + |\nabla u|^m) \leq C \left(1 + \int_{A'_{j_0} \cap \Omega} (|\nabla u|^m + f(x, u)u) |\nabla \psi_{j_0}| \right).$$

Thus, according to (19), we obtain

$$\int_{B(y, a_{j_0}) \cap \Omega} (f(x, u)u + |\nabla u|^m) \leq C(1 + i(u))^{\delta+1}.$$

By covering argument (see (10)), we get finally

$$\int_{\Omega} |\nabla u|^m + \int_{\Omega} f(x, u)u \leq C(i(u) + 1)^{\alpha}, \text{ where } \alpha = \delta + 1.$$

Therefore, Theorem 1.1 follows. $\square$

Proof of Theorem 1.2.

By Gagliardo–Nirenberg–Sobolev inequality, that's there exists a positive constant C dependent only on N such that

$$\|u\|_{L^{\frac{mN}{N-m}}(\Omega)} \leq C\|u\|_{W_0^{1,m}(\Omega)}, \quad \forall u \in W_0^{1,m}(\Omega).$$

Thus, in view of (19), we have

$$\left(\int_{\Omega} |u|^{\frac{mN}{N-m}} \right)^{\frac{N-m}{N}} \leq C \int_{\Omega} |\nabla u|^m \leq c(1 + i(u))^{\frac{m(m+\mu)}{\mu} + 1}. \quad (22)$$

Multiplying (1) by $|u|^{q-1}u$, where $q > 1$ and integrating over Ω , we find

$$\frac{qm^m}{(q-1+m)^m} \int_{\Omega} \left| \nabla \left(|u|^{\frac{q-1}{m}} u \right) \right|^m = \int_{\Omega} f(x, u) |u|^{q-1} u.$$

Since $u \in C_{loc}^{1,\alpha}(\Omega)$ and $u = 0$ on $\partial\Omega$, then by Gagliardo–Nirenberg–Sobolev inequality, there exists $C > 0$ such that

$$\left(\int_{\Omega} |u|^{\frac{N(q-1+m)}{N-m}} \right)^{\frac{N-m}{N}} \leq C \int_{\Omega} |f(x, u)| |u|^q. \quad (23)$$

For $\epsilon > 0$, set

$$a_\epsilon = \epsilon(1 + i(u))^{-(\frac{m(m+\mu)}{\mu}+1)(\frac{m}{N-m})} \quad \text{and} \quad s_\epsilon = a_\epsilon^{-\frac{1}{\theta}}.$$

Hence we have

$$\frac{|u|^{\frac{Nm}{N-m}-(1+\theta)}}{|u|^{\frac{Nm}{N-m}-1}} = |u|^{-\theta} < a_\epsilon \quad \text{for } |u| > s_\epsilon. \quad (24)$$

Define $A_\epsilon = \{x \in \Omega; |u| > s_\epsilon\}$. From (8) and (24), there exists $C > 0$ such that

$$\begin{aligned} \int_{\Omega} |f(x, u)| |u|^q &\leq C \left(1 + \int_{A_\epsilon} |u|^{\frac{mN}{N-m}-(1+\theta)+q} + \int_{A_\epsilon^c} |u|^{\frac{mN}{N-m}-(1+\theta)+q} \right) \\ &\leq C \left(1 + a_\epsilon \int_{A_\epsilon} |u|^{\frac{Nm}{N-m}-1+q} + \int_{A_\epsilon^c} |u|^{\frac{mN}{N-m}-(1+\theta)+q} \right) \\ &\leq C \left(1 + a_\epsilon \int_{A_\epsilon} |u|^{\frac{m^2}{N-m}} |u|^{q+m-1} + \int_{A_\epsilon^c} |u|^{\frac{mN}{N-m}-(1+\theta)+q} \right). \end{aligned}$$

Hence, using Hölder's inequality, then according to the definitions of a_ϵ and s_ϵ , we get

$$\begin{aligned} \int_{\Omega} |f(x, u)| |u|^q &\leq C a_\epsilon \left(\int_{\Omega} |u|^{\frac{mN}{N-m}} \right)^{\frac{m}{N}} \cdot \left(\int_{\Omega} |u|^{q+m-1 \frac{N}{N-m}} \right)^{\frac{N-m}{N}} \\ &\quad + C s_\epsilon^{(\frac{mN}{N-m}-1+q)} + C \\ &\leq C \epsilon \left(\int_{\Omega} |u|^{(q+m-1) \frac{N}{N-m}} \right)^{\frac{N-m}{N}} \\ &\quad + C \epsilon (1 + i(u))^{\frac{m}{(N-m)\theta} (\frac{m(m+\mu)}{\mu}+1)(\frac{mN}{N-m}-1+q)} + C. \end{aligned}$$

From (23), we get

$$\begin{aligned} C(1 - \epsilon) \left(\int_{\Omega} |u|^{(q+m-1) \frac{N}{N-m}} \right)^{\frac{N-m}{N}} \\ \leq C \epsilon (1 + i(u))^{\frac{m}{(N-m)\theta} (\frac{m(m+\mu)}{\mu}+1)(\frac{mN}{N-m}-1+q)} + C. \end{aligned}$$

Choose $\epsilon > 0$ sufficiently small, we can conclude that

$$\left(\int_{\Omega} |u|^{(q+m-1) \frac{N}{N-m}} \right)^{\frac{N-m}{N}} \leq C \epsilon (1 + i(u))^{\frac{m}{(N-m)\theta} (\frac{m(m+\mu)}{\mu}+1)(\frac{mN}{N-m}-1+q)}. \quad (25)$$

On the other hand, by L^p -elliptic regularity, Sobolev embedding and the fact that $\frac{4N}{3} > N$, we find

$$\|u\|_{L^\infty}^{\frac{4N}{3}} \leq \int_{\Omega} |f(x, u)|^{\frac{4N}{3}} \leq C \left(\int_{\Omega} |u|^{\frac{4N}{3} \frac{N(1-\theta)+m(1+\theta)}{(N-m)(1+\theta)}} + 1 \right).$$

Choose $q = \frac{4N(1-\theta)+(m+3)(\theta+1)}{3(1+\theta)}$, so that $\frac{N(q+m-1)}{N-m} = \frac{4N}{3} \cdot \frac{N(1-\theta)+m(1+\theta)}{(N-m)(1+\theta)}$ and using (25), we obtain

$$\begin{aligned} \|u\|_{L^\infty}^{\frac{4N}{3} \cdot \frac{N-m}{N}} &\leq \left(\int_{\Omega} |u|^{(q+m-1) \frac{N}{N-m}} \right)^{\frac{N-m}{N}} \\ &\leq C \epsilon (1 + i(u))^{\frac{m}{(N-m)\theta} \left(\frac{m(m+\mu)}{\mu} + 1 \right) \left(\frac{mN}{(N-m)} - 1 + q \right)}. \end{aligned}$$

Therefore, there holds $\|u\|_{L^\infty(\Omega)} \leq C(1+i(u))^{\beta'}$, with β' given in Theorem 1.2. $\square$

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