Locally Strong Majorization and Commutativity in C^* -Algebras with Applications

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Abstract. In this paper, we introduce the notion of locally strong majorization for self-adjoint operators in a C^* -algebra. This allows, by using a Sherman type theorem for operators, to prove a Hardy–Littlewood–Pólya–Karamata like theorem. We show the role of commutativity of self-adjoint operators in such problems. We study operator inequalities of Moslehian–Micić–Kian, Mercer and Dragomir types.

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1. Preliminaries

Throughout the paper $\mathbb{B}(H)$ stands for the C^* -algebra of all bounded linear operators on a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. For self-adjoint operators $A, B \in \mathbb{B}(H)$, we write $B \leq A$ provided that $\langle Bh, h \rangle \leq \langle Ah, h \rangle$ for all $h \in H$. We say that A is *positive*, and write $0 \leq A$, if $0 \leq \langle Ah, h \rangle$ for all $h \in H$.

In the sequel, $\mathcal{A}$ and $\mathcal{B}$ are unital C^* -algebras with unities $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$, respectively. The symbol $\text{id}_{\mathcal{A}}$ stands for the identity map on $\mathcal{A}$. By $\mathcal{A}_{sa}$ and $\mathcal{B}_{sa}$ we denote the real spaces of self-adjoint operators in $\mathcal{A}$ and $\mathcal{B}$, respectively.

We say that a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ between C^* -algebras $\mathcal{A}$ and $\mathcal{B}$ is *positive*, in symbols $\Phi \geq 0$, if for $A \in \mathcal{A}_{sa}$,

$$0 \leq A \quad \text{implies} \quad 0 \leq \Phi(A).$$

For a self-adjoint operator $A \in \mathcal{A}$ we denote by $C^*(A)$ the C^* -algebra generated by A and $I_{\mathcal{A}}$.

The Gelfand map ϕ is a $*$ -isometric isomorphism between the algebra of continuous complex functions defined on the spectrum of $A \in \mathcal{A}_{sa}$ and the C^* -algebra $C^*(A)$ generated by A and $I_{\mathcal{A}}$ such that $\phi(f_0) = I_{\mathcal{A}}$ and $\phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for t running over the spectrum of A .

Let $A \in \mathcal{A}_{sa}$ and f be a continuous complex function defined on the spectrum of A . We define

$$f(A) = \phi(f).$$

A continuous function $f : J \rightarrow \mathbb{R}$ defined on an interval $J \subset \mathbb{R}$ is said to be *operator convex* if for any $\lambda \in [0, 1]$ and any self-adjoint operators A, B with spectra in J , it holds that

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B).$$

Definition 1 [14]. We say that an $m \times n$ matrix $\mathbf{S} = (S_{ij})$ with linear maps $S_{ij} : \mathcal{A} \rightarrow \mathcal{B}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is *column stochastic* if $S_{ij} \geq 0$ for $i = 1, \dots, m$, $j = 1, \dots, n$, and $\sum_{i=1}^m S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{B}}$ for $j = 1, \dots, n$.

We say that the matrix $\mathbf{S} = (S_{ij})$ is *row stochastic* if $S_{ij} \geq 0$ for $i = 1, \dots, m$, $j = 1, \dots, n$, and $\sum_{j=1}^n S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{B}}$ for $i = 1, \dots, m$.

Definition 2 [14]. Let $\mathcal{A} \subset \mathcal{B}$ and $m = n$.

An $m \times m$ matrix $\mathbf{S} = (S_{ij})$ is said to be *strongly row stochastic*, if $S_{ij} \geq 0$ for $i, j = 1, \dots, m$, and $\sum_{j=1}^m S_{ij} = \text{id}_{\mathcal{A}}$ for $i = 1, \dots, m$, where $\text{id}_{\mathcal{A}}$ is the identity map on $\mathcal{A}$.

An $m \times m$ matrix $\mathbf{S} = (S_{ij})$ is called *strongly doubly stochastic*, if $\mathbf{S}$ is column stochastic and strongly row stochastic, i.e., $S_{ij} \geq 0$ for $i, j = 1, \dots, m$, and $\sum_{i=1}^m S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{A}}$ for $j = 1, \dots, m$, and $\sum_{j=1}^m S_{ij} = \text{id}_{\mathcal{A}}$ for $i = 1, \dots, m$.

Definition 3 [14]. We say that an n -tuple $\mathbf{B} = (B_1, B_2, \dots, B_n)$ with operators $B_j \in \mathcal{B}_{sa}$, $j = 1, \dots, n$, is *right pre-majorized* by an m -tuple $\mathbf{A} = (A_1, A_2, \dots, A_m)$ with operators $A_i \in \mathcal{A}_{sa}$, $i = 1, \dots, m$, written as $\mathbf{B} \prec_{rp} \mathbf{A}$, if there exists an $m \times n$ column stochastic matrix $\mathbf{S} = (S_{ij})$ such that

$$B_j = S_{1j}(A_1) + S_{2j}(A_2) + \dots + S_{mj}(A_m) \quad \text{for } j = 1, \dots, n \quad (1)$$

(see [13], cf. [3, 15]).

It is convenient to write

$$(B_1, B_2, \dots, B_n) \prec_{rp} (A_1, A_2, \dots, A_m) \quad \text{by } \mathbf{S}$$

in place of (1).

Definition 4 [14]. We say that an m -tuple $\mathbf{B} = (B_1, B_2, \dots, B_m)$ with $B_j \in \mathcal{A}_{sa}$, $j = 1, \dots, m$, is *strongly majorized* by an m -tuple $\mathbf{A} = (A_1, A_2, \dots, A_m)$ with $A_i \in \mathcal{A}_{sa}$, $i = 1, \dots, m$, written as $\mathbf{B} \prec_{str} \mathbf{A}$, if there exists an $m \times m$ strongly doubly stochastic matrix $\mathbf{S} = (S_{ij})$ such that

$$B_j = S_{1j}(A_1) + S_{2j}(A_2) + \dots + S_{mj}(A_m) \quad \text{for } j = 1, \dots, m$$

(see [13], cf. [3, 15]).

The following result is a precursor for a Hardy–Littlewood–Pólya–Karamata (HLPK) type theorem for (pre-)majorization on tuples of operators in C^* -algebras (cf. [15, 16]). (See [8, 9] for Majorization Theorem for scalar tuples in $\mathbb{R}^n$.)

Theorem A ([14, Corollary 3.4]). *Let $f : J \rightarrow \mathbb{R}$ be an operator convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -algebras. Suppose that A_i and B_j are self-adjoint operators in $\mathcal{A}$ and $\mathcal{B}$, respectively, with spectra in J for $i = 1, \dots, m$, $j = 1, \dots, n$.*

Assume that

$$(B_1, B_2, \dots, B_n) \prec_{rp} (A_1, A_2, \dots, A_m) \text{ by } \mathbf{S} \quad (2)$$

for some $m \times n$ column stochastic matrix $\mathbf{S} = (S_{ij})$ with positive linear maps $S_{ij} : \mathcal{A} \rightarrow \mathcal{B}$.

Then the following inequality holds

$$\sum_{j=1}^n f(B_j) \leq \sum_{i=1}^m \Phi_i(f(A_i)), \quad (3)$$

where $\Phi_i = \sum_{j=1}^n S_{ij}$ is the i -th row sum of the matrix $\mathbf{S}$.

Theorem A contains a collection of important operator inequalities, as special cases. In order to explain this phenomenon, in the rest of this expository section, we demonstrate the usefulness of a unified deriving method for some operator inequalities.

- (I) Let $Z \in \mathcal{A}$ be a contraction (i.e., $Z^*Z \leq I_{\mathcal{A}}$). Let $f : [a, b] \rightarrow \mathbb{R}$ be an operator convex function with the property $0 \in [a, b]$ and $f(0) \leq 0$. Let $A \in \mathcal{A}$ be a self-adjoint operator with its spectrum contained in $[a, b]$. Then, by Hansen–Pedersen’s inequality (see [1, 5–7]),

$$f(Z^*AZ) \leq Z^*f(A)Z. \quad (4)$$

To establish (4) we consider the maps

$$S_{11} = Z^*(\cdot)Z \quad \text{and} \quad S_{21} = (I_{\mathcal{A}} - Z^*Z)^{1/2}(\cdot)(I_{\mathcal{A}} - Z^*Z)^{1/2}$$

on $\mathcal{A}$. Then the map matrix

$$\mathbf{S} = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix}$$

is column stochastic, because S_{11} and S_{21} are positive linear maps on $\mathcal{A}$, and

$$S_{11}(I_{\mathcal{A}}) + S_{21}(I_{\mathcal{A}}) = I_{\mathcal{A}}.$$

Clearly, the row sums of $\mathbf{S}$ are S_{11} and S_{21} . Moreover,

$$Z^*AZ = S_{11}A + S_{21}0$$

with the zero operator $0 \in \mathcal{A}$.

In consequence, by making use of Theorem **A**, via $f(0) \leq 0$ and $S_{21}f(0) \leq S_{21}0 = 0$, one obtains

$$f(Z^*AZ) = f(S_{11}A + S_{21}0) \leq S_{11}f(A) + S_{21}f(0) \leq S_{11}f(A) + S_{21}0 = Z^*f(A)Z,$$

which gives (4).

(II) Now, assume that $f : [0, \infty) \rightarrow \mathbb{R}$ is an operator convex function on $[0, \infty)$ with the property $f(0) \leq 0$.

Let $0 < A, B \in \mathcal{A}$ and $Z = (A + B)^{-1/2}A^{1/2}$ and $W = (A + B)^{-1/2}B^{1/2}$ (see [1]). Then $A = Z^*(A + B)Z$ and $B = W^*(A + B)W$.

In addition, Z and W are contractions. In fact, we have $0 < A \leq A + B$ and therefore $(A + B)^{-1} \leq A^{-1}$. Hence $A^{1/2}(A + B)^{-1}A^{1/2} \leq A^{1/2}A^{-1}A^{1/2} = I_{\mathcal{A}}$. In other words,

$$Z^*Z = A^{1/2}(A + B)^{-1/2}(A + B)^{-1/2}A^{1/2} \leq I_{\mathcal{A}}.$$

Similarly,

$$W^*W = B^{1/2}(A + B)^{-1/2}(A + B)^{-1/2}B^{1/2} \leq I_{\mathcal{A}}.$$

We take the maps

$$S_{11} = Z^*(\cdot)Z \quad \text{and} \quad S_{21} = (I_{\mathcal{A}} - Z^*Z)^{1/2}(\cdot)(I_{\mathcal{A}} - Z^*Z)^{1/2} \quad \text{and} \quad S_{31} = S_{41} = 0$$

and

$$S_{12} = S_{22} = 0 \quad \text{and} \quad S_{32} = W^*(\cdot)W \quad \text{and} \quad S_{42} = (I_{\mathcal{A}} - W^*W)^{1/2}(\cdot)(I_{\mathcal{A}} - W^*W)^{1/2}$$

on $\mathcal{A}$. We put

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \\ S_{41} & S_{42} \end{pmatrix}.$$

Then $\mathbf{S}$ is column stochastic, and

$$A = S_{11}(A + B) + S_{21}0 + S_{31}(A + B) + S_{41}0$$

and

$$B = S_{12}(A + B) + S_{22}0 + S_{32}(A + B) + S_{42}0.$$

The row sums of $\mathbf{S}$ are S_{11} , S_{21} , S_{32} and S_{42} .

Finally, by Theorem **A** we get

$$f(A) + f(B) \leq S_{11}(f(A + B)) + S_{21}(f(0)) + S_{32}(f(A + B)) + S_{42}(f(0)). \quad (5)$$

Since $f(0) \leq 0$ and S_{21} and S_{42} are positive, we have $S_{21}(f(0)) \leq S_{21}(0) = 0$ and $S_{42}(f(0)) \leq S_{42}(0) = 0$. Therefore (5) implies

$$f(A) + f(B) \leq S_{11}(f(A + B)) + S_{32}(f(A + B)),$$

that is,

$$f(A) + f(B) \leq A^{1/2}(A+B)^{-1/2}f(A+B)(A+B)^{-1/2}A^{1/2} \\ + B^{1/2}(A+B)^{-1/2}f(A+B)(A+B)^{-1/2}B^{1/2}.$$

Likewise, if $f : [0, \infty) \rightarrow \mathbb{R}$ is an operator concave function on $[0, \infty)$ with $0 \leq f(0)$, then

$$f(A) + f(B) \geq A^{1/2}(A+B)^{-1/2}f(A+B)(A+B)^{-1/2}A^{1/2} \\ + B^{1/2}(A+B)^{-1/2}f(A+B)(A+B)^{-1/2}B^{1/2}.$$

This inequality was first given in [17] (see also [1]).

(III) A characterization of operator convexity is as follows (see [12, Corollary 3.3], cf. [4, Theorem 3.1]). For a continuous function $f : J \rightarrow \mathbb{R}$ on an interval J , the following two conditions are equivalent:

- (i) f is an operator convex function on an interval J .
- (ii) It holds that

$$T(f(T^{-1}\Phi(A))) - \Psi f(B) \leq f(\Phi(A) - \Psi(B)) \quad (6)$$

for all self-adjoint operators $A, B \in \mathcal{A}$ with spectra $\sigma(A), \sigma(B), \sigma(\Phi(A) - \Psi(B))$ included in the interval J , and for all positive linear maps Φ, Ψ, T from $\mathcal{A}$ to $\mathcal{B}$ satisfying $\Phi(I_{\mathcal{A}}) - \Psi(I_{\mathcal{A}}) = I_{\mathcal{A}}$ and $\Phi(I_{\mathcal{A}}) = T(I_{\mathcal{A}})$ with invertible T and positive inverse T^{-1} .

Inequality (6) can be viewed as corollary to Theorem A. To see this, note that

$$T^{-1}\Phi(A) = T^{-1}\Psi(B) + T^{-1}(\Phi(A) - \Psi(B)).$$

In addition, the map matrix

$$\mathbf{S} = \begin{pmatrix} T^{-1}\Psi \\ T^{-1} \end{pmatrix}$$

is column stochastic with row sums $T^{-1}\Psi$ and T^{-1} . Indeed, both the maps are positive linear, and

$$T^{-1}\Psi(I_{\mathcal{A}}) + T^{-1}(I_{\mathcal{A}}) = T^{-1}(\Psi(I_{\mathcal{A}}) + I_{\mathcal{A}}) = T^{-1}(\Phi(I_{\mathcal{A}})) = I_{\mathcal{A}}.$$

In conclusion, by Theorem A, we get

$$f(T^{-1}\Phi(A)) \leq T^{-1}\Psi f(B) + T^{-1}f(\Phi(A) - \Psi(B)),$$

completing the proof of (6).

Here is a summary of the paper. Theorem A gives us a motivation to define a *locally strongly doubly stochastic matrix* as well as leads us to the notion of *locally strong majorization* for tuples of self-adjoint operators in a C^* -algebra (see Sect. 2). In this context, we prove a Hardy–Littlewood–Pólya–Karamata type theorem for such kind of majorization (see Theorem 1). Next, we employ the property of commutativity of self-adjoint operators to establish certain generalizations of a Moslehian–Micić–Kian type inequality [11] (see Sect. 3). This allows to replace the standard assumption $mI_{\mathcal{A}} \leq C \leq MI_{\mathcal{A}}$

with real scalars $m < M$ by $A \leq C \leq B$ with mutually commuting operators A, B, C . We also study some operator inequalities of Mercer and Dragomir [2] types.

2. Locally Strong Majorization for Tuples of Self-Adjoint Operators

We start this section by introducing some relevant notions.

Definition 5. Let $\mathcal{A} \subset \mathcal{B}$. Let $\mathbf{A} = (A_1, A_2, \dots, A_m)$ with operators $A_i \in \mathcal{A}_{sa}$, $i = 1, \dots, m$.

We say that an $m \times n$ matrix $\mathbf{S} = (S_{ij})$ with linear maps $S_{ij} : \mathcal{A} \rightarrow \mathcal{B}$ is *locally strongly row stochastic at $\mathbf{A}$* if $S_{ij} \geq 0$ for $i = 1, \dots, m, j = 1, \dots, n$, and the restriction of the map $\sum_{j=1}^n S_{ij}$ to the C^* -algebra $C^*(A_i)$ is the identity map $\text{id}_{C^*(A_i)}$ on $C^*(A_i)$ for $i = 1, \dots, m$, i.e.,

$$\sum_{j=1}^n S_{ij}x = x \quad \text{for all } x \in C^*(A_i) \text{ and } i = 1, \dots, m. \quad (7)$$

Observe that for the $m \times n$ matrix $\mathbf{S} = (S_{ij})$ with linear maps $S_{ij} : \mathcal{A} \rightarrow \mathcal{B}$, condition (7) forces the inclusion $\mathcal{A} \subset \mathcal{B}$.

Definition 6. Let $\mathcal{A} \subset \mathcal{B}$ and $m = n$. Let $\mathbf{A} = (A_1, A_2, \dots, A_m)$ with operators $A_i \in \mathcal{A}_{sa}$, $i = 1, \dots, m$.

We say that an $m \times m$ matrix $\mathbf{S} = (S_{ij})$ with linear maps $S_{ij} : \mathcal{A} \rightarrow \mathcal{A}$ is *locally strongly doubly stochastic at $\mathbf{A}$* , if $\mathbf{S}$ is column stochastic and locally strongly row stochastic at $\mathbf{A}$.

In the last definition, it must hold that $m = n$ whenever $\mathcal{A} = \mathcal{B}$. In fact, if (7) is satisfied then for $x = I_{\mathcal{A}} \in C^*(A_i)$ with $i = 1, \dots, m$, we get

$$\sum_{j=1}^n S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{A}} \quad \text{for } i = 1, \dots, m,$$

which means that $\mathbf{S}$ is row stochastic. In consequence,

$$\sum_{i=1}^m \sum_{j=1}^n S_{ij}(I_{\mathcal{A}}) = \sum_{i=1}^m I_{\mathcal{A}} = mI_{\mathcal{A}}.$$

Moreover, in Definition 6 matrix $\mathbf{S}$ is column stochastic. So,

$$\sum_{i=1}^m S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{A}} \quad \text{for } j = 1, \dots, n.$$

For this reason,

$$\sum_{j=1}^n \sum_{i=1}^m S_{ij}(I_{\mathcal{A}}) = \sum_{j=1}^n I_{\mathcal{A}} = nI_{\mathcal{A}}.$$

Finally, $mI_{\mathcal{A}} = nI_{\mathcal{A}}$, which gives $m = n$, as claimed.

It is clear that the strong double stochasticity (SDS) of $\mathbf{S} = (S_{ij})$ implies its local strong double stochasticity (LSDS). Also, LSDS implies double stochasticity (DS) of $\mathbf{S}$. That is,

$$\text{SDS} \Rightarrow \text{LSDS} \Rightarrow \text{DS}. \quad (8)$$

On the other hand, the implications (8) can't be reversed, that is, in general,

$$\text{DS} \not\Rightarrow \text{LSDS} \not\Rightarrow \text{SDS}. \quad (9)$$

Remark 1. Under the meaning that a scalar c is identified with the operator cid , in the case of scalar tuples (in place of operator tuples) all properties SDS, LSDS and DS are equivalent mutually.

In the forthcoming example we present an instance of a locally strongly doubly stochastic matrix, and illustrate the validity the second statement of (9).

Example 1. Let $\mathcal{A} = \mathcal{B}$ and $m = n = 2$. Assume that $\mathcal{A}_1$ is a C^* -subalgebra of $\mathcal{A}$ with $I_{\mathcal{A}} \in \mathcal{A}_1$. Denote by Φ_1 the orthoprojector from $\mathcal{A}$ onto $\mathcal{A}_1$. Let $V_1 \oplus V_2 = \mathcal{A}_1$ be an orthogonal decomposition of $\mathcal{A}_1$ with orthogonal linear subspaces V_1 and V_2 . For $i = 1, 2$, denote by S_i the orthoprojector from $\mathcal{A}$ onto V_i . Then $S_1 + S_2 = \Phi_1$. Hence $S_1x + S_2x = \Phi_1x = x$ for any $x \in \mathcal{A}_1$. In particular, $S_1(I_{\mathcal{A}}) + S_2(I_{\mathcal{A}}) = I_{\mathcal{A}}$, since $I_{\mathcal{A}} \in \mathcal{A}_1$.

It is interesting that the map matrix

$$\mathbf{S} = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$$

is locally strongly doubly stochastic at $\mathbf{A} = (A, A)$ for any $A \in \mathcal{A}_1$. Obviously, the row sums of $\mathbf{S}$ are Φ_1 and Φ_1 . However, $\mathbf{S}$ is not strongly row stochastic, because $\Phi_1 \neq \text{id}_{\mathcal{A}}$ (whenever $\mathcal{A}_1 \subsetneq \mathcal{A}$). Thus $\mathbf{S}$ is not strongly doubly stochastic.

Definition 7. Let $\mathcal{A} \subset \mathcal{B}$ and $m = n$.

An n -tuple $\mathbf{B} = (B_1, B_2, \dots, B_m)$ with operators $B_j \in \mathcal{B}_{sa}$, $j = 1, \dots, m$, is said to be *locally strongly majorized* by $\mathbf{A} = (A_1, A_2, \dots, A_m)$ with operators $A_j \in \mathcal{A}_{sa}$, $j = 1, \dots, m$, written as $\mathbf{B} \prec_{loc} \mathbf{A}$, if

$$B_j = S_{1j}(A_1) + S_{2j}(A_2) + \dots + S_{mj}(A_m) \quad \text{for } j = 1, \dots, m \quad (10)$$

for some $m \times m$ locally strongly doubly stochastic matrix $\mathbf{S} = (S_{ij})$ at $\mathbf{A}$.

Usually, in place of (10) we write

$$(B_1, B_2, \dots, B_m) \prec_{loc} (A_1, A_2, \dots, A_m) \quad \text{by } \mathbf{S}.$$

Remark 2. It is interesting that the relation of local majorization

$$(B_1, B_2, \dots, B_m) \prec_{loc} (A_1, A_2, \dots, A_m) \quad (11)$$

implies the equality

$$B_1 + \cdots + B_m = A_1 + \cdots + A_m. \quad (12)$$

In other words, (12) is a necessary condition for (11) to hold.

In fact, (11) says that

$$B_j = \sum_{i=1}^m S_{ij}(A_i) \quad \text{for } j = 1, \dots, m$$

for some $m \times m$ locally strongly doubly stochastic matrix $\mathbf{S} = (S_{ij})$ at $\mathbf{A} = (A_1, \dots, A_m)$.

Therefore

$$\sum_{j=1}^m B_j = \sum_{j=1}^m \sum_{i=1}^m S_{ij}(A_i) = \sum_{i=1}^m \left(\sum_{j=1}^m S_{ij} \right) (A_i). \quad (13)$$

Moreover, for each $i = 1, \dots, m$, $\Phi_i = \sum_{j=1}^m S_{ij}$ is the i th row sum of $\mathbf{S}$. In consequence,

$$\left(\sum_{j=1}^m S_{ij} \right) (A_i) = \Phi_i(A_i) = A_i \quad (14)$$

[see (7)]. Now, by combining (13) and (14) we obtain (12).

As shown in [14], the strong majorization $(B_1, \dots, B_m) = (A_1, \dots, A_m)\mathbf{S}$ (with the aid of a strongly doubly stochastic matrix $\mathbf{S}$) is a sufficient condition for HLPK inequality (16) to hold for operators. However, usual majorization (based on a doubly stochastic matrix $\mathbf{S}$) need not be sufficient for operators. Therefore, a motivation for introducing the notion of local majorization is to ensure the validity of the HLPK inequality (16) in a wide context.

Theorem 1 (Local majorization theorem for operators). *Let $f : J \rightarrow \mathbb{R}$ be an operator convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -algebras with $\mathcal{A} \subset \mathcal{B}$ and $m = n$. Suppose that A_i and B_j are self-adjoint operators in $\mathcal{A}$ and $\mathcal{B}$, respectively, with spectra in J for $i, j = 1, \dots, m$.*

If

$$(B_1, B_2, \dots, B_m) \prec_{loc} (A_1, A_2, \dots, A_m), \quad (15)$$

then the following HLPK type inequality holds

$$\sum_{j=1}^m f(B_j) \leq \sum_{i=1}^m f(A_i). \quad (16)$$

Proof. It follows from hypotheses and definitions that

$$(B_1, B_2, \dots, B_m) \prec_{rp} (A_1, A_2, \dots, A_m) \quad \text{by } \mathbf{S} \quad (17)$$

for some $m \times m$ locally strongly doubly stochastic matrix $\mathbf{S} = (S_{ij})$ with positive linear maps $S_{ij} : \mathcal{A} \rightarrow \mathcal{B}$.

We employ the positive linear maps $\Phi_i : \mathcal{A} \rightarrow \mathcal{B}$, $i = 1, \dots, m$, defined by

$$\Phi_i = \sum_{j=1}^m S_{ij}. \quad (18)$$

Obviously, Φ_i is the i -th row sum of $\mathbf{S}$.

By Theorem A, the following inequality holds

$$\sum_{j=1}^m f(B_j) \leq \sum_{i=1}^m \Phi_i(f(A_i)). \quad (19)$$

On the other hand, by definition,

$$f(A_i) \in C^*(A_i) \quad \text{for } i = 1, \dots, m.$$

Therefore (7) and (18) give

$$\Phi_i(f(A_i)) = \sum_{j=1}^m S_{ij}f(A_i) = f(A_i) \quad \text{for } i = 1, \dots, m. \quad (20)$$

Now, combining (19) and (20) yields (16), completing the proof. $\square$

Remark 3. It is evident that

$$(B_1, B_2, \dots, B_m) \prec_{str} (A_1, A_2, \dots, A_m) \Rightarrow (B_1, B_2, \dots, B_m) \prec_{loc} (A_1, A_2, \dots, A_m). \quad (21)$$

Therefore the above Theorem 1 is an extension of [14, Theorem 4.1].

In the interest of convenience, we now introduce the notion of *right multiplication* $(\cdot)_r$ for tuples (matrices) of operators and maps.

Let $\mathbf{S} = \begin{pmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \cdots & S_{mn} \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} T_{11} & \cdots & T_{1k} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nk} \end{pmatrix}$. We define

$$\mathbf{S} \cdot_r \mathbf{T} = \begin{pmatrix} R_{11} & \cdots & R_{1k} \\ \vdots & \ddots & \vdots \\ R_{m1} & \cdots & R_{mk} \end{pmatrix} = (R_{il})_{i=1, \dots, m; l=1, \dots, k},$$

where

$$R_{il} = \sum_{j=1}^n T_{jl} S_{ij} \quad \text{for } i = 1, \dots, m; l = 1, \dots, k.$$

We continue studying special cases of Theorem 1.

The following is an extension of [14, Corollary 5.1] with a more conceptual proof.

Corollary 1 (cf. [11, Theorem 2.1]) *Let $f : J \rightarrow \mathbb{R}$ be an operator convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ be a unital C^* -algebra. Suppose that A_i and B_j are self-adjoint operators in $\mathcal{A}$ with spectra in J for $i, j = 1, 2$.*

If

$$(B_1, B_2) \prec_{loc} (A_1, A_2) \quad (22)$$

then

$$f(\Psi(B_1)) + f(\Psi(B_2)) \leq \Psi(f(A_1)) + \Psi(f(A_2)), \quad (23)$$

where $\Psi : \mathcal{A} \rightarrow \mathcal{A}$ is an unital positive linear map.

Proof. In light of (22), there exists a 2×2 locally strongly doubly stochastic matrix $\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ at $\mathbf{A} = (A_1, A_2)$ with $S_{ij} : \mathcal{A} \rightarrow \mathcal{A}$, $i, j = 1, 2$, such that

$$(B_1, B_2) = (A_1, A_2) \cdot_r \mathbf{S}. \quad (24)$$

We take

$$\mathbf{S}_0 = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}.$$

On account of (24) one has

$$(\Psi(B_1), \Psi(B_2)) = (B_1, B_2) \cdot_r \mathbf{S}_0 = ((A_1, A_2) \cdot_r \mathbf{S}) \cdot_r \mathbf{S}_0 = (A_1, A_2) \cdot_r (\mathbf{S} \cdot_r \mathbf{S}_0), \quad (25)$$

where

$$\mathbf{S} \cdot_r \mathbf{S}_0 = \begin{pmatrix} \Psi S_{11}, & \Psi S_{12} \\ \Psi S_{21}, & \Psi S_{22} \end{pmatrix}$$

is column stochastic, because Ψ is unital and $\mathbf{S}$ is locally strongly doubly (hence, column) stochastic.

In other words, by (25),

$$(\Psi(B_1), \Psi(B_2)) \prec_{rp} (A_1, A_2) \quad \text{by } \mathbf{S} \cdot_r \mathbf{S}_0. \quad (26)$$

On the other hand, for $i = 1, 2$ the i th row sum of $\mathbf{S} \cdot_r \mathbf{S}_0$ is

$$\Phi_i = \Psi S_{i1} + \Psi S_{i2} = \Psi(S_{i1} + S_{i2}).$$

Therefore, by Theorem A,

$$f(\Psi(B_1)) + f(\Psi(B_2)) \leq \Phi_1 f(A_1) + \Phi_2 f(A_2). \quad (27)$$

The restriction to $C^*(A_i)$ of Φ_i is

$$\Phi_i|_{C^*(A_i)} = \Psi(S_{i1} + S_{i2})|_{C^*(A_i)} = \Psi \operatorname{id}_{C^*(A_i)},$$

because $\mathbf{S}$ is locally strongly doubly stochastic.

However, by definition, $f(A_i) \in C^*(A_i)$ for $i = 1, 2$. So, we obtain

$$\Phi_i f(A_i) = \Phi_i|_{C^*(A_i)} f(A_i) = \Psi \operatorname{id}_{C^*(A_i)} f(A_i) = \Psi f(A_i).$$

Consequently, (27) reduces to

$$f(\Psi(B_1)) + f(\Psi(B_2)) \leq \Psi f(A_1) + \Psi f(A_2),$$

as claimed. $\square$

3. Inequalities for Pairs of Operators

In order to give further applications of Theorem A and Theorem 1, we shall employ commutativity of certain elements in C^* -algebras to construct the required maps S_{ij} of 2×2 column (resp. strongly doubly) stochastic matrix $\mathbf{S} = (S_{ij})$ leading to the notion of locally strong majorization.

Theorem 2. *Let $f : J \rightarrow \mathbb{R}$ be an operator convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ be a unital C^* -algebra. Suppose that A, B, C, D are self-adjoint operators in $\mathcal{A}$ with spectra in J .*

If A, B, C mutually commute and A, B, D mutually commute and

$$A \leq C \leq B \quad \text{and} \quad A \leq D \leq B \quad \text{with} \quad A < B,$$

then

$$\begin{aligned} f(\Psi(C)) + f(\Psi(D)) &\leq \Psi((B-A)^{-1}(2B-C-D)f(A)) \\ &\quad + \Psi((B-A)^{-1}(C+D-2A)f(B)), \end{aligned} \quad (28)$$

where $\Psi : \mathcal{A} \rightarrow \mathcal{A}$ is an unital positive linear map.

If in addition

$$A + B = C + D, \quad (29)$$

then (28) reduces to

$$f(\Psi(C)) + f(\Psi(D)) \leq \Psi(f(A)) + \Psi(f(B)). \quad (30)$$

Proof. Take any mutually commuting elements $A, B, C \in \mathcal{A}$. We shall show that

$$A < B \quad \text{and} \quad A \leq C \leq B \quad \text{imply} \quad C \prec_{rp} (A, B)$$

in the sense $C = S_1(A) + S_2(B)$ for some column stochastic matrix $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ with positive linear maps $S_i : \mathcal{A} \rightarrow \mathcal{A}$, $i = 1, 2$.

We derive

$$\begin{aligned} C(B-A) &= (B-C)A + (C-A)B \\ &= (B-C)^{1/2}A(B-C)^{1/2} + (C-A)^{1/2}B(C-A)^{1/2}. \end{aligned} \quad (31)$$

Hence, by $0 < B - A$,

$$\begin{aligned} C &= (B-C)^{1/2}A(B-C)^{1/2}(B-A)^{-1} + (C-A)^{1/2}B(C-A)^{1/2}(B-A)^{-1} \\ &= (B-A)^{-1/2}(B-C)^{1/2}A(B-C)^{1/2}(B-A)^{-1/2} \\ &\quad + (B-A)^{-1/2}(C-A)^{1/2}B(C-A)^{1/2}(B-A)^{-1/2}. \end{aligned} \quad (32)$$

We define positive linear maps $S_1 : \mathcal{A} \rightarrow \mathcal{A}$ and $S_2 : \mathcal{A} \rightarrow \mathcal{A}$ by

$$S_{11}(\cdot) = S_1(\cdot) = (B - A)^{-1/2}(B - C)^{1/2}(\cdot)(B - C)^{1/2}(B - A)^{-1/2}, \quad (33)$$

$$S_{21}(\cdot) = S_2(\cdot) = (B - A)^{-1/2}(C - A)^{1/2}(\cdot)(C - A)^{1/2}(B - A)^{-1/2}. \quad (34)$$

Equation (32) says that

$$C = S_1(A) + S_2(B). \quad (35)$$

Hence

$$\Psi C = \Psi S_1(A) + \Psi S_2(B). \quad (36)$$

Moreover, it is not hard to check that

$$S_1(I_{\mathcal{A}}) + S_2(I_{\mathcal{A}}) = I_{\mathcal{A}}, \quad (37)$$

which gives the column stochasticity of the matrix $\begin{pmatrix} \Psi S_1 \\ \Psi S_2 \end{pmatrix}$.

So, in light of Theorem A and (35)–(37), we see that

$$f(\Psi(C)) \leq \Phi_1 f(A) + \Phi_2 f(B), \quad (38)$$

where

$$\Phi_1 = \Psi S_1 \quad \text{and} \quad \Phi_2 = \Psi S_2.$$

Therefore,

$$\begin{aligned} \Phi_1 f(A) &= \Psi S_1 f(A) = \Psi \left((B - A)^{-1/2}(B - C)^{1/2} f(A)(B - C)^{1/2}(B - A)^{-1/2} \right) \\ &= \Psi((B - A)^{-1}(B - C)f(A)) \end{aligned}$$

and

$$\begin{aligned} \Phi_2 f(B) &= \Psi S_2 f(B) = \Psi \left((B - A)^{-1/2}(C - A)^{1/2} f(B)(C - A)^{1/2}(B - A)^{-1/2} \right) \\ &= \Psi((B - A)^{-1}(C - A)f(B)). \end{aligned}$$

Finally, inequality (38) takes the form

$$f(\Psi(C)) \leq \Psi((B - A)^{-1}(B - C)f(A)) + \Psi((B - A)^{-1}(C - A)f(B)). \quad (39)$$

In a similar manner, by introducing the maps

$$S_{12}(\cdot) = (B - A)^{-1/2}(B - D)^{1/2}(\cdot)(B - D)^{1/2}(B - A)^{-1/2}, \quad (40)$$

$$S_{22}(\cdot) = (B - A)^{-1/2}(D - A)^{1/2}(\cdot)(D - A)^{1/2}(B - A)^{-1/2}, \quad (41)$$

we show that

$$f(\Psi(D)) \leq \Psi((B - A)^{-1}(B - D)f(A)) + \Psi((B - A)^{-1}(D - A)f(B)). \quad (42)$$

By combining (39) and (42) we get

$$\begin{aligned}
 & f(\Psi(C)) + f(\Psi(D)) \\
 & \leq \Psi((B-A)^{-1}(B-C)f(A)) + \Psi((B-A)^{-1}(C-A)f(B)) \\
 & \quad + \Psi((B-A)^{-1}(B-D)f(A)) + \Psi((B-A)^{-1}(D-A)f(B)) \\
 & = \Psi((B-A)^{-1}(2B-C-D)f(A)) + \Psi((B-A)^{-1}(C+D-2A)f(B)).
 \end{aligned} \tag{43}$$

This completes the proof of (28).

If in addition

$$A + B = C + D,$$

then (43) ensures that

$$\begin{aligned}
 & f(\Psi(C)) + f(\Psi(D)) \\
 & \leq \Psi((B-A)^{-1}(2B-A-B)f(A)) + \Psi((B-A)^{-1}(A+B-2A)f(B)) \\
 & = \Psi(f(A)) + \Psi(f(B)),
 \end{aligned}$$

which proves (30). $\square$

We now present some consequences of Theorem 2.

Corollary 2. *Let $f : [m, M] \rightarrow \mathbb{R}$ be an operator convex function on interval $[m, M] \subset \mathbb{R}$ with $m < M$. Let $\mathcal{A}$ be a unital C^* -algebra. Suppose that C, D are self-adjoint operators in $\mathcal{A}$ with spectra in $[m, M]$.*

Then

$$f(\Psi(C)) + f(\Psi(D)) \leq \frac{f(m)}{M-m} \Psi(2MI_{\mathcal{A}} - C - D) + \frac{f(M)}{M-m} \Psi(C + D - 2mI_{\mathcal{A}}). \tag{44}$$

If in addition

$$(m + M)I_{\mathcal{A}} = C + D$$

then (44) reduces to

$$f(\Psi(C)) + f(\Psi(D)) \leq (f(m) + f(M))I_{\mathcal{A}}. \tag{45}$$

Proof. Apply Theorem 2 for $A = mI_{\mathcal{A}}$ and $B = MI_{\mathcal{A}}$. $\square$

Corollary 3. *Let $f : J \rightarrow \mathbb{R}$ be an operator convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ be a unital C^* -algebra. Suppose that A, B, C, D are self-adjoint operators in $\mathcal{A}$ with spectra in J .*

If A, B, C mutually commute and A, B, D mutually commute and

$$A \leq C \leq B \quad \text{and} \quad A \leq D \leq B \quad \text{with} \quad A < B,$$

then

$$f(C) + f(D) \leq (B-A)^{-1}(2B-C-D)f(A) + (B-A)^{-1}(C+D-2A)f(B). \tag{46}$$

In particular, if A, B, C mutually commute and $A \leq C \leq B$ with $A < B$ then the following Mercer like inequality holds

$$f(A + B - C) \leq f(A) + f(B) - f(C). \quad (47)$$

Proof. Apply Theorem 2 for $\Psi = \text{id}_{\mathcal{A}}$. Observe that for $D = A + B - C$, the requirement $A \leq D \leq B$ is satisfied as a consequence of the assumption $A \leq C \leq B$. The mutual commutativity of A, B, D follows from the mutual commutativity of A, B, C . Next, use (46) for $D = A + B - C$. $\square$

Corollary 4 (cf. [10, Theorem 1]) *Let $f : [m, M] \rightarrow \mathbb{R}$ be an operator convex function on interval $[m, M] \subset \mathbb{R}$ with $m < M$. Let $\mathcal{A}$ be a unital C^* -algebra. Suppose that C is a self-adjoint operator in $\mathcal{A}$ with spectrum in $[m, M]$.*

Then the following Mercer inequality holds

$$f(mI_{\mathcal{A}} + MI_{\mathcal{A}} - C) \leq (f(m) + f(M))I_{\mathcal{A}} - f(C). \quad (48)$$

Proof. Since the spectrum of C lies in $[m, M]$, we are guaranteed that $mI_{\mathcal{A}} \leq C \leq MI_{\mathcal{A}}$. By setting

$$A = mI_{\mathcal{A}} \quad \text{and} \quad B = MI_{\mathcal{A}},$$

we are sure that A, B, C mutually commute.

So, it is sufficient to utilize Corollary 3 and inequality (47). $\square$

It was proved in [2, Corollary 2] that if $f : J \rightarrow \mathbb{R}$ is an operator convex function on interval $J \subset \mathbb{R}$ and A, B are two distinct self-adjoint operators with spectra in J , and $C \in [A, B] = \{tA + (1-t)B : t \in [0, 1]\}$, then

$$f(C) + f\left(\frac{A+B}{2}\right) \leq f\left(\frac{A+C}{2}\right) + f\left(\frac{C+B}{2}\right). \quad (49)$$

We supplement this result as follows.

Corollary 5. *Let $f : J \rightarrow \mathbb{R}$ be an operator convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ be a unital C^* -algebra. Suppose that A, B, C are self-adjoint operators in $\mathcal{A}$ with spectra in J .*

If A, B, C mutually commute and $A \leq C \leq B$ with $A < B$, then

$$f(\Psi(C)) + f\left(\Psi\left(\frac{A+B}{2}\right)\right) \leq \Psi\left(f\left(\frac{A+C}{2}\right)\right) + \Psi\left(f\left(\frac{C+B}{2}\right)\right), \quad (50)$$

where $\Psi : \mathcal{A} \rightarrow \mathcal{A}$ is a unital positive linear map.

In particular, if $\Psi = \text{id}_{\mathcal{A}}$ then inequality (49) holds.

Proof. From $A \leq C \leq B$ we have $\frac{1}{2}A \leq \frac{1}{2}C \leq \frac{1}{2}B$, and next $\frac{A+C}{2} \leq C \leq \frac{C+B}{2}$ and $\frac{A+C}{2} \leq \frac{A+B}{2} \leq \frac{C+B}{2}$. In addition,

$$C + \frac{A+B}{2} = \frac{A+C}{2} + \frac{C+B}{2}.$$

Furthermore, we deduce that $\frac{A+C}{2}, C, \frac{C+B}{2}$ mutually commute and $\frac{A+C}{2}, \frac{A+B}{2}, \frac{C+B}{2}$ mutually commute.

Therefore we infer from Theorem 2 that inequality (50) is valid.

In the case $\Psi = \text{id}_{\mathcal{A}}$ inequality (50) becomes (49), as required. $\square$

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