



Moment Functions on Affine Groups

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Abstract. Moments of probability measures on a hypergroup can be obtained from so-called (generalized) moment functions of a given order. The aim of this paper is to characterize generalized moment functions on a non-commutative affine group. We consider a locally compact group G and its compact subgroup K . First we recall the notion of the double coset space $G//K$ of a locally compact group G and introduce a hypergroup structure on it. We present the connection between K -spherical functions on G and exponentials on the double coset hypergroup $G//K$. The definition of the generalized moment functions and their connection to the spherical functions is discussed. We study an important class of double coset hypergroups: specifying K as a compact subgroup of the group of invertible linear transformations on a finitely dimensional linear space V we consider the affine group $\text{Aff } K$. Using the fact that in the finitely dimensional case $(\text{Aff } K, K)$ is a Gelfand pair we give a description of exponentials on the double coset hypergroup $\text{Aff } K//K$ in terms of K -spherical functions. Moreover, we give a general description of generalized moment functions on $\text{Aff } K$ and specific examples for $K = SO(n)$, and on the so-called $ax + b$ -group.

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1. Introduction

In this paper we shall consider the generalized moment functions on some type of hypergroups called *affine groups* (see [3] for details). By a *hypergroup* we mean a locally compact hypergroup. The identity element of the hypergroup K is denoted by o . For x in K the symbol δ_x denotes the point mass with

support $\{x\}$. In the sequel \mathbb{C} denotes the set of complex numbers. The classical monograph on hypergroups is [3]. A comprehensive discussion on the subject is in [14] and reference therein.

Let K be a hypergroup. We begin with recalling the notion of exponential, additive and m -sine functions on hypergroups which are strongly connected with the generalized moment functions. The non-identically zero continuous function $m : K \rightarrow \mathbb{C}$ is called an *exponential* on K if m satisfies $m(x * y) = m(x)m(y)$ for each x, y in K . The continuous function $a : K \rightarrow \mathbb{C}$ is called *additive function* if it satisfies $a(x * y) = a(x) + a(y)$ for each x, y in K . The description of exponentials and additive functions on some types of commutative hypergroups can be found in [14]. The continuous function $f : K \rightarrow \mathbb{C}$ is called an *m -sine function* if it satisfies

$$f(x * y) = f(x)m(y) + f(y)m(x) \quad (1)$$

for each x, y in K . The function f is called a *sine function* if f is an m -sine function for some exponential m . Clearly, every sine function f satisfies $f(o) = 0$. For a given exponential m all m -sine functions form a linear space. Obviously, $m \equiv 1$ is an exponential on any hypergroup, and 1-sine functions are exactly the additive functions. The description of sine functions on some types of hypergroups can be found in [5].

Let G be a locally compact group with identity e and K a compact subgroup with the normalized Haar measure ω . As K is unimodular ω is left and right invariant, and it is also inversion invariant. For each x in G we define the *double coset* of x as the set

$$KxK = \{kxl : k, l \in K\}.$$

Let

$$L := G//K = \{KxK : x \in G\}$$

be the set of all double cosets. We introduce a hypergroup structure on the set L of all double cosets: the topology of L is the factor topology, which is locally compact. The identity is the coset $K = KeK$ itself and the involution is defined by

$$(KxK)^\vee = Kx^{-1}K$$

for x in G . Finally, the convolution of δ_{KxK} and δ_{KyK} is defined by

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_{KxkyK} d\omega(k),$$

where x and y are in G . It is known that this gives a hypergroup structure on L (see [3], p. 12.), which is non-commutative, in general. If K is a normal subgroup, then L is isomorphic to the hypergroup arising from the factor group G/K .

In [8] representations of double coset hypergroups are investigated. Such representations can be canonically obtained from those of the group in question. Nevertheless, not every representation arises in this way. However, our results show that on some affine groups, basic representing functions, like exponentials, additive functions and quadratic functions are closely related to the corresponding functions on the original group.

We note that continuous functions on L can be identified with those continuous functions on G which are K -invariant: $f(x) = f(kxl)$ for each x in G and k, l in K . Hence, for a continuous function $f : L \rightarrow \mathbb{C}$ the simplified – and somewhat loose – notation $f(x)$ can be used for the function value $f(KxK)$. Using this convention we can write for each continuous function $f : L \rightarrow \mathbb{C}$ and for each x, y in G :

$$f(x * y) = \int_K f(xky) d\omega(k).$$

The following theorem exhibits a close connection between exponentials on double coset hypergroups and spherical functions on locally compact groups. Following the terminology of [3] (see also [4]) we recall the concept of spherical functions. The continuous bounded K -invariant function $f : G \rightarrow \mathbb{C}$ is called a K -spherical function if $f(e) = 1$ and

$$\int_K f(xky) d\omega(k) = f(x)f(y) \quad (2)$$

holds for each x, y in G . A *generalized K -spherical function* on G is the same as above without the boundedness hypothesis. For the sake of simplicity in this paper we use the term *spherical function* for continuous functions satisfying (2) without the boundedness assumption. The following theorem, which is an immediate consequence of the previous considerations gives the link between spherical functions and exponentials of double coset hypergroups.

Theorem 1. *Let G be a locally compact group, and $K \subseteq G$ a compact subgroup of G . Then a nonzero continuous complex valued function m is a K -spherical function on G if and only if it is an exponential on the double coset hypergroup $G//K$. In particular, K -spherical functions on G can be identified with the multiplicative functions on $G//K$.*

Here a multiplicative function means an exponential map $m : G//K \rightarrow \mathbb{C}$ such that $m(e) = 1$. See [3, Def 1.4.32]

It is obvious, that if G is a locally compact Abelian group, and K is a compact subgroup of G , then K -spherical functions are exactly the exponentials of the (locally compact) Abelian group G/K .

Given the locally compact group G and the compact subgroup K the space $\mathcal{M}_c(G)$ of compactly supported regular complex Borel measures on G is identified with the topological dual of the locally convex topological vector space $\mathcal{C}(G)$ of all continuous complex valued functions on G when equipped

with the topology of uniform convergence on compact sets. The pairing between $\mathcal{C}(G)$ and $\mathcal{M}_c(G)$ will be denoted by $\langle \cdot, \cdot \rangle$, hence

$$\langle \mu, f \rangle = \int_G f d\mu$$

holds for each μ in $\mathcal{M}_c(G)$ and f in $\mathcal{C}(G)$. All K -invariant functions in $\mathcal{C}(G)$ form a closed subspace, which is identified with $\mathcal{C}(G//K)$, the locally convex topological vector space of all continuous complex valued functions on the double coset space $G//K$. The topological dual of $\mathcal{C}(G//K)$ is identified with the space $\mathcal{M}_c(G//K)$ of all K -invariant measures on G – the measure μ in $\mathcal{M}_c(G)$ is called K -invariant if

$$\langle \mu, f \rangle = \int_G \int_K \int_K f(kxl) d\omega(k) d\omega(l) d\mu(x)$$

holds for each f in $\mathcal{C}(G)$. Using the hypergroup structure on $L = G//K$ we see easily that $\mathcal{M}_c(G//K)$ is a complex unital topological algebra with unit δ_e , the point mass with support $\{e\}$. We say that (G, K) is a *Gelfand pair* if this algebra is commutative (see e.g. [4, 16]). It turns out that (G, K) is a Gelfand pair if and only if the double coset hypergroup L is commutative ([16]).

A sequence $(g_n)_{1 \leq n \leq N}$ of continuous K -invariant functions on G is a *generalized K -moment function sequence* if they satisfy

$$\int_K g_n(xky) d\omega(k) = \sum_{j=0}^n \binom{n}{j} g_j(x) g_{n-j}(y) \tag{3}$$

for $n = 0, 1, \dots, N$ and x, y in G . In fact, this is exactly the case if $(g_n)_{1 \leq n \leq N}$ is a generalized moment function sequence on the double coset hypergroup $G//K$. In particular, g_0 is a K -spherical function on G , or what is the same, g_0 is an exponential on $G//K$. In the case $K = \{e\}$ it is clear that this concept coincides with the concept of a *generalized moment function sequence*. In the case $g_0 \equiv 1$ the first order moment functions are exactly the additive functions, which, for general K , can be called *K -additive functions*. In the generalized moment function sequence $(g_n)_{1 \leq n \leq N}$ with $N \geq 1$ the function g_1 is a g_0 -sine function, according to the terminology introduced in [7] (see also [5]), which, in the general case can be called *K -sine functions* associated with the K -spherical function g_0 . Generalized moment functions and generalized moment function sequences have been described on different types of commutative hypergroups (see e.g. [11–14]). Nevertheless, non-commutative hypergroups have not yet been considered in this context.

2. Affine Groups

An important type of double coset hypergroups arises from the concept of affine groups. Let V be an n -dimensional vector space over the field \mathbb{K} and let $GL(V)$

denote the *general linear group* of V , the invertible linear transformations on V . For each subgroup H of $GL(V)$ we form the *semidirect product*

$$\text{Aff } H = H \ltimes V$$

in the following way: we equip the set $H \times V$ with the following multiplication:

$$(x, u) \cdot (y, v) = (x \cdot y, x \cdot v + u)$$

for each x, y in H and u, v in V . Here $x \cdot v$ is the image of v under the linear mapping x . Then $\text{Aff}(H)$ is a group with identity $(id, 0)$, where id is the identity operator and 0 is the zero of the vector space V . The inverse of (x, u) is

$$(x, u)^{-1} = (x^{-1}, -x^{-1} \cdot u)$$

for each x in H and u in V . We note that, in general, $\text{Aff } H$ is non-commutative, even if H is commutative. In any case V – as a commutative group – is isomorphic to the normal subgroup of $\text{Aff } H$ consisting of all elements of the form (id, u) with u in V the isomorphism provided by the mapping $u \mapsto (id, u)$. Indeed, we have

$$(x, u) \cdot (id, v) \cdot (x, u)^{-1} = (id, x \cdot v),$$

which proves that the image of V , which we identify with V , is normal. The set of all elements of the form $(x, 0)$ with x in H is a subgroup of $\text{Aff } H$ isomorphic to H , and it will be identified with H . Affine groups play an important role in geometry and physics. For instance, the *Poincaré group* $\text{Aff } O(1, 3)$ is the affine group of the *Lorentz group* $O(1, 3)$: $O(1, 3) \ltimes \mathbb{R}^{1,3}$, where $O(1, 3)$ is the *isometry group* of the real vector space $\mathbb{R}^{1,3} = \mathbb{R} \oplus \mathbb{R}^3$ equipped with the quadratic form

$$\langle v, w \rangle = \sum_{t=1}^p v_t w_t - \sum_{t=p+1}^{p+q} v_t w_t,$$

where $v = (v_1, \dots, v_p, v_{p+1}, \dots, v_{p+q})$ and $w = (w_1, \dots, w_p, w_{p+1}, \dots, w_{p+q})$. For more about affine groups and geometry see e.g. [2, 10].

In the case $V = \mathbb{K}^n$ we have $GL(V) = GL(\mathbb{K}, n)$, which is a matrix group. With the usual Euclidean topology V and $GL(V)$ are locally compact topological groups. If H is a subgroup of $GL(V)$ a function $f : V \rightarrow \mathbb{C}$ is called *H-invariant* if $f(h \cdot v) = f(v)$ holds for each h in H and v in V . Suppose that H is a closed subgroup of $GL(V)$ and K is a compact subgroup in H . Then K – identified with the set of all elements of the form $(x, 0)$ with x in K – is a compact subgroup in $H \ltimes V$. Let L denote the double coset hypergroup $\text{Aff } H // K$. As we have seen above K -invariant continuous functions on $\text{Aff } H$ can be identified with the continuous functions on L – we use the notation $f(x, u)$ for $f(K(x, u)K)$. Using this notation K -invariance means $f(kx, lu) = f(x, u)$ for each k, l in K , x in H and u in V . If $*$ denotes convolution in L then we have

$$f((x, u) * (y, v)) = \int_K f(xky, xkv + u) \, d\omega(k),$$

for each x, y in H and u, v in V , where ω is the normalized Haar measure on K .

A basic observation about affine groups is the following theorem.

Theorem 2. *Let V be a finite dimensional vector space and K a compact subgroup of $GL(V)$. Then $(\text{Aff } K, K)$ is a Gelfand pair.*

For the sake of completeness we present here the proof of this statement which is based on the following theorem of Gelfand (see e.g. [4, 17]).

Theorem 3. (Gelfand) *Let G be a locally compact group and K a compact subgroup of G . Suppose that there exists a continuous involutive automorphism $\theta : G \rightarrow G$ such that $\theta(x)$ is in $Kx^{-1}K$ holds for each x in G . Then (G, K) is a Gelfand pair.*

Proof (Proof of Theorem 2). We define $\theta : \text{Aff } K \rightarrow \text{Aff } K$ as

$$\theta(k, u) = (k, -u)$$

for each k in K and u in V . Then clearly θ is a continuous involutive automorphism of $\text{Aff } K$. On the other hand, we can write

$$(k, -u) = (k, 0) \cdot (k^{-1}, -k^{-1}u) \cdot (k, 0),$$

as it is easy to verify. □

Our purpose is to describe generalized moment functions on double coset hypergroups. We shall consider the case of double coset hypergroups of affine groups. Recall that a continuous K -invariant function on V satisfying

$$\int_K \varphi(k \cdot u + v) \, d\omega(k) = \varphi(u)\varphi(v) \tag{4}$$

for each u, v in V is called K -spherical function on V .

The next theorem is a generalization of [6, Theorem 2.1].

Theorem 4. *Let $(\text{Aff } H, K)$ be a Gelfand pair. Then a continuous K -invariant function $m : \text{Aff } H \rightarrow \mathbb{C}$ is an exponential on $\text{Aff } H$ if and only if it has the form*

$$m(x, u) = e(x)\varphi(u)$$

for each x in H and u in V , where $e : H \rightarrow \mathbb{C}$ and $\varphi : V \rightarrow \mathbb{C}$ are continuous K -invariant functions satisfying

$$\int_K e(xky)\varphi(xkv + u) \, d\omega(k) = e(x)e(y)\varphi(u)\varphi(v) \tag{5}$$

for each x, y in H and u, v in V . In particular, e is a K -spherical function on H :

$$\int_K e(xky) \, d\omega(k) = e(x)e(y) \text{ for each } x, y \in H,$$

and $\varphi : V \rightarrow \mathbb{C}$ is a K -spherical functions on V .

Proof. Observe that

$$(x, u) * (y, v) = \int_K \delta_{(x,u)} * \delta_{(k,0)} * \delta_{(y,v)} d\omega(k) = \int_K \delta_{(xky, xkv+u)} d\omega(k)$$

for each x, y in H and u, v in V , hence for every continuous K -invariant function $f : \text{Aff } H \rightarrow \mathbb{C}$ it follows

$$\begin{aligned} \int_K f(xky, xkv + u) d\omega(k) &= f((x, u) * (y, v)) \\ &= f((y, v) * (x, u)) = \int_K f(ykx, yku + v) d\omega(k) \end{aligned} \tag{6}$$

whenever x, y is in H and u, v is in V . In particular, putting $y = id$ and $u = 0$ in (6) we have

$$\int_K f(x, xkv) d\omega(k) = f(x, v) \tag{7}$$

for each K -invariant continuous function $f : L \rightarrow \mathbb{C}$ and x in H , v in V .

Now assume that m is an exponential on L . Then it satisfies

$$\int_K m(xky, xkv + u) d\omega(k) = m(x, u) \cdot m(y, v) \tag{8}$$

whenever x, y are in H and u, v are in V . For each k in K we have

$$m(x, 0)m(id, u) = m(x, 0) \cdot m(id, ku) = m(x, xku),$$

and, by integrating

$$m(x, 0)m(id, u) = \int_K m(x, xku) d\omega(k) = m(x, u), \tag{9}$$

by (7). Denoting $e(x) = m(x, 0)$ and putting $u = v = 0$ in (8) we have

$$\int_K e(xky) d\omega(k) = e(x)e(y)$$

as it was stated. Similarly, denoting $\varphi(u) = m(id, u)$ and putting $x = y = id$ in (8) we conclude

$$\int_K \varphi(kv + u) d\omega(k) = \varphi(u)\varphi(v).$$

Finally, substitution into (9) and (8) gives (5).

Conversely, if m has the desired form then (5) implies that m is exponential. □

Lemma 1. *Let V be a finite dimensional \mathbb{K} -vector space and K a compact subgroup of $GL(V)$. Then a continuous function $f : \text{Aff } K \rightarrow \mathbb{C}$ is K -invariant if and only if it has the form $f(k, u) = \varphi(u)$ for each k in K and u in V , where $\varphi : V \rightarrow \mathbb{C}$ is a continuous K -invariant function, i.e. $\varphi(u) = \varphi(k \cdot u)$ holds for each k in K and u in V .*

Functions φ with this property are called *K-radial functions* on V .

Proof. By the K -invariance of f we have for each (k, u) in $\text{Aff } K$ and l in K

$$f(k, u) = f[(l^{-1}, 0) \cdot (id, l \cdot u) \cdot (lk, 0)] = f(id, l \cdot u),$$

as $(l^{-1}, 0)$ and $(lk, 0)$ are in K . □

Corollary 1. *Let V be a finite dimensional \mathbb{K} -vector space and K a compact subgroup of $GL(V)$. Then a continuous K -invariant function $m : \text{Aff } K \rightarrow \mathbb{C}$ is an exponential on $\text{Aff } K/K$ if and only if it has the form*

$$m(k, u) = \varphi(u)$$

for each k in K and u in V , where $\varphi : V \rightarrow \mathbb{C}$ is a K -spherical function.

As an example we consider the case $V = \mathbb{R}^n$ and $K = SO(n)$, the *special orthogonal group* on \mathbb{R}^n . It is known (see e.g. [17]) that the $SO(n)$ -spherical functions on \mathbb{R}^n are exactly the radial eigenfunctions of the Laplacian in \mathbb{R}^n , hence they can be described in terms of the Bessel functions J_λ in the form

$$\varphi_\lambda(u) = J_\lambda(\|u\|)$$

for each u in \mathbb{R}^n and complex number λ , where

$$J_\lambda(r) = \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{\infty} \frac{\lambda^k}{k! \Gamma(k + \frac{n}{2})} \left(\frac{r}{2}\right)^{2k}.$$

It follows that the function $(k, u, \lambda) \mapsto \Phi(k, u, \lambda)$ defined by

$$\Phi(k, u, \lambda) = J_\lambda(\|u\|)$$

represents an *exponential family* for the double coset hypergroup

$$L = \text{Aff } SO(n) // SO(n).$$

In particular, for $n = 3$ we have

$$\Phi(k, u, \lambda) = \frac{\sinh \lambda \|u\|}{\lambda \|u\|}$$

whenever u is in \mathbb{R}^3 and λ is in \mathbb{C} , where in the case $\lambda \|u\| = 0$ we take the limit, which gives the spherical function identically 1.

Suppose that V is a finite dimensional \mathbb{K} -vector space, H is a closed subgroup of $GL(V)$ and K is a compact subgroup of H with the normalized Haar measure ω . Specializing the concept of generalized K -moment function sequence to the case $G = H \times V$ with the compact subgroup $\{(k, 0) : k \in K\} \cong K$ we can rewrite the system of Eq. (3) in the following way:

$$\int_K g_n(xky, xk \cdot v + u) d\omega(k) = \sum_{j=0}^n \binom{n}{j} g_j(x, u) g_{n-j}(y, v) \tag{10}$$

which holds for $n = 0, 1, \dots, N$ and for each $(x, u), (y, v)$ in G . Here we suppose that the functions $g_n : G \rightarrow \mathbb{C}$ are continuous and K -invariant for

$n = 0, 1, \dots, N$. In terms of the double coset hypergroup $L = G//K$ this is exactly the system of Eq. (3).

In the case $H = K$ we have the following theorem.

Theorem 5. *Let V be a finite dimensional \mathbb{K} -vector space and let K be a compact subgroup of $GL(V)$. Then a sequence $(g_n)_{0 \leq n \leq N}$ of K -invariant continuous complex functions on $\text{Aff } K$ is a generalized moment function sequence if and only if it has the form*

$$g_n(x, u) = \varphi_n(u) \quad (n = 0, 1, \dots, N) \tag{11}$$

for each (x, u) in $\text{Aff } K$, where $\varphi_n : V \rightarrow \mathbb{C}$ is continuous and K -invariant, further we have

$$\int_K \varphi_n(k \cdot u + v) d\omega(k) = \sum_{j=0}^n \binom{n}{j} \varphi_j(u) \varphi_{n-j}(v) \tag{12}$$

for $n = 0, 1, \dots, N$ and for each u, v in V .

Proof. By Lemma 1, we have that $g_n(x, u) = \varphi_n(u)$ holds for each (x, u) in $\text{Aff } K$, where $\varphi_n : V \rightarrow \mathbb{C}$ is continuous and K -invariant ($n = 0, 1, \dots, N$). Substitution into (10) gives

$$\int_K \varphi_n(xk \cdot v + u) d\omega(k) = \sum_{j=0}^n \binom{n}{j} \varphi_j(u) \varphi_{n-j}(v) \tag{13}$$

for each x in K and u, v in V . By the translation invariance of ω and interchanging u and v , we obtain the system (12).

The converse statement follows by direct calculation. □

In general, it is not easy to describe all solutions of the functional equation system (12). In the next section we shall give an example where this is possible. Now we are going to use the form of K -spherical functions for a certain compact subgroup K of $GL(\mathbb{R}^n)$. More about spherical functions on Euclidean spaces from the viewpoint of Riemannian symmetric spaces can be found in [18] and reference therein. Using results from [18] one can see that similar examples can be performed using any compact subgroup of $O(n)$ which is transitive on the spheres around the origin. We restrict ourselves to the case of $V = \mathbb{R}^n$ and $K = SO(n)$. We have seen above that the K -spherical functions, that is the generalized K -moment functions can be expressed in terms of the Bessel functions in the form

$$\varphi_0(u, \lambda) = J_\lambda(\|u\|)$$

for each u in \mathbb{R}^n , where λ is a complex number. On the other hand, by Theorem 3.4 and Corollary 2 in [15], it follows that φ_n is a linear combination of the derivatives of $\lambda \mapsto \varphi_0(u, \lambda)$ up to the n -th order. In other words, we have

Theorem 6. *Let N be a natural number and let $\varphi_l : \mathbb{R}^n \rightarrow \mathbb{C}$ ($l = 0, 1, \dots, N$) be a generalized $SO(n)$ -moment function sequence on the double coset hypergroup $\text{Aff } SO(n)/SO(n)$. Then for each u in \mathbb{R}^n we have*

$$\varphi_0(u) = J_\lambda(\|u\|),$$

and

$$\varphi_l(u) = \sum_{j=1}^l c_{l,j} \frac{d^j}{d\lambda^j} J_\lambda(\|u\|),$$

where $\lambda, c_{l,j}$ are complex numbers for $l = 1, 2, \dots, N, j = 1, 2, \dots, l$.

The particular value of the coefficients $c_{l,j}$ can be determined by substitution into (13).

3. An Example

Now we are going to give an explicit example of solutions of (12). We consider the following group G : it is the multiplicative group of matrices of the form

$$\begin{pmatrix} x & u \\ 0 & 1 \end{pmatrix}$$

where x, u are complex numbers with $x \neq 0$. All these matrices form a subgroup of $GL(2, \mathbb{C})$ which can be identified with a subset of \mathbb{C}^2 and it is a locally compact topological group G when equipped with the topology inherited from \mathbb{C}^2 . As we have

$$\begin{pmatrix} x & u \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xy & xv + u \\ 0 & 1 \end{pmatrix},$$

where x, y are nonzero complex numbers and u, v are in \mathbb{C} , we can describe the group operation on the set $G = \{(x, u) : x, u \in \mathbb{C}, x \neq 0\}$ in the following way:

$$(x, u) \cdot (y, v) = (xy, xv + u)$$

where (x, u) and (y, v) are in G . Let K denote the set of all matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

where z is a complex number with $|z| = 1$. Then K is topologically isomorphic to the complex unit circle group with multiplication. Clearly, K is a compact subgroup of G . Finally, G is topologically isomorphic to the affine group $\text{Aff}(K) = K \rtimes \mathbb{C}$. For more about this group see e.g. [9, p. 201] or [5].

The normalized Haar measure on K is given by

$$\int_K \varphi(u, 0) du = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}, 0) dt$$

for each continuous function $\varphi : K \rightarrow \mathbb{C}$. It is easy to check that K is not a normal subgroup, hence the hypergroup structure on the double coset space $G//K$ is not induced by a group operation. The function $f : G \rightarrow \mathbb{C}$ is K -invariant if and only if it satisfies the compatibility condition

$$f(x, u) = f(e^{it}x, e^{is}u)$$

for each (x, u) in G and t, s in \mathbb{R} .

Therefore, for each n in \mathbb{N} by K -invariance of φ_n we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi_n(xe^{ik}y) dk = \varphi_n(xy)$$

for all x, y in $\mathbb{C} \setminus \{0\}$ and

$$\varphi(xe^{it}) = \varphi(x)$$

for all x in $\mathbb{C} \setminus \{0\}$ and $t \in \mathbb{R}$. Now Eq. (12) becomes a binomial type equation

$$\varphi_n(xy) = \sum_{j=0}^n \binom{n}{j} \varphi_j(x) \varphi_{n-j}(y), \tag{14}$$

for x, y in $\mathbb{C} \setminus \{0\}$, where $\varphi_n : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ is continuous and K -invariant. By K -invariance we obtain that φ_k for $k = 0, \dots, n$ depends only on $|x|$, that is $\varphi(x) = \varphi(|x|)$ for each $x \neq 0$ in \mathbb{C} . There exist u, v in \mathbb{R} such that $|x| = e^u$ and $|y| = e^v$. Let ψ_k for $k = 0, \dots, n$ be given by $\varphi_k \circ \exp : \mathbb{R} \rightarrow \mathbb{C}$ for $k = 0, \dots, n$. Then

$$\psi_n(u + v) = \sum_{j=0}^n \binom{n}{j} \psi_j(v) \psi_{n-j}(u), \tag{15}$$

for u and v in \mathbb{R} . Equation (15) is well known and it has been considered e.g. in [1]. Using the results from [1] we infer that there exist complex constants $\alpha_1, \dots, \alpha_n$ such that

$$\psi_n(u) = n! \sum_{j_1+2j_2+\dots+nj_n=n} \prod_{m=1}^n \left(\frac{\alpha_n u}{m!} \right)^{j_m} / j_m!$$

for u in \mathbb{R} . Therefore

$$\varphi_n(x) = n! \sum_{j_1+2j_2+\dots+nj_n=n} \prod_{m=1}^n \left(\frac{\alpha_n \ln(|x|)}{m!} \right)^{j_m} / j_m!$$

for $x \neq 0$ in \mathbb{C} .

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