The Form of Multi-additive Symmetric Functions

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Abstract. In this paper we show that every n -additive symmetric function, between quite general structures like abelian groups and semigroups, can be factorized into a composition of additive function with the product of additive functions. We also show that every two n -additive functions, defined on a product of groups, with equal counterimages of the positive half-line must be proportional.

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Introduction

Multi-additive symmetric functions play an important role in the theory of functional equations: they naturally generalize additive functions, their diagonalizations are regarded as generalizations of monomial functions, and they appear frequently as solutions (or components of solutions) of functional equations. The general construction of multi-additive symmetric functions is well known (see eg. [1]). Some known facts about multi-additive symmetric functions can be found eg. in [3, 5, 7, 8, 11]. The main result of this paper is the characterization of multi-additive symmetric functions in the form of an additive function of the product of additive functions.

The structure of the paper is as follows. In the first section we recall some well known definitions and theorems from the theory of groups and semigroups but also state and prove a few results concerning extensions of a multi-additive symmetric functions defined on a semigroup. The section ends with a simple, but important lemma connecting notions of an algebraical and linear independence. The second section contains main results of the paper – we show

how a multi-additive symmetric function can be expressed in terms of additive functions in several settings. The third part contains mainly examples showing the importance of assumptions made in main results. At the very end of the paper we had put a result which may be considered as independent of the rest of the paper, bound up with the theorem of Maksa and Ratz [6] about the proportionality of the additive functions with equal counterimages.

1. Multi-additive Symmetric Function on Groups and Semigroups

We start by recalling here some notions and results from the theory of groups and semigroups. Unless stated otherwise we refer to [4, Appendix A]) and [2].

Definition 1. An abelian group G is torsion-free if every element except the identity has the infinite order.

Definition 2. A subset L of a group G is said to be independent if it does not contain the identity 0 and for all distinct $x_1, \dots, x_k \in L$ we have

$$\bigwedge_{n_1, \dots, n_k \in \mathbb{Z}} (n_1 x_1 + \dots + n_k x_k = 0 \implies n_1 x_1 = \dots = n_k x_k = 0).$$

Let G be an abelian group, $\mathcal{A}$ be the family of all independent sets L in G consisting only of elements whose order is infinite or a power of a prime and such that L is maximal with respect to these properties. Similarly, let $\mathcal{A}_0$ [$\mathcal{A}_p$] be the family of independent sets L_0 [L_p] in G consisting of elements whose order is infinite [a power of p] and such that L_0 [L_p] is maximal with respect to these properties; p denotes any prime integer. The cardinal number of any set in $\mathcal{A}$ [resp. $\mathcal{A}_0$, $\mathcal{A}_p$] is called the rank of G [resp. the torsion-free rank, the p -rank] and is denoted by $r(G)$ [resp. $r_0(G)$, $r_p(G)$] (all the sets in $\mathcal{A}$ [resp. $\mathcal{A}_0$, $\mathcal{A}_p$] have the same cardinal number).

Let p be a prime number. The Prüfer p -group is the unique p -group in which every element has p different p -th roots. Alternatively we can write $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$, where $\mathbb{Z}[1/p] = \left\{ \frac{m}{p^n} \in \mathbb{Q} : m \in \mathbb{Z}, n \in \mathbb{N}_0 \right\}$.

Definition 3. Let A_i , $i \in I$, be groups. The direct sum $\bigoplus_{i \in I} A_i$ is the set of tuples $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ such that $a_i \neq 0$ for finitely many $i \in I$.

Definition 4. A semigroup S is said to be divisible if

$$\bigwedge_{x \in S} \bigwedge_{n \in \mathbb{N}} \bigvee_{y \in S} x = ny.$$

Theorem 1. Let D be a divisible abelian group. Then D is the direct sum of groups each of which is isomorphic either to $\mathbb{Z}(p^\infty)$ or to $\mathbb{Q}$, i.e.

$$D = \bigoplus_{p \in \mathbb{P}} \bigoplus_{i_p \in I_p} \mathbb{Z}(p^\infty) \oplus \bigoplus_{i \in I} \mathbb{Q},$$

where $\mathbb{P}$ denotes the set of all prime numbers.

Theorem 2. Let G be an abelian group. Then there exist a minimal divisible abelian group D such that G is isomorphic with some subgroup of D . Moreover D/G is a torsion group and $r_0(D) = r_0(G)$, $r_p(D) = r_p(G)$ for all prime numbers p .

Definition 5. An abelian semigroup $(S, +)$ is called cancellative if

$$\bigwedge_{x,y,z \in S} (x + z = y + z \implies x = y).$$

Theorem 3. Let S be a cancellative abelian semigroup. Then there exists an abelian group G such that $S \leq G$ and $G = S - S$.

Theorem 4. Let $(S, +)$ be an abelian semigroup. Then a relation $\sim$ given by

$$\bigwedge_{x,y \in S} \left(x \sim y \iff \bigvee_{z \in S} x + z = y + z \right) \quad (1)$$

is the equivalence relation, $S/\sim$ with the operation $+: (S/\sim)^2 \rightarrow S/\sim$ defined by

$$[x]_{\sim} + [y]_{\sim} = [x + y]_{\sim}, \quad x, y \in S, \quad (2)$$

is the cancellative abelian semigroup and the function $\varkappa: S \rightarrow S/\sim$ given by

$$\varkappa(x) = [x]_{\sim}, \quad x \in S, \quad (3)$$

is the semigroup homomorphism. Moreover, if S is divisible, then $S/\sim$ and G are divisible, where G is an abelian group such that $S/\sim \leq G$ and $G = S/\sim - S/\sim$.

Proof. It is easy to see that relation $\sim$ is reflexive and symmetric. Let $x, y, z \in S$, $x \sim y$, $y \sim z$. Then there exist $u, v \in S$ such that $x + u = y + u$ and $y + v = z + v$. Now we notice that

$$x + u + v = y + u + v = y + v + u = z + v + u = z + u + v,$$

which shows that $x \sim z$, and hence relation $\sim$ is transitive.

We proceed to show that operation $+$ defined by (2) is well-defined. Let $x_1, x_2, y_1, y_2 \in S$ be such that $x_1 \sim x_2$ and $y_1 \sim y_2$. Then there exist $u, v \in S$ such that $x_1 + u = x_2 + u$ and $y_1 + v = y_2 + v$. Hence we have $x_1 + y_1 + u + v = x_1 + u + y_1 + v = x_2 + u + y_2 + v = x_2 + y_2 + u + v$, which means that $x_1 + y_1 \sim x_2 + y_2$.

It is also easily seen that $S/\sim$ with the operation $+$ given by (2) is the abelian semigroup and the function $\varkappa$ given by (3) is a homomorphism. To show that $S/\sim$ is cancellative, let $x, y, z \in S$ be such that $[x]_{\sim} + [z]_{\sim} = [y]_{\sim} + [z]_{\sim}$. Then we have $[x + z]_{\sim} = [y + z]_{\sim}$, which gives us $x + z \sim y + z$, and there exists $u \in S$ such that $x + z + u = y + z + u$. It follows that $x \sim y$, which means $[x]_{\sim} = [y]_{\sim}$. This shows that $S/\sim$ is cancellative.

Assume that S is divisible, and let G be an abelian group such that $G = S/\sim - S/\sim$. Let $x \in S$, $m \in \mathbb{N}$. Then there exist $y \in S$ such that $x = my$,

and thus $\varkappa(x) = \varkappa(my) = m\varkappa(y)$, hence $S/\sim$ is divisible. Let further $x \in G$, $m \in \mathbb{N}$. Then $x = \varkappa(u) - \varkappa(v)$ for some $u, v \in S$, and moreover there exist $y, z \in S$ such that $u = my$ and $v = mz$. It follows that

$$m(\varkappa(y) - \varkappa(z)) = \varkappa(my) - \varkappa(mz) = \varkappa(u) - \varkappa(v) = x,$$

and hence G is divisible. $\square$

For semigroups we define a cardinal number with a meaning similar to the rank of a group.

Definition 6. Let S be an abelian semigroup. The cardinal number $r_0(S)$ is equal to the torsion-free rank $r_0(G)$, where G is an abelian group such that $S/\sim \leq G$, $G = S/\sim - S/\sim$ and relation $\sim$ is defined by (1).

After these preparations we may now pass to multi-additive functions. By $\text{Perm}(n)$ we denote the set of all bijections of the set $\{1, \dots, n\}$.

Definition 7. Let S be an abelian semigroup, H be an abelian group. The function $A_n: S^n \rightarrow H$ is called n -additive symmetric if

$$\begin{aligned} A_n(x + y, x_2, \dots, x_n) &= A_n(x, x_2, \dots, x_n) + A_n(y, x_2, \dots, x_n), \\ A_n(x_1, \dots, x_n) &= A_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \end{aligned}$$

for all $x, y, x_1, \dots, x_n \in S$ and $\sigma \in \text{Perm}(n)$. If A_n is an n -additive symmetric, then by $A_n^*: S \rightarrow H$ we denote its diagonalization

$$A_n^*(x) = A_n(x, \dots, x), \quad x \in S.$$

Such diagonalizations will be called generalized monomials of degree n .

Let us quote here the theorem of Székelyhidi about extending n -additive symmetric functions from subgroup to the whole group, that we use later. For our further purposes we also need a similar result on extending n -additive symmetric functions from semigroup S to group G generated by S in the sense of Theorem 3, which is stated and proved below.

Theorem 5 ([13, Theorem 2]). *Let G be an abelian group, D be a divisible abelian group, $n \in \mathbb{N}$. Furthermore, let H be a subgroup of G , and $A_n: H^n \rightarrow D$ an n -additive symmetric function. Then A_n can be extended to an n -additive symmetric mapping of G^n into D , that is, there exists an n -additive symmetric function $\mathcal{A}_n: G^n \rightarrow D$ such that*

$$A_n(h_1, \dots, h_n) = \mathcal{A}_n(h_1, \dots, h_n), \quad h_1, \dots, h_n \in H.$$

Theorem 6. *Let S be a cancellative abelian semigroup, H be an abelian group, $n \in \mathbb{N}$. Furthermore, let $A_n: S^n \rightarrow H$ be an n -additive symmetric function. Then, for an abelian group G such that $S \leq G$ and $G = S - S$, the function A_n can be uniquely extended to an n -additive symmetric mapping of G^n into H , that is, there exists an n -additive symmetric function $\mathcal{A}_n: G^n \rightarrow H$ such that*

$$\mathcal{A}_n(h_1, \dots, h_n) = A_n(h_1, \dots, h_n), \quad h_1, \dots, h_n \in S.$$

Proof. We define $\mathcal{A}_n: G^n \rightarrow H$ by the formula

$$\begin{aligned} & \mathcal{A}_n(g_1 - h_1, \dots, g_n - h_n) \\ &= \sum_{x_1 \in \{g_1, h_1\}} \dots \sum_{x_n \in \{g_n, h_n\}} \varepsilon(x_1, \dots, x_n) A_n(x_1, \dots, x_n), \end{aligned} \quad (4)$$

where $g_1, \dots, g_n, h_1, \dots, h_n \in S$ and

$$\varepsilon(x_1, \dots, x_n) = \begin{cases} 1, & |\{i \in \{1, \dots, n\} : x_i = h_i\}| \in 2\mathbb{N}, \\ -1, & |\{i \in \{1, \dots, n\} : x_i = h_i\}| \notin 2\mathbb{N}. \end{cases}$$

We next show that $\mathcal{A}_n$ is well-defined. Let $g_i, h_i, u_i, v_i \in S$ for all $i \in \{1, \dots, n\}$ be such that $g_i - h_i = u_i - v_i$ for all $i \in \{1, \dots, n\}$. Let further $y_2, \dots, y_n \in S$. We have

$$\begin{aligned} & A_n(g_1, y_2, \dots, y_n) + A_n(v_1, y_2, \dots, y_n) \\ &= A_n(g_1 + v_1, y_2, \dots, y_n) = A_n(u_1 + h_1, y_2, \dots, y_n) \\ &= A_n(u_1, y_2, \dots, y_n) + A_n(h_1, y_2, \dots, y_n), \end{aligned}$$

and thus

$$\begin{aligned} & A_n(g_1, y_2, \dots, y_n) - A_n(h_1, y_2, \dots, y_n) \\ &= A_n(u_1, y_2, \dots, y_n) - A_n(v_1, y_2, \dots, y_n). \end{aligned}$$

We now observe that

$$\begin{aligned} & \mathcal{A}_n(g_1 - h_1, \dots, g_n - h_n) \\ &= \sum_{x_1 \in \{g_1, h_1\}} \dots \sum_{x_n \in \{g_n, h_n\}} \varepsilon(x_1, \dots, x_n) A_n(x_1, \dots, x_n) \\ &= \sum_{x_2 \in \{g_2, h_2\}} \dots \sum_{x_n \in \{g_n, h_n\}} \varepsilon(g_1, x_2, \dots, x_n) (A_n(g_1, x_2, \dots, x_n) \\ &\quad - A_n(h_1, x_2, \dots, x_n)) \\ &= \sum_{x_2 \in \{g_2, h_2\}} \dots \sum_{x_n \in \{g_n, h_n\}} \varepsilon(u_1, x_2, \dots, x_n) (A_n(u_1, x_2, \dots, x_n) \\ &\quad - A_n(v_1, x_2, \dots, x_n)) \\ &= \sum_{x_1 \in \{u_1, v_1\}} \sum_{x_2 \in \{g_2, h_2\}} \dots \sum_{x_n \in \{g_n, h_n\}} \varepsilon(x_1, \dots, x_n) A_n(x_1, \dots, x_n) \\ &= \mathcal{A}_n(u_1 - v_1, g_2 - h_2, \dots, g_n - h_n). \end{aligned}$$

By the induction we obtain

$$\mathcal{A}_n(g_1 - h_1, \dots, g_n - h_n) = \mathcal{A}_n(u_1 - v_1, \dots, u_n - v_n),$$

which shows that $\mathcal{A}_n$ is well-defined. It is immediate that $\mathcal{A}_n$ is n -additive symmetric.

On the other hand, if $\mathcal{A}_n$ is an extension of A_n , then equation (4) holds which implies the uniqueness of $\mathcal{A}_n$. $\square$

The following elementary result will be indispensable in the proof of one of the corollaries given in the next section.

Theorem 7. *Let S be an abelian semigroup, relation $\sim$ be defined by (1), H be an abelian group. Furthermore, let $n \in \mathbb{N}$ and $A_n: S^n \rightarrow H$ be an n -additive symmetric function. Then $\mathcal{A}_n: (S/\sim)^n \rightarrow H$ given by the formula*

$$\mathcal{A}_n([h_1]_{\sim}, \dots, [h_n]_{\sim}) = A_n(h_1, \dots, h_n), \quad h_1, \dots, h_n \in S,$$

is the well-defined n -additive symmetric function.

Proof. Let $g_1, \dots, g_n, h_1, \dots, h_n \in S$ be such that $[g_i]_{\sim} = [h_i]_{\sim}$, $i \in \{1, \dots, n\}$. Then there exist $x_1, \dots, x_n \in S$ such that $g_i + x_i = h_i + x_i$, $i \in \{1, \dots, n\}$. We prove by the induction that

$$\begin{aligned} A_n(g_1, \dots, g_{k-1}, g_k, y_{k+1}, \dots, y_n) \\ = A_n(h_1, \dots, h_{k-1}, h_k, y_{k+1}, \dots, y_n) \end{aligned} \quad (5)$$

for all $y_k, \dots, y_n \in S$, $1 \leq k \leq n$. For $k = 1$ and $y_2, \dots, y_n \in S$ we have

$$\begin{aligned} A_n(g_1, y_2, \dots, y_n) &= A_n(g_1 + x_1, y_2, \dots, y_n) - A_n(x_1, y_2, \dots, y_n) \\ &= A_n(h_1 + x_1, y_2, \dots, y_n) - A_n(x_1, y_2, \dots, y_n) \\ &= A_n(h_1, y_2, \dots, y_n). \end{aligned}$$

Assume that (5) holds for some $k \in \{1, \dots, n-1\}$. For $y_{k+1}, \dots, y_n \in S$ we have

$$\begin{aligned} A_n(g_1, \dots, g_{k-1}, g_k, y_{k+1}, \dots, y_n) \\ &= A_n(g_1, \dots, g_{k-1}, g_k + x_k, y_{k+1}, \dots, y_n) \\ &\quad - A_n(g_1, \dots, g_{k-1}, x_k, y_{k+1}, \dots, y_n) \\ &= A_n(h_1, \dots, h_{k-1}, g_k + x_k, y_{k+1}, \dots, y_n) \\ &\quad - A_n(h_1, \dots, h_{k-1}, x_k, y_{k+1}, \dots, y_n) \\ &= A_n(h_1, \dots, h_{k-1}, h_k + x_k, y_{k+1}, \dots, y_n) \\ &\quad - A_n(h_1, \dots, h_{k-1}, x_k, y_{k+1}, \dots, y_n) \\ &= A_n(h_1, \dots, h_{k-1}, h_k, y_{k+1}, \dots, y_n), \end{aligned}$$

which ends the proof of (5). In particular, we have

$$A_n(g_1, \dots, g_n) = A_n(h_1, \dots, h_n),$$

hence the function $\mathcal{A}_n$ is well-defined. We see at once that $\mathcal{A}_n$ is n -additive symmetric. $\square$

In the last part of this section we will state a simple, yet crucial observation concerning the notions of algebraical and linear independence. We start with some facts from the theory of fields. Let E be a subfield of a field F . The set $S \subset F$ is said to be algebraically independent over E if $f(t_1, \dots, t_n) \neq 0$ for all non-zero polynomials $f \in E[X_1, \dots, X_n]$ and all distinct $t_1, \dots, t_n \in S$, $n \in \mathbb{N}$. It is known that every extension F/E has a maximal algebraically

independent subset which is called a transcendence base for F/E . The cardinality of each transcendence base for F/E is the same and it is denoted by $\text{trdeg}(F/E)$.

Let $\mathbb{K}$ be a field of characteristic 0 and let $\mathbb{T} \subset \mathbb{K}$ be an algebraically independent set over the field $\mathbb{Q}$ of rational numbers. Define the sets

$$\mathbb{T}^n = \left\{ \prod_{i=1}^n t_i : t_i \in \mathbb{T}, i \in \{1, \dots, n\} \right\}, n \in \mathbb{N}.$$

The following statement is elementary and its proof is left to the reader.

Lemma 1. *For every $n \in \mathbb{N}$, the set $\mathbb{T}^n$ is linearly independent over $\mathbb{Q}$ and its elements are determined uniquely up to the order of multiplication, i.e.:*

$$\begin{aligned} \bigwedge_{n \in \mathbb{N}} \bigwedge_{x_1, \dots, x_n \in \mathbb{T}} \bigwedge_{y_1, \dots, y_n \in \mathbb{T}} & \left(x_1 \cdot \dots \cdot x_n = y_1 \cdot \dots \cdot y_n \right. \\ \implies & \bigvee_{\sigma \in \text{Perm}(n)} \bigwedge_{i \in \{1, \dots, n\}} y_i = x_{\sigma(i)} \Big). \end{aligned}$$

In particular, $\mathbb{T}^n$ can be extended to the basis of $\mathbb{K}$ over $\mathbb{Q}$.

2. Main Results

We are now ready to present main results of the paper. The first theorem presents a characterization of n -additive symmetric functions as an additive function of a product of additive functions. The following corollaries presents the same factorization in a slightly different settings.

Theorem 8. *Let G be a divisible abelian group, H be an abelian group, $n \in \mathbb{N}$, $A_n: G^n \rightarrow H$ an n -additive symmetric function. Furthermore, if $n = 1$, then let G or H be torsion-free. Then for every field $\mathbb{K}$ of characteristic 0 and $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(G)$ there exist additive maps $\phi: G \rightarrow \mathbb{K}$, $\psi: \mathbb{K} \rightarrow H$ such that*

$$A_n(x_1, \dots, x_n) = \psi(\phi(x_1) \cdot \dots \cdot \phi(x_n)), x_1, \dots, x_n \in G. \quad (6)$$

Proof. In view of Theorem 1 we have (up to isomorphism)

$$G = \bigoplus_{p \in \mathbb{P}} \bigoplus_{i_p \in I_p} \mathbb{Z}(p^\infty) \oplus \bigoplus_{i \in I} \mathbb{Q}.$$

Let

$$G_0 = \bigoplus_{p \in \mathbb{P}} \bigoplus_{i_p \in I_p} \mathbb{Z}(p^\infty) = \{x \in G : \bigvee_{k \in \mathbb{N}} kx = 0\} \quad \text{and} \quad G_1 = \bigoplus_{i \in I} \mathbb{Q},$$

hence we have $G = G_0 \oplus G_1$. We now show that

$$G_0 \subset \{x_1 \in G : \bigwedge_{x_2, \dots, x_n \in G} A_n(x_1, \dots, x_n) = 0\}.$$

Indeed, when $n = 1$ and G is torsion-free, then $G_0 = \{0\}$. When $n = 1$ and H is torsion-free we have that for $x \in G$ of the finite order, the order of $A_1(x)$ divides the order of x , hence we obtain $A_1(x) = 0$. If $n \geq 2$, $x_1, \dots, x_n \in G$ and $kx_1 = 0$ for some $k \in \mathbb{N}$, then since G is divisible, there exists $y_2 \in G$ such that $x_2 = ky_2$. Hence

$$\begin{aligned} A_n(x_1, \dots, x_n) &= A_n(x_1, ky_2, x_3, \dots, x_n) = kA_n(x_1, y_2, x_3, \dots, x_n) \\ &= A_n(kx_1, y_2, x_3, \dots, x_n) = A_n(0, y_2, x_3, \dots, x_n) = 0. \end{aligned}$$

We have $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(G_1) = r_0(G) = |I|$, hence there exists an algebraically independent set $\mathbb{T} \subset \mathbb{K}$ over $\mathbb{Q}$ of the cardinality $|I|$. Let $f: I \rightarrow \mathbb{T}$ be a bijection. We will denote by 1_i an element of G_1 such that it is equal to 1 on the i -th coordinate and equal to 0 on all remaining coordinates. We now define the function $\phi: G \rightarrow \mathbb{K}$ by the formula

$$\phi\left(\sum_{i \in I_0} q_i 1_i + g_0\right) = \sum_{i \in I_0} q_i f(i), \quad q_i \in \mathbb{Q}, \quad i \in I_0 \subset I, \quad |I_0| < \infty, \quad g_0 \in G_0.$$

It is easily seen that ϕ is additive and $\ker \phi = G_0$. In view of Lemma 1 there exists a basis $\mathbb{H}$ of $\mathbb{K}/\mathbb{Q}$ such that $\mathbb{T}^n \subset \mathbb{H}$. We define the map $\psi_0: \mathbb{H} \rightarrow H$ by the formula

$$\psi_0(x) = \begin{cases} A_n(h_1, \dots, h_n), & x = \phi(h_1) \cdot \dots \cdot \phi(h_n) \in \mathbb{T}^n, \\ 0, & x \in \mathbb{H} \setminus \mathbb{T}^n. \end{cases}$$

This function is well-defined. Indeed, if $x = \phi(g_1) \cdot \dots \cdot \phi(g_n) = \phi(h_1) \cdot \dots \cdot \phi(h_n)$ for some $g_1, \dots, g_n, h_1, \dots, h_n \in G$, then in view of Lemma 1 there exists $\sigma \in \text{Perm}(n)$ such that

$$\bigwedge_{i \in \{1, \dots, n\}} \phi(h_i) = \phi(g_{\sigma(i)}),$$

which give us $h_i = g_{\sigma(i)} + z_i$, for $i \in \{1, \dots, n\}$, where $z_1, \dots, z_n \in G_0$, and thus $A_n(g_1, \dots, g_n) = A_n(h_1, \dots, h_n)$. Obviously, we can uniquely extend ψ_0 to the additive map $\psi: \mathbb{K} \rightarrow H$.

Finally, we show that Eq. (6) holds. Let $x_1, \dots, x_n \in G$. Then

$$x_k = \sum_{i_k \in I_0} q_{k, i_k} 1_{i_k} + g_{k, 0},$$

where $q_{k, i_k} \in \mathbb{Q}$, $i_k \in I_0 \subset I$, $|I_0| < \infty$, $g_{k, 0} \in G_0$, $k \in \{1, \dots, n\}$. Since $f(i_k) = \phi(1_{i_k})$, then

$$\psi(f(i_1) \cdot \dots \cdot f(i_n)) = \psi(\phi(1_{i_1}) \cdot \dots \cdot \phi(1_{i_n})) = A(1_{i_1}, \dots, 1_{i_n}),$$

and hence we have

$$\begin{aligned} \psi(\phi(x_1) \cdot \dots \cdot \phi(x_n)) &= \psi\left(\prod_{k=1}^n \sum_{i_k \in I_0} q_{k, i_k} f(i_k)\right) \\ &= \sum_{i_1, \dots, i_n \in I_0} q_{1, i_1} \cdot \dots \cdot q_{n, i_n} \psi(f(i_1) \cdot \dots \cdot f(i_n)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i_1, \dots, i_n \in I_0} q_{1, i_1} \cdot \dots \cdot q_{n, i_n} A_n(1_{i_1}, \dots, 1_{i_n}) \\
&= \sum_{i_1, \dots, i_n \in I_0} A_n(q_{1, i_1} 1_{i_1}, \dots, q_{n, i_n} 1_{i_n}) \\
&= A_n \left(\sum_{i_1 \in I_0} q_{1, i_1} 1_{i_1}, \dots, \sum_{i_n \in I_0} q_{n, i_n} 1_{i_n} \right) \\
&= A_n \left(\sum_{i_1 \in I_0} q_{1, i_1} 1_{i_1} + g_{1, 0}, \dots, \sum_{i_n \in I_0} q_{n, i_n} 1_{i_n} + g_{n, 0} \right) \\
&= A_n(x_1, \dots, x_n),
\end{aligned}$$

which ends the proof. $\square$

Remark 1. In the above theorem the choice of a function ϕ depends only of the group G .

Corollary 1. *Let G, H be abelian groups, G or H be divisible, $n \in \mathbb{N}$, $A_n: G^n \rightarrow H$ an n -additive symmetric function. Furthermore, if $n = 1$, then let G or H be torsion-free. Then for every field $\mathbb{K}$ of characteristic 0 and $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(G)$ there exist additive maps $\phi: G \rightarrow \mathbb{K}$, $\psi: \mathbb{K} \rightarrow H$ such that (6) holds.*

Proof. When G is divisible, then it is exactly Theorem 8. Assume that H is divisible. In view of Theorem 2 we can extend the group G to the divisible group G_1 such that $r_0(G_1) = r_0(G)$. Next, applying Theorem 5 we can extend A_n to the n -additive symmetric map $\mathcal{A}_n: G_1^n \rightarrow H$. By Theorem 8 for every field $\mathbb{K}$ of characteristic 0 and $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(G_1) = r_0(G)$ there exist additive maps $\phi_0: G_1 \rightarrow \mathbb{K}$, $\psi: \mathbb{K} \rightarrow H$ such that

$$\mathcal{A}_n(x_1, \dots, x_n) = \psi(\phi_0(x_1) \cdot \dots \cdot \phi_0(x_n)), \quad x_1, \dots, x_n \in G_1.$$

Taking $\phi = \phi_0|_G$ we find

$$\begin{aligned}
A_n(x_1, \dots, x_n) &= \mathcal{A}_n(x_1, \dots, x_n) \\
&= \psi(\phi(x_1) \cdot \dots \cdot \phi(x_n)), \quad x_1, \dots, x_n \in G,
\end{aligned}$$

which ends the proof. $\square$

Corollary 2. *Let S be an abelian semigroup, H be an abelian group, S or H be divisible, $n \in \mathbb{N}$, $A_n: S^n \rightarrow H$ an n -additive symmetric function. Furthermore, if $n = 1$, then let H be torsion-free. Then for every field $\mathbb{K}$ of characteristic 0 and $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(S)$ there exist additive maps $\phi: S \rightarrow \mathbb{K}$, $\psi: \mathbb{K} \rightarrow H$ such that*

$$A_n(x_1, \dots, x_n) = \psi(\phi(x_1) \cdot \dots \cdot \phi(x_n)), \quad x_1, \dots, x_n \in S.$$

Proof. Let G be an abelian group such that $S/\sim \leq G$, $G = S/\sim - S/\sim$. We combine Theorems 7, 4 and 6 to get the n -additive symmetric map $\mathcal{A}_n: G^n \rightarrow H$ such that

$$\mathcal{A}_n(\varkappa(x_1), \dots, \varkappa(x_n)) = A_n(x_1, \dots, x_n), \quad x_1, \dots, x_n \in S.$$

In view of Corollary 1 for every field $\mathbb{K}$ of characteristic 0 and $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(S)$ there exist additive maps $\phi_0: G \rightarrow \mathbb{K}$, $\psi: \mathbb{K} \rightarrow H$ such that

$$\mathcal{A}_n(x_1, \dots, x_n) = \psi(\phi_0(x_1) \cdot \dots \cdot \phi_0(x_n)), \quad x_1, \dots, x_n \in G.$$

Let $\phi: S \rightarrow \mathbb{K}$ be given by $\phi = \phi_0 \circ \varkappa$. Then

$$\begin{aligned} A_n(x_1, \dots, x_n) &= \mathcal{A}_n(\varkappa(x_1), \dots, \varkappa(x_n)) \\ &= \psi(\phi(x_1) \cdot \dots \cdot \phi(x_n)), \quad x_1, \dots, x_n \in S, \end{aligned}$$

which ends the proof. $\square$

At the end of this section we are obliged to notice that the assumptions of Theorem 8 are certainly fulfilled in the real numbers world, that is when $G = H = \mathbb{K} = \mathbb{R}$, hence we have the following result.

Corollary 3. *Function $A_n: \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is n -additive symmetric if, and only if there exist an additive maps $\phi: \mathbb{R} \rightarrow \mathbb{R}$, $\psi: \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$A_n(x_1, \dots, x_n) = \psi(\phi(x_1) \cdot \dots \cdot \phi(x_n)), \quad x_1, \dots, x_n \in \mathbb{R}.$$

3. Examples and Auxiliary Results

Let us first notice that in Theorem 8, in the case when $H = \mathbb{K}$, $n \geq 2$, we cannot write A_n in the form

$$A_n(x_1, \dots, x_n) = \phi(x_1) \cdot \dots \cdot \phi(x_n), \quad x_1, \dots, x_n \in G.$$

A suitable example is given in [9]. Moreover, in the case when $G = \mathbb{K}$, $n \geq 2$, we are not able to write A_n in the form

$$A_n(x_1, \dots, x_n) = \psi(x_1 \cdot \dots \cdot x_n), \quad x_1, \dots, x_n \in G, \quad (7)$$

as the following example shows.

Example 1. Let $n \geq 2$, $G = H = \mathbb{R}$, and take a discontinuous additive function $a: \mathbb{R} \rightarrow \mathbb{R}$. Consider an n -additive symmetric function

$$A_n(x_1, \dots, x_n) = a(x_1) \cdot a(x_2) \cdot x_3 \cdot \dots \cdot x_n, \quad x_1, \dots, x_n \in \mathbb{R},$$

and suppose that (7) holds for some additive function $\psi: \mathbb{R} \rightarrow \mathbb{R}$. Notice that diagonalization $A_n^*(x) = a(x)^2 x^{n-2}$, $x \in \mathbb{R}$, is discontinuous and $A_n^*(x) \geq 0$ for $x \geq 0$. From (7) it follows that $A_n^*(x) = \psi(x^n)$, hence ψ is also discontinuous and positive on the positive real axis, which is a contradiction to well known property of additive functions.

The following example shows that the field $\mathbb{K}$ in Theorem 8 cannot be too small.

Example 2. Take $G = \mathbb{Q}^2$ and $H = \mathbb{Q}$. Let $a, b \in \mathbb{Q}$ be such that $a < 0 < b$ and define $A_2: G^2 \rightarrow H$ by

$$A_2((q_1; q_2), (r_1; r_2)) = aq_1r_1 + bq_2r_2, \quad q_1, q_2, r_1, r_2 \in \mathbb{Q}.$$

It is straightforward to check that A_2 is biadditive symmetric. Suppose that there exist additive maps $\phi: \mathbb{Q}^2 \rightarrow \mathbb{Q}$, $\psi: \mathbb{Q} \rightarrow \mathbb{Q}$ such that

$$A_2((q_1; q_2), (r_1; r_2)) = \psi(\phi(q_1; q_2)\phi(r_1; r_2)), \quad q_1, q_2, r_1, r_2 \in \mathbb{Q}.$$

Certainly $\psi(x) = x\psi(1)$ for $x \in \mathbb{Q}$. Combining two above formulas for A_2 we obtain

$$0 > aq^2 = A_2((q; 0), (q; 0)) = \phi(q; 0)^2\psi(1), \quad q \in \mathbb{Q} \setminus \{0\},$$

implying $\psi(1) < 0$, but on the other hand

$$0 < bq^2 = A_2((0; q), (0; q)) = \phi(0; q)^2\psi(1), \quad q \in \mathbb{Q} \setminus \{0\},$$

hence $\psi(1) > 0$, which leads to a contradiction.

Let us also notice that Theorem 8 does not hold for $n = 1$ and G, H which are divisible but not torsion-free.

Example 3. Let $G = H$ be a divisible, but not torsion-free group, and let $A: G \rightarrow H$ be an identity map, which obviously is additive. Suppose that there exist a field $\mathbb{K}$ of characteristic 0 and additive functions $\phi: G \rightarrow \mathbb{K}$, $\psi: \mathbb{K} \rightarrow H$ such that $A = \psi \circ \phi$. For $n \in \mathbb{N}$ and $y \in G$, $y \neq 0$, such that $ny = 0$ we have $0 = \phi(ny) = n\phi(y)$, hence $\phi(y) = 0$, but this implies $y = A(y) = \psi(\phi(y)) = \psi(0) = 0$, a contradiction.

Dealing with generalized polynomial functions, we often assume that the domain or the codomain is uniquely divisible by $n!$. It is not enough for Corollary 1 to hold.

Example 4. Let $n \in \mathbb{N}$, $G = \{m/(n!)^k : m \in \mathbb{Z}, k \in \mathbb{N}\}$. It is easily seen that G (with the usual addition) is an abelian group uniquely divisible by $n!$. Let $A: G \rightarrow G$ be an identity map, which obviously is additive. Suppose that there exist a field $\mathbb{K}$ of characteristic 0 and additive functions $\phi: G \rightarrow \mathbb{K}$, $\psi: \mathbb{K} \rightarrow G$ such that $A = \psi \circ \phi$. Let $x \in \mathbb{K}$. Then $\psi(x) = m/(n!)^k$ for some $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. Hence

$$(n! + 1)^j \psi(x/(n! + 1)^j) = \psi(x) = \frac{m}{(n!)^k}, \quad j \in \mathbb{N},$$

and thus we have $\psi(x/(n! + 1)^j) = m/(n!)^k / (n! + 1)^j$ for $j \in \mathbb{N}$. Since G is not divisible by $(n! + 1)$, then $m = 0$, which means that $\psi(x) = 0$. We conclude that $A = \psi \circ \phi = 0$, which give us a contradiction.

Finally we can apply the results given above to generalized polynomial functions.

Definition 8. Let S be an abelian semigroup, H be an abelian group. The function $f: S \rightarrow H$ is called generalized polynomial function of degree at most n if

$$f(x) = A_n^*(x) + \cdots + A_1^*(x) + A_0, \quad x \in S,$$

where $A_0 \in H$ and $A_k^*: S \rightarrow H$ is a generalized monomial of degree k , for $k \in \{1, \dots, n\}$.

It is known (see e.g. [12]) that f is a generalized polynomial function of degree at most n iff f satisfies the Fréchet equation:

$$\Delta_h^{n+1} f(x) = 0, \quad x, h \in S. \quad (8)$$

Corollary 4. Let S be an abelian semigroup, H be a torsion-free abelian group, S or H be divisible, $n \in \mathbb{N}$, $f: S \rightarrow H$. Then f satisfies the Fréchet equation (8) iff for every field $\mathbb{K}$ of characteristic 0 and $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(S)$ there exist additive maps $\phi: S \rightarrow \mathbb{K}$, $\psi_1, \dots, \psi_n: \mathbb{K} \rightarrow H$ and a constant $A_0 \in H$ such that

$$f(x) = \psi_n(\phi(x)^n) + \cdots + \psi_1(\phi(x)) + A_0, \quad x \in S. \quad (9)$$

Proof. ($\Rightarrow$) Assume that f satisfies (8). Then there exist k -additive symmetric functions $A_k: S^n \rightarrow H$, $k \in \{1, \dots, n\}$, $A_0 \in H$ such that

$$f(x) = A_n(x, \dots, x) + \cdots + A_1(x) + A_0, \quad x \in S.$$

In view of Corollary 2 and Remark 1 for every field $\mathbb{K}$ of characteristic 0 and $\text{trdeg}(\mathbb{K}/\mathbb{Q}) \geq r_0(S)$ there exist additive maps $\phi: S \rightarrow \mathbb{K}$, $\psi_1, \dots, \psi_n: \mathbb{K} \rightarrow H$ such that

$$A_k(x, \dots, x) = \psi_k(\phi(x)^k), \quad x \in S, \quad k \in \{1, \dots, n\}.$$

Hence we have (9).

($\Leftarrow$) Assume that (9) holds. Since for each $k \in \{1, \dots, n\}$ the function $\psi_k(\phi(x_1) \cdots \phi(x_k))$ is k -additive symmetric, then f is a polynomial function of degree at most n . Hence f satisfies the Fréchet equation (8). $\square$

4. Counterimages and Proportionality

As it was noticed by Maksa and Rätz [6], and generalized by Sablik [10], for any group G , each two additive functions $f, g: G \rightarrow \mathbb{R}$, with the property

$$f^{-1}((0, \infty)) = g^{-1}((0, \infty))$$

are proportional, i.e. there exists a real number $r > 0$, such that $f = rg$.

Unfortunately similar result does not hold in the whole generality for generalized monomial functions of degree greater than 1. Indeed, if we take functions A_n^* defined in the Example 1, and $P_n(x) = x^k$, then $(A_n^*)^{-1}(0, \infty) = (0, \infty) = (P_n)^{-1}(0, \infty)$ for every $n \in \mathbb{N} \setminus 1$, but surely these functions are not proportional. The possible reason, for which such a generalization does not hold, may be the fact, that diagonal is too small comparing to the whole

domain, and its intersection with the counterimage of the positive half-line does not carry enough information about the function.

However the result quoted above may be positively extended to n -additive (not necessarily symmetric) functions. To justify this, we end with the following theorem.

Theorem 9. *Let $G_1, \dots, G_n$ be groups. Any two n -additive functions $f, g: G_1 \times \dots \times G_n \rightarrow \mathbb{R}$, with the property $f^{-1}((0, \infty)) = g^{-1}((0, \infty))$ are proportional, that is, there exists positive real number $r > 0$, such that $f = rg$.*

Proof. By theorem of Sablik we can assume that $n > 1$. For any $i \in \{1, \dots, n\}$ let us denote $G^{(i)} = G_1 \times \dots \times G_{i-1} \times G_{i+1} \times \dots \times G_n$, and let $\bar{G} = G_1 \times \dots \times G_n$. Moreover, for arbitrary function $h: \bar{G} \rightarrow \mathbb{R}$ let us denote by $h_i: G^{(i)} \rightarrow \text{Hom}(G_i, \mathbb{R})$ the function defined as

$$h_i(x^{(i)})(\cdot) = h(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n),$$

where $x^{(i)} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in G^{(i)}$. Obviously $h_i(x^{(i)})(x_i) = h(x)$ for all $x = (x_1, \dots, x_n) \in \bar{G}$, and thus for every fixed $i \in \{1, \dots, n\}$ and $x \in \bar{G}$ we have

$$\begin{aligned} x_i \in (f_i(x^{(i)}))^{-1}((0, \infty)) &\iff f_i(x^{(i)})(x_i) > 0 \iff f(x) > 0 \\ &\iff x \in f^{-1}((0, \infty)) \iff x \in g^{-1}((0, \infty)) \iff g(x) > 0 \\ &\iff g_i(x^{(i)})(x_i) > 0 \iff x_i \in (g_i(x^{(i)}))^{-1}((0, \infty)). \end{aligned}$$

Hence, by theorem of Sablik:

$$\forall_{i \in \{1, \dots, n\}} \forall_{x^{(i)} \in G^{(i)}} \exists_{\gamma_i(x^{(i)}) \in \mathbb{R}} \forall_{x_i \in G_i} f_i(x^{(i)})(x_i) = \gamma_i(x^{(i)})g_i(x^{(i)})(x_i).$$

By putting $r_i(x) = \gamma_i(x^{(i)})$, $i \in \{1, \dots, n\}$, we can rewrite the condition given above as

$$\forall_{i \in \{1, \dots, n\}} \forall_{x \in \bar{G}} f(x) = r_i(x)g(x).$$

But this, and the fact that function r_i is constant with respect to the i -th variable, implies that all these functions are equal and constant:

$$r_1 = r_2 = \dots = r_n \equiv r > 0.$$

□

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