The Cayley Property of Some Distant Graphs and Relationship with the Stern–Brocot Tree

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Abstract. One of the graphs associated with any ring R is its distant graph $G(R, \Delta)$ with points of the projective line $\mathbb{P}(R)$ over R as vertices. We prove that the distant graph of any commutative, Artinian ring is a Cayley graph. The main result is the fact that $G(\mathbb{Z}, \Delta)$ is a Cayley graph of a non-artinian commutative ring. We indicate two non-isomorphic subgroups of $PSL_2(\mathbb{Z})$ corresponding to this graph.

Mathematics Subject Classification. Primary 05C25, Secondary 51C05.

Keywords. Cayley graphs, distant graphs, projective lines over rings, Stern–Brocot tree.

1. Introduction

Let R be an associative ring with an identity. The automorphism group of the left R -module R^2 is the group $GL_2(R)$ of invertible 2×2 matrices with entries in R . The projective line over R is the orbit $\mathbb{P}(R) = R(1, 0)^{GL_2(R)}$ of the free cyclic submodule $R(1, 0)$ under the action of $GL_2(R)$. The projective line $\mathbb{P}(R)$ is endowed with the symmetric relation Δ (“distant”).

Two points $p, q \in \mathbb{P}(R)$ are *distant* if there exists $\gamma \in GL_2(R)$ such that $R(1, 0)^\gamma = p$, $R(0, 1)^\gamma = q$.

A pair $(a, b) \in R^2$ is called *admissible* if there exists $(a', b') \in R^2$ (not necessary unique) such that

$$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \in GL_2(R).$$

It is clear that every point of the projective line can be represented as $R(a, b)$ with the admissible pair (a, b) . If $p = R(a, b)$, $q = R(c, d) \in \mathbb{P}(R)$ with admissible (a, b) , $(c, d) \in R^2$ then

$$p\Delta q \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R).$$

The distant relation Δ defines the distant graph $G(R, \Delta)$ of the ring R , with the vertex set equal to the projective line of R . The notion of the distant relation was introduced in the nineties of the previous century (see e.g. [9]). Blunck and Havlicek explored this notion later in [2–4] and [5]. They gave for instance the necessary and sufficient condition for the distant graph to be connected. In [7] Blunck and Herzer consider $G(R, \Delta)$ as a special case of so called a distant space.

In the recently developing algebraic combinatorics there are definitions of some graphs attached to rings, for example comaximal graphs, unit graphs, zero-divisor graphs and total graphs [10]. The distant graph is yet another way of assigning a graph to a ring. It is a locally homogenous, regular of the degree $|R|$, vertex transitive graph with the transitive action of the group $PGL_2(R)$.

On the other hand, if $\mathcal{G}$ is a group and $\mathcal{S}$ is any set of its generators such that $1 \notin \mathcal{S}$, $\mathcal{S} = \mathcal{S}^{-1}$, the undirected Cayley graph $\Gamma(\mathcal{G}, \mathcal{S})$ of the group $\mathcal{G}$ is the graph with vertex set $\mathcal{G}$, where two vertices $g, h \in \mathcal{G}$ are connected iff $g^{-1}h \in \mathcal{S}$. It is also a locally homogenous, regular, vertex transitive graph with an automorphism subgroup isomorphic to $\mathcal{G}$.

Hence the natural question is whether a given distant graph may be a Cayley one.

In Sect. 2 we prove that if the distant graph of $\bar{R} = R/J$ (where J is the Jacobson radical of R) is a Cayley graph then the distant graph of R is so. Then for rings with R/J isomorphic to product of skew-fields we show the Cayley property. As a corollary we get the Cayley property for commutative rings.

In Sect. 3 we present some examples of rings with Cayley distant graphs: $\mathbb{Z}_n$ for any natural $n > 1$, lower (or upper) triangular matrices over a skew-field and the noncommutative ring of square matrices $M_2(p^n)$ over the finite Galois field F_{p^n} . By the way, we are pointing out the only planar distant graphs.

In the last section we prove that the distant graph $G(\mathbb{Z}, \Delta)$ of the ring of integers is a Cayley graph and we show that it can be seen as the Cayley graph of two non-isomorphic subgroups of $PSL_2(\mathbb{Z})$. Since $PSL_2(\mathbb{Z})$ is the free product $Z_2 * Z_3$, we can use the Kurosh subgroup theorem about the structure of subgroups of free products. The Cayley property of the distant graph $G(\mathbb{Z}, \Delta)$ allows us to find an alternative solution of the shortest path problem presented in the [11].

2. Cayley Representation of Some Special Rings

In this section we consider rings with identity, not necessary commutative, with the following property:

$$R/J \text{ is a product of skew-fields,} \quad (2.1)$$

where $J = J(R)$ is the Jacobson radical of R . As an example of such a ring may serve any semi-local commutative one. Indeed, if $\{M_i : i = 1, \dots, k\}$ is the family of maximal ideals in R , then

$$R/J = \prod_{i=1}^k R/M_i,$$

which easily follows from the Chinese Remainder Theorem.

First we describe points of $\mathbb{P}(R)$ in terms of members of J and points of $\mathbb{P}(\overline{R})$. Let $\pi : \mathbb{P}(R) \longrightarrow \mathbb{P}(\overline{R})$ denote the epimorphism induced by the canonical epimorphism $R \longrightarrow \overline{R}$. The following is essentially proved in [5].

Lemma 2.1. *There exists a surjective map $\rho : \mathbb{P}(R) \longrightarrow J$ such that the following map*

$$\mathbb{P}(R) \ni p \xrightarrow{D} (\rho(p), \bar{p}) \in J \times \mathbb{P}(\overline{R})$$

is a bijection.

Proof. First we show that

$$\pi^{-1}(\overline{R}(\overline{1}, \overline{0})) = \{R(1, j) : j \in J\}.$$

Obviously $R(1, j) \neq R(1, j')$ for $j, j' \in J, j \neq j'$. Further, if

$$R(a, b) \in \pi^{-1}(\overline{R}(\overline{1}, \overline{0})), \text{ then } R(a, b) = R(1 + i, j')$$

for some $i, j' \in J$. Since $1 + i \in R^*$,

$$R(a, b) = R(1, (1 + i)^{-1}j') = R(1, j),$$

where $j = (1 + i)^{-1}j' \in J$. Define

$$\pi^{-1}(\overline{R}(\overline{1}, \overline{0})) \ni R(a, b) \longmapsto \tilde{\rho}(R(a, b)) = j \in J.$$

Observe that $\tilde{\rho}$ is a bijection. Now consider an arbitrary section $\sigma : \mathbb{P}(\overline{R}) \longrightarrow \mathbb{P}(R)$ of π , $\pi \circ \sigma = Id_{\overline{R}}$. Define

$$\mathbb{P}(R) \ni p \longmapsto \rho(p) = \tilde{\rho}(p^\gamma) \in J,$$

where γ is an element of $GL_2(R)$ that carries $\sigma(\bar{p})$ to $R(1, 0)$: $\sigma(\bar{p})^\gamma = R(1, 0)$. Since $\gamma(\pi^{-1}(\bar{p})) = \pi^{-1}(\overline{R}(\overline{1}, \overline{0}))$ and $\tilde{\rho}$ is a bijection, ρ is a bijection on each of the cosets of π .

Since ρ is obviously surjective, D is so. Now D is bijective since $\rho|_{\pi^{-1}(\bar{p})}$ is bijective for every $p \in \mathbb{P}(R)$. $\square$

Corollary 2.2. *If the distant graph of $\overline{R} = R/J$ is a Cayley graph then the distant graph of R is so.*

Proof. Assume that $(\overline{\mathcal{G}}, \overline{\mathcal{S}})$ is an appropriate Cayley group with $\overline{\mathcal{G}}$ written additively.

We will show that $(J \times \overline{\mathcal{G}}, J \times \overline{\mathcal{S}})$ is a Cayley group of R (here J is understood as the additive group of the Jacobson radical of R).

Let $\mathbb{P}(\overline{R}) \ni \overline{p} \mapsto g_{\overline{p}} \in \overline{\mathcal{G}}$ be an appropriate bijection. Using Lemma 2.1 we get a bijection (being a composition of two bijections)

$$\mathbb{P}(R) \ni p \mapsto D(p) = (\rho(p), \overline{p}) \mapsto (\rho(p), g_{\overline{p}}) \in J \times \overline{\mathcal{G}}.$$

Obviously we have

$$p \triangle q \Leftrightarrow \overline{p} \triangle \overline{q} \Leftrightarrow (\rho(p), g_{\overline{p}}) - (\rho(q), g_{\overline{q}}) = (\rho(p) - \rho(q), g_{\overline{p}} - g_{\overline{q}}) \in J \times \overline{\mathcal{S}},$$

for every $p, q \in \mathbb{P}(R)$, which finishes the proof. $\square$

Theorem 2.3. *If R is a ring satisfying (2.1) then its distant graph is a Cayley graph.*

Proof. Assume that $\overline{R} \simeq \prod_{\lambda \in \Lambda} F_{\lambda}$, where F_{λ} are skew-fields.

The distant graph of F_{λ} is a complete graph of order $\text{card}(F_{\lambda}) + 1$. Hence, using any group $\mathcal{G}_{\lambda}$ of this order and $\mathcal{S}_{\lambda} = \mathcal{G}_{\lambda} \setminus \{1_{\lambda}\}$, where 1_{λ} is a neutral element of $\mathcal{G}_{\lambda}$, one obtains the distant graph as associated Cayley graph.

By our assumption we have

$$\mathbb{P}(\overline{R}) \simeq \prod_{\lambda \in \Lambda} \mathbb{P}(F_{\lambda}).$$

It is not difficult to see that the distant graph of a product of rings is a tensor product of appropriate graphs. On the other hand, it is easy to see that one of the Cayley representations of a tensor product of Cayley graphs can be obtain by taking the product of the Cayley groups of the factors with the set of generators equal to the product of the appropriate sets of generators. Therefore the distant graph of $\mathbb{P}(\overline{R})$ has a Cayley group equal to

$$\left(\prod_{\lambda \in \Lambda} \mathcal{G}_{\lambda}, \prod_{\lambda \in \Lambda} \mathcal{S}_{\lambda} \right).$$

Now we may use Corollary 2.2. $\square$

It is well known that every artinian commutative ring is a direct sum of local rings (see i.e. [1]). From this and Theorem 2.3 we have the following.

Corollary 2.4. *A distant graph of every artinian commutative ring is a Cayley graph.*

3. Examples

Example. Let $n \geq 1$ be a natural number and consider the ring $R = \mathbb{Z}_p^n$, where p is an arbitrary prime number. This is a local ring with a precisely one maximal ideal $M = J(R) \simeq \mathbb{Z}_{p^{n-1}}$. Therefore $\overline{R} \simeq \mathbb{Z}_p$ and the Cayley group of the distant graph of R is equal to $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p+1}$ with a generating set equal to $\mathbb{Z}_{p^{n-1}} \times (\mathbb{Z}_{p+1} \setminus \{0\})$.

Remark 3.1. *If a finite ring R has more then 4 elements then its distant graph is not planar.*

FIGURE 1. Distant graphs of **a** $\mathbb{Z}_2$, **b** $\mathbb{Z}_3$, **c** $\mathbb{Z}_4$ and $F_2[x]/(x^2)$

Proof. It is a well known fact that for a simple, finite, connected, planar graph with $v \geq 3$ vertices and e edges one has $e \leq 3v - 6$. Assume that $r = |R| \geq 6$. The degree of a vertex in a tensor product is equal to the product of degrees of its components. Since degree of each vertex is equal to r , we have: $v \geq r \geq 6$ and $e = \frac{rv}{2}$. Therefore $\frac{r}{2} \geq 3 > 3 - \frac{6}{v}$, which means that $e > 3v - 6$, and hence our graph is not planar.

There is only one finite ring with unity of order 5, namely F_5 (here and in the sequel by F_q we denote a finite field of order $q \geq 2$). Its distant graph is the non-planar graph K_6 . $\square$

Corollary 3.2. *Let R be a finite ring with unity. The distant graph of R is planar iff*

$$R \in \{F_2, F_3, \mathbb{Z}_4, F_2[x]/(x^2)\}.$$

Proof. If $|R| = p$ where $p = 2$ or 3 , then $R = F_p$ and its distant graph is the planar graph K_{p+1} , as is depicted on Fig. 1a, b.

If $|R| = 4$ then we have four rings to consider: $\mathbb{Z}_4$, $F_2[x]/(x^2)$, $F_2 \times F_2$ and F_4 .

Two former ones have the same planar graph depicted on Fig. 1c. Here we can use the proof of Theorem 2.3 to see that the Cayley group of the distant graph of $\mathbb{Z}_4$ is equal to $\mathbb{Z}_2 \times \mathbb{Z}_3$ with a generating set $\{(0, 1), (0, 2), (1, 1), (1, 2)\}$.

The ring F_4 has the non-planar graph K_5 .

The distant graph of the remaining ring $F_2 \times F_2$ is equal to $K_3 \otimes K_3$. In this case we cannot use the necessary condition $e \leq 3v - 6$ since for this graph we have $e = 18 < 21 = 3v - 6$. Nevertheless $K_3 \otimes K_3$ has the graph K_5 as its minor. Indeed, let $\{u_j, v_j, w_j : j = 1, 2, 3\}$ be the set of vertices of $K_3 \otimes K_3$ and let us decide that $a_i \triangle b_j$, where $a, b \in \{u, v, w\}$ and $i, j \in \{1, 2, 3\}$, iff simultaneously $a \neq b$ and $i \neq j$. The graph K_5 may be obtain as a minor of $K_3 \otimes K_3$ after contracting the following edges: (u_1, v_2) , (u_2, w_3) , (u_3, v_1) and

FIGURE 2. Contracting $K_3 \otimes K_3$ to K_5

(v_3, w_2) (see Fig. 2). Therefore $K_3 \otimes K_3$ is not a planar graph in virtue of Wagner's theorem. $\square$

Example. Let $R = T_n(K)$ be the ring of lower $n \times n$ triangular matrices over the skew-field K . The Jacobson radical is $J(R) = \{A \in R : a_{ii} = 0, 1 \leq i \leq n\}$. The only maximal ideals of $T_n(K)$ are $M_i = \{A \in R : a_{ii} = 0\}$, and $J(R) = \bigcap_{i=1}^n M_i$. Hence $R/J(R) \simeq \prod_{i=1}^n K_i$, where $K_i \simeq K$. Therefore R is a ring with the Cayley group of its distant graph equal to $J(R) \oplus (\bigoplus_{i=1}^n G_i)$, where $G_i \simeq G$ is any Cayley group corresponding to the Cayley group of K .

The algebra R contains the subalgebra $R' = \{A \in R : a_{11} = a_{22} = \dots = a_{nn}\}$. It is a local ring and $C' = J(R') \oplus G$ is a Cayley representation of its distant graph.

From the proof of Theorem 2.3 it follows that $\mathcal{S}$ is equal to $\{(g_1, \dots, g_n) : g_j \neq 1_G\}$ in the former case and $G \setminus \{1_G\}$ in the latter one.

Example. This example shows that the condition (2.1) is sufficient but not necessary for R to have a distant graph with Cayley's property. Let $R = M_2(F_q)$, $q \geq 2$. This is a non-commutative ring with no ideals. Nevertheless we show that it has a Cayley distant graph. It is known that:

- the projective line of R has $m = (q^2 + 1)(q^2 + q + 1)$ elements.

Moreover, from Result 7 and Theorem 1 of [6] it can be deduced that

- the distant graph consists of $q^2 + q + 1$ disjoint maximal cliques, each of cardinality of $q^2 + 1$.

Now we immediately have that

- each vertex has $(q^4 - q^2)/(q^2 + q) = q^2 - q$ adjacent vertices in every maximal clique it does not belong to.

Those properties characterize the graph. Let $\mathcal{G} = \mathbb{Z}_m$ and

$$\mathcal{S} = \left\{ \pm 1, \dots, \pm \frac{(q^2 - q)(q^2 + q + 1)}{2} \right\} \\ \cup \left\{ \pm k(q^2 + q + 1) : k = \frac{q^2 - q}{2} + 1, \dots, \left\lfloor \frac{q^2 + 1}{2} \right\rfloor \right\}.$$

FIGURE 3. Subgraph of the distant graph of $M_2(F_2)$ consisting of two disjoint maximal cliques

The Cayley graph $\Gamma(\mathcal{G}, \mathcal{S})$ has the following disjoint maximal cliques:

$$C_j = \left\{ \pm k(q^2 + q + 1) : k = 0, \dots, \left\lfloor \frac{q^2 + 1}{2} \right\rfloor \right\} + j, \quad j = 0, \dots, q^2 + q.$$

It may be checked by inspection that the Cayley graph of $(\mathcal{G}, \mathcal{S})$ satisfies the remaining properties described in the previous paragraph as well. Therefore $(\mathcal{G}, \mathcal{S})$ is a Cayley group of the distant graph of R . See Fig. 3, where a subgraph consisting of two disjoint maximal cliques is depicted in case $q = 2$.

4. The Distant Graph of the Ring of Integers

It can be shown that $PGL(2, \mathbb{Z})$ is isomorphic to the automorphism group of the distant graph of the ring of integers. We omit the proof since it is not crucial for our considerations. On the other hand the group serving as a Cayley representation is a subgroup of the automorphism group. Therefore we will seek for such a representation in $PSL(2, \mathbb{Z}) < PGL(2, \mathbb{Z})$.

One can draw the distant graph G of $\mathbb{Z}$ using the Stern–Brocot tree ([12]). In fact, to build G one has to perform the procedure twice. First starting with two points $[1, 0]$ and $[0, 1]$, then starting with $[-1, 0]$ and $[0, 1]$. After obtaining two Stern–Brocot trees we identify their nodes $[k, l]$ with vertices $\mathbb{Z}(k, l)$ of G , see Fig. 4. Recall that $\mathbb{Z}(k, l)$ belongs to $\mathbb{P}(\mathbb{Z})$ iff the greatest common divisor of k and l is 1.

In this section we will show that the distant graph of the ring of integers is a Cayley one. We find two non-isomorphic Cayley representations of this graph showing that such a representation is not an isomorphism invariant. In order to do this we use the approach developed by Hall in his proof of the Kurosh subgroup theorem [8].

FIGURE 4. Distant graph of $\mathbb{Z}$

Having a group which is a free product

$$\mathcal{G} = \prod_{\nu}^* A_{\nu}$$

write

$$g = a_1 a_2 \dots a_t$$

as the reduced form of the element of $\mathcal{G}$. This means that the identity is the void product; and for $g \neq 1$ each a_i is an element $\neq 1$ of one of the free factors A_{ν} , and no two consecutive terms a_i, a_{i+1} ($i = 1, \dots, t-1$) belong to the same free factor A_{ν} . The length $|g|$ of an element g is defined as zero for $g = 1$, and for $g \neq 1$ as the number t of terms in its reduced form. In fact Hall uses the so called semi-alphabetical ordering but in our setup we need only the partial ordering by length. Having a subgroup $\mathcal{H} < \mathcal{G}$ let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ denote a set of all elements $k \in \mathcal{H} \setminus \{1\}$ such that k does not belong to the group generated by the elements of $\mathcal{H}$ which precede k in the semi-alphabetical ordering.

Consider two independent generators

$$J = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

$J^2 = T^3 = Id$ of $\text{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$. Now put

$$K_0^{(i)} = \begin{cases} J : i = 0; \\ T : i = 1; \end{cases}$$

and for $n \geq 1$

$$K_{2n}^{(i)} = \begin{cases} (JT)^{-n+1} T J T^{-1} (JT)^{n-1} & : i = 0; \\ (TJ)^{-n+1} J_1 (TJ)^{n-1} & : i = 1, \end{cases}$$

where $J_1 = (JT)J(JT)^{-1}$, and define

$$\mathcal{K}_i = \{K_{2n}^{(i)} : n \geq 0\}.$$

Now we show that both sets $\mathcal{K}_i$ satisfy the following property (P):

An arbitrary finite product of elements of $\mathcal{K}_i$ has length greater than each of its factor.

Proof. We have to split calculations into separate cases (in both we will write $K_{2n} = K_{2n}^{(i)}$).

Case $i = 0$. Let $K = K_{2n_1} \dots K_{2n_t}$, $t \geq 2$. First observe that trivially we have

$$|K_{2n_1} \dots K_{2n_t}| = |K_{2n_1} \dots K_{2n_{j-1}}| + |K_{2n_{j+1}} \dots K_{2n_t}| + 1 > 1 = |K_{2n_j}|$$

if $n_j = 0$. Therefore it is enough to show (P) with the assumption $n_j > 0$. To end this note that

$$K_{2n_j} K_{2n_{j+1}} = (JT)^{-n_j+1} F (JT)^{n_{j+1}-1},$$

where

$$F = \begin{cases} T J T^{-1} (JT)^{n_j-n_{j+1}-1} (JT^{-1})^2 & : n_j > n_{j+1}; \\ (TJ)^2 (T^{-1}J)^{n_{j+1}-n_j-1} T J T^{-1} & : n_j < n_{j+1}, \end{cases}$$

which yields

$$|K_{2n_j} K_{2n_{j+1}}| = |K_{2n_j}| + |K_{2n_{j+1}}| - 4(n_j \wedge n_{j+1}) + 3.$$

(Having two numbers a and b we denote $a \wedge b = \min(a, b)$.) From this it follows the general formula

$$|K| = \sum_{j=1}^t |K_{n_j}| + 3(t-1) - 4 \sum_{j=1}^{t-1} (n_j \wedge n_{j+1}).$$

Now (P) follows from the fact that $|K_{2n}| = 4n - 1$ and

$$\begin{aligned} \sum_{j \neq i} n_j - \sum_{j=1}^{t-1} (n_j \wedge n_{j+1}) &= \sum_{j=1}^{i-1} (n_j - (n_j \wedge n_{j+1})) \\ &\quad + \sum_{j=i+1}^t (n_j - (n_{j-1} \wedge n_j)) \geq 0. \end{aligned} \quad (4.1)$$

Case $i = 1$. Similar considerations show that in this case

$$|K| = \sum_{j=1}^t |K_{n_j}| + (t-1) - 4 \sum_{j=1}^{t-1} (n_j \wedge n_{j+1}).$$

Now (P) follows from $|K_{2n}| = 4n + 1$ and (4.1). $\square$

Now it is obvious that K_{2n} , $n > 1$, does not belong to the subgroup generated by $\{K_{2m} : 0 \leq m < n\}$. From this and the property (P) of $\mathcal{K}_i$ we conclude that

$$\mathcal{K}_i = \mathcal{K}(\langle \mathcal{K}_i \rangle), \quad i = 0, 1.$$

Now $\mathcal{K}_0$ consists of involutions itself, and $\mathcal{K}_1$ contains only involutions with the exception of one element $K_0^{(1)}$ that is of order 3. Denote $\mathcal{C}_i := \langle \mathcal{K}_i \rangle$, $i = 0, 1$. From Hall's proof of the Kurosh subgroup theorem it follows that

$$\mathcal{C}_0 \simeq \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \dots \quad \text{and} \quad \mathcal{C}_1 \simeq \mathbb{Z}_3 * \mathbb{Z}_2 * \mathbb{Z}_2 * \dots,$$

hence $\mathcal{C}_0 \not\simeq \mathcal{C}_1$.

As a byproduct of the proof of the property (P) we get that

$$(JT)^n \in \mathcal{C}_i \quad \text{iff} \quad n = 0. \quad (4.2)$$

Now we will replace the sets of generators $\mathcal{K}_i$ by other sets of generators $\mathcal{S}_i$ in such a way that $\Gamma(\mathcal{C}_i, \mathcal{S}_i)$ become Cayley representations of G .

For $n \geq 0$ put

$$H_{2n}^{(i)} = \begin{cases} K_0^{(i)} K_2^{(i)} \cdots K_{2n}^{(i)} : i = 0; \\ K_2^{(i)} \cdots K_{2n}^{(i)} K_0^{(i)} : i = 1, \end{cases}$$

and

$$J_{2n+1}^{(i)} = H_{2n}^{(i)} (H_{2(n+1)}^{(i)})^{-1}, \quad i = 0, 1.$$

Observe that already the set $\{H_{2n}^{(i)} : n \geq 0\}$ generates $\mathcal{C}_i$. Finally define

$$S_n^{(i)} = \begin{cases} H_n^{(i)} & : \text{even } n \geq 0; \\ J_n^{(i)} & : \text{odd } n \geq 0; \\ (H_{-2(n+i)}^{(i)})^{-1} & : n < 0, \end{cases}$$

and put

$$\mathcal{S}_i = \{S_n^{(i)} : n \in \mathbb{Z}\}.$$

Obviously we have $\text{Id} \notin \mathcal{S}_i$, $\mathcal{C}_i = \langle \mathcal{S}_i \rangle$ and since $J_n^{(i)}$ are involutions, $\mathcal{S}_i = \mathcal{S}_i^{-1}$.

Lemma 4.1. *The following formulas hold*

$$S_{2n}^{(i)} = (JT)^{2n} J (JT)^{n+i} \quad \text{and} \quad S_{2n+1}^{(i)} = (JT)^{2n+1} J (JT)^{-2n-1}$$

for $n \geq 0$.

Proof. To show the left hand side equality in the even case we use induction on n . For $n = 0$ this follows from the definitions. Assume that this formula holds for some $n \geq 0$. Then in the case $i = 0$ we have

$$\begin{aligned} S_{2(n+1)}^{(0)} &= (K_0^{(0)} \cdots K_{2n}^{(0)}) \cdot K_{2(n+1)}^{(0)} = H_{2n}^{(0)} K_{2(n+1)}^{(0)} \\ &= (JT)^{2n} J(JT)^n (JT)^{-n} T J T^{-1} (JT)^n \\ &= (JT)^{2n+1} J T^{-1} (JT)^n \\ &= (JT)^{2(n+1)} T^{-1} J J T^{-1} T^{-1} J (JT)^{n+1} \\ &= (JT)^{2(n+1)} J (JT)^{n+1}. \end{aligned}$$

For $i = 1$ we have

$$\begin{aligned} S_{2(n+1)}^{(1)} &= K_2^{(1)} \cdots K_{2n}^{(1)} \cdot K_{2(n+1)}^{(1)} K_0^{(1)} \\ &= (K_2^{(1)} \cdots K_{2n}^{(1)} K_0^{(1)}) \cdot (K_0^{(1)})^{-1} K_{2(n+1)}^{(1)} K_0^{(1)} \\ &= H_{2n}^{(1)} (K_0^{(1)})^{-1} K_{2(n+1)}^{(1)} K_0^{(1)} \\ &= (JT)^{2n} J(JT)^{n+1} T^{-1} (TJ)^{-n} J T J T^{-1} J (TJ)^n T \\ &= (JT)^{2n} T (JT)^{n-1} J (TJ)^{-n} J T J T^{-1} (JT)^{n+1} \\ &= (JT)^{2n} (TJ)^n (TJ)^{-n} J T J T^{-1} (JT)^{n+1} \\ &= (JT)^{2n+1} J T^{-1} (JT)^{n+1} \\ &= (JT)^{2n+2} T^{-1} J J T^{-1} T^{-1} J (JT)^{n+2} \\ &= (JT)^{2(n+1)} J (JT)^{n+2}. \end{aligned}$$

The odd case follows from the definitions and just proved equalities. $\square$

We intend to show that $S_k^{(i)} (S_l^{(i)})^{-1} \in \mathcal{S}_i$ iff $|k - l| = 1$. In order to do this we define two involutions $\iota^{(i)} : \mathbb{Z} \longrightarrow \mathbb{Z}$, $i = 0, 1$ by

$$\iota^{(i)}(n) = \begin{cases} n & : n \geq 0, n - \text{odd}; \\ -\frac{n}{2} - i & : n \geq 0, n - \text{even}; \\ -2n - 2i & : n < 0. \end{cases}$$

From Lemma 4.1 we immediately have (with superscript omitted)

$$S_n^{-1} = S_{\iota(n)}. \quad (4.3)$$

Lemma 4.2. *The involutions ι satisfy*

$$\iota(\iota(n) - 1) = \iota(n + 1) + 1.$$

Proof. Let $n \geq 1$ be odd. Then

$$\begin{aligned} \iota(\iota(n) - 1) &= \iota(n - 1) = -\frac{n-1}{2} - i = -\frac{n+1}{2} - i + 1 \\ &= \iota(n + 1) + 1. \end{aligned}$$

If $n \geq 0$ is even then

$$\iota(\iota(n + 1) + 1) = \iota(n + 2) = -\frac{n+2}{2} - i = -\frac{n}{2} - i - 1 = \iota(n) - 1.$$

Let $i = 0$ and $n \leq -1$ or $i = 1$ and $n < -1$. Then

$$\begin{aligned} \iota(\iota(n) - 1) &= \iota(-2(n + i) - 1) \\ &= -2(n + i) - 1 = -2(n + 1 + i) + 1 = \iota(n + 1) + 1. \end{aligned}$$

In the remaining case $i = 1$ and $n = -1$ the formula follows directly from the definition. $\square$

To simplify notation we omitted the superscript „ (i) ” in the above lemma since the formula in this lemma is the same in both cases. In the sequel we will consistently omit this superscript.

Lemma 4.3. *The members of $\mathcal{S}$ satisfy*

$$S_k S_l^{-1} \in \mathcal{S} \quad \text{iff} \quad |k - l| = 1.$$

Proof. Necessity is obvious.

Assume that $k = l \pm 1$. Choosing the representation with the left lower entry equal to 1 we may write

$$S_k^{-1} S_{k \pm 1} = \begin{pmatrix} \iota(k) \pm 1 * & \\ 1 & -(\iota(k \mp 1) \mp 1) \end{pmatrix}.$$

The statement now follows from Lemma 4.2. $\square$

Now we are in a position to show that the Cayley graphs $\Gamma(\mathcal{C}, \mathcal{S})$ are both isomorphic to our distant graph G . In order to do this let us define a map $\phi : \mathcal{C} \longrightarrow V(G)$,

$$\phi(A) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

If $\phi(A) = \phi(A')$ for some $A, A' \in \mathcal{C}_i$ then $A^{-1}A' = \pm(JT)^n$ for some integer n . By (4.2), $A = A'$. We have shown that ϕ is 1-1.

Observe that if $A \in \mathcal{C}$ has representation with the left lower entry equal to 1 then $A \in \mathcal{S}$. Indeed, if this representation has the form $\begin{pmatrix} n & mn - 1 \\ 1 & m \end{pmatrix}$ then

$S_{\iota(n)}A = (JT)^{m+\iota(n)}$, hence (4.2) and (4.3) yield $A = S_n \in \mathcal{S}$. From this it follows that for arbitrary $A, A' \in \mathcal{C}$ we have

$$A^{-1}A' \in \mathcal{S} \quad \text{iff} \quad |\det(\phi(A), \phi(A'))| = 1 \quad \text{iff} \quad \phi(A) \triangle \phi(A').$$

We have shown that ϕ is a graph injection.

It is left to show that ϕ is an epimorphism. For a vertex y in some graph F denote a neighborhood of y in F by $\mathcal{S}_y = \{x \in V(F) : x \triangle y\}$. With such a notation it is enough to show that $\mathcal{S}_v \subset \phi(\mathcal{C})$ for each $v \in \phi(\mathcal{C})$ since $\pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \phi(\mathcal{C})$ and our distant graph is connected.

Let $\phi(A) = v$. We may enumerate neighborhoods $S_A = \{A_n : n \in \mathbb{Z}\}$ and $\mathcal{S}_v = \{v_n : n \in \mathbb{Z}\}$ in such a way that $A_k \triangle A_l$ and $v_k \triangle v_l$ iff $|k - l| = 1$. This follows from the definitions or may be checked by inspection in Fig. 4. Since ϕ is an injection, $\phi(S_A) \subset \mathcal{S}_v$ and if $\phi(A_k) = v_{n_k}$ then $v_{n_k} \triangle v_{n_l}$ iff $|k - l| = 1$. It follows that

$$\phi(S_A) = \{v_{n_k} : k \in \mathbb{Z}\} = \{v_n : n \in \mathbb{Z}\} = \mathcal{S}_v \subset \phi(\mathcal{C}).$$

Let us denote: $\mathcal{F}_k$ - the free group with k free generators; $\mathcal{D}_l$ - the free product of l copies of $\mathbb{Z}_2$; $\mathcal{T}_m$ - the free product of m copies of $\mathbb{Z}_3$, $k, l, m \in \mathbb{N}_0 \cup \{\aleph_0\}$. With such notations we obtained the following result.

Theorem 4.4. *The distant graph of the ring of integers is a Cayley one. Moreover, among its Cayley representations one can find two non-isomorphic groups $\mathcal{D}_{\aleph_0}$ and $\mathcal{T}_1 * \mathcal{D}_{\aleph_0}$.*

□

We are able to show even more (but it is not a subject of this paper), namely:

Every group of the form

$$\mathcal{F}_k * \mathcal{D}_l * \mathcal{T}_m < PSL(2, \mathbb{Z}),$$

where $k \in \{0, 2\}$, $k + l + m = \aleph_0$, can serve as a Cayley representation of the distant graph of integers.

□

We conjecture that:

A subgroup of $PSL(2, \mathbb{Z})$ is a group of Cayley representation of the distant graph of integers if and only if it is of the form

$$\mathcal{F}_k * \mathcal{D}_l * \mathcal{T}_m,$$

where $k \in (\mathbb{N}_0 \setminus \{1\}) \cup \{\aleph_0\}$, $k + l + m = \aleph_0$.

□

This will be a subject of a forthcoming paper.

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Received: August 2, 2018.

Accepted: September 27, 2018.