Solvability of Abstract Semilinear Equations by a Global Diffeomorphism Theorem

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Abstract. In this work we develop a method towards unique solvability of abstract semilinear equations. We use a global diffeomorphism theorem for which we provide a simplified proof. Applications to second order partial differential equations are given. Some additional technical tools about the properties of the Niemytskij operator are also given.

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1. Introduction

The aim of this work is to consider unique solvability of semilinear abstract equations with further applications to elliptic Dirichlet boundary value problems. In order to fulfill this task we derive some new methodology based on the global diffeomorphism theorem given in [8, Theorem 3.1] for which we provide a somewhat simplified and more intuitive proof. Up to now Theorem 2 and related global implicit function theorems from [9] have been applied to various first order integro-differential problems which cover also the so called fractional case (with the fractional derivative and also the fractional Laplacian) and correspond to Urysohn and Volterra type equations, see [4, 10, 12]. There was also an attempt to examine second order Dirichlet problems for O.D.E. in [2], but for certain specific problems and without any abstract scheme which allows for considering boundary value problems in some unified manner. Results for the continuous problem in [2] are related to the existence result obtained in [19]; although the methods are different, both yield the existence with similar assumptions. This suggests that it is possible to obtain an abstract framework based on a global invertibility result. Our applications are meant for partial differential equations and thus do not have their counterparts in [19, 21].

We consider the following abstract framework, see also [14]. Let $(H, \langle \cdot | \cdot \rangle)$ be a real Hilbert space with a norm denoted by $\|\cdot\|$ and let A be a self-adjoint operator on H with domain $D(A)$. Recall that $(D(A), \langle \cdot | \cdot \rangle_A)$ is a real Hilbert space. The scalar product is defined by

$$\langle \cdot | \cdot \rangle_A = \langle \cdot | \cdot \rangle + \langle A \cdot | A \cdot \rangle.$$

By $\|\cdot\|_A$ we denote the associated norm, i.e. the graph norm of A . Let $(B, \|\cdot\|_B)$ be a real Banach space and let

$$N : (B, \|\cdot\|_B) \rightarrow (H, \langle \cdot | \cdot \rangle)$$

be an operator which is not necessarily linear. In this framework we shall study in $D(A)$ the following equation

$$Ax = N(x) \tag{1}$$

In order to consider (1) we will make the following assumptions:

- (A1) $D(A) \subset B \subset H$ and the embedding $(D(A), \langle \cdot | \cdot \rangle_A) \hookrightarrow (B, \|\cdot\|_B)$ is compact;
- (A2) there exists a constant $\alpha > 0$ such that $\langle Au | u \rangle \geq \alpha \|u\|^2$ for all $u \in D(A)$;
- (N1) N is of class C^1 with $N'(u)$ symmetric, i.e. $\langle N'(u)x | y \rangle = \langle x | N'(u)y \rangle$ for all $u, x, y \in B$;
- (N2) there exist constants $0 < \beta < 1$, $0 < \gamma < \alpha$, $\delta > 0$, such that:
 - (i) $\|N(u)\| \leq \beta \|Au\| + \delta$ for all $u \in D(A)$;
 - (ii) $\langle N'(u)h | h \rangle \leq \gamma \|h\|^2$ for all $u, h \in D(A)$.

Our main abstract result reads as follows.

Theorem 1. *Assume that (A1)–(A2) and (N1)–(N2) are satisfied. Then equation (1) has a unique solution in $D(A)$. Moreover, the operator F given by $F(u) = Au - N(u)$ is a diffeomorphism.*

In the above Theorem we may replace assumption (A1) by the following one:

- (A1') $(B, \|\cdot\|_B) = (H, \langle \cdot | \cdot \rangle)$ and $A : D(A) \rightarrow H$ is a self-adjoint operator with purely discrete spectrum.

If the above holds then it follows by [20, Proposition 5.12] that the embedding $(D(A), \langle \cdot | \cdot \rangle_A) \hookrightarrow (H, \langle \cdot | \cdot \rangle)$ is compact.

Remark. While all spaces which we consider are over $\mathbb{R}$, the theory developed in [20] works for spaces over $\mathbb{C}$. Nevertheless, results which we use (namely: Propositions 3.10, 5.12 and 10.19) extend to the setting of a space over $\mathbb{R}$ using direct calculation based on Brezis book [5]. Moreover, the theorem of Kato-Rellich works in spaces over $\mathbb{R}$. The proof of this fact could be found in [7].

We will further apply our abstract tools in order to study the following nonlinear Dirichlet problem

$$\begin{cases} -\Delta u = f(x, u), \\ u|_{\partial\Omega} = 0. \end{cases} \quad (2)$$

Here $\Omega \subset \mathbb{R}^m$ is an open and bounded set of class C^2 and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -Caratheodory function, i.e. for a.e. $x \in \Omega$, $f(x, \cdot)$ is of class C^1 and for all $u \in \mathbb{R}$, $f(\cdot, u)$, $f'_u(\cdot, u)$ are measurable.

Our results towards the abstract approach were inspired by some recent abstract approaches developed in [6, 17] which were based on the variational framework due to [17] and which utilize relations between critical points to action functionals and fixed points of certain mappings. Nevertheless, our approach towards solvability is different and relies on different abstract tools.

The paper is organized as follows. Firstly we provide some remarks on the main abstract tool. Next we consider the solvability of abstract equations which is then followed by applications to elliptic problems. Finally we provide an appendix in which we address differentiability of some Niemytskij type operator in our applied case. We provide the proof since we have not found such a result in the literature.

2. Remarks on a Global Diffeomorphism Theorem

Theorem 2 proposes an approach to the existence of unique solutions to nonlinear equations which is variational in spirit, i.e. concerns the use of certain functionals which are different from the classical energy (Euler type) action functional. Moreover, it allows one to obtain uniqueness of solutions without any notion of convexity, again contrary to what is known in the application of a direct method, see for example [13, Corollary 1.3].

Theorem 2 [8, Theorem 3.1]. *Let X be a real Banach space and let H be a real Hilbert space. Suppose that $F : X \rightarrow H$ is a C^1 mapping such that:*

- (D1) *for every $y \in H$ the functional $\varphi_y : X \rightarrow \mathbb{R}$ given by $\varphi_y(x) := \frac{1}{2} \|F(x) - y\|^2$ satisfies the Palais–Smale condition, i.e. every sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that $(\varphi_y(x_n))_{n \in \mathbb{N}}$ is bounded and $\varphi'_y(x_n) \rightarrow \mathbf{0}_{X^*}$ admits a convergent subsequence;*
- (D2) *for every $x \in X$ the operator $F'(x)$ is bijective.*

Then the operator F is a diffeomorphism.

The proof of Theorem 2 is based on the application of the celebrated Mountain Pass Theorem due to Ambrosetti and Rabinowitz, see [1], and relies in checking that the functional φ satisfies the mountain geometry. Precisely the fact that F is onto is reached through the classical Ekeland's Variational Principle. The injectivity part is obtained by contradiction by using the Mountain Pass Theorem. The most difficult part of the proof is the estimation of φ

on some sphere around $\mathbf{0}$. However, we will show using some ideas from [18] that the proof can be performed in a different and more readable manner thus simplifying the arguments from [8].

For the proof of Theorem 2 we need the following

Theorem 3 [18, Theorem 2]. *Let X be a Banach space and let $J : X \rightarrow \mathbb{R}$ be a C^1 functional satisfying the Palais–Smale condition with $\mathbf{0}_X$ its strict local minimum. If there exists $e \neq \mathbf{0}_X$ such that $J(e) \leq J(\mathbf{0}_X)$, then there is a critical point $\bar{x}$ of J , with $J(\bar{x}) > J(\mathbf{0}_X)$, which is not a local minimum.*

Proof of Theorem 2. The proof that operator F is „onto” is taken from the original proof of Theorem 2 and we provide it for reader’s convenience. Fix $y \in H$. As F is of class C^1 , $\varphi_y(x) = \frac{1}{2} \|F(x) - y\|^2$ is of the same type and its differential $\varphi'_y(x)$ at $x \in X$ is given by

$$\varphi'_y(x)h = \langle F(x) - y \mid F'(x)h \rangle.$$

for all $h \in X$. Clearly, φ_y is bounded from below and it satisfies the Palais–Smale condition, by D1. Hence, φ_y has a critical point (see [13, Chapter 3, Corollary 3.3]). In other words, there exists $x_0 \in X$ such that $\langle F(x_0) - y \mid F'(x_0)h \rangle = 0$ for all $h \in X$. Since $F'(x_0)$ is surjective, $F(x_0) - y = 0$ and so $F(x_0) = y$.

Now we show that F is “one-to-one”. Aiming for a contradiction, suppose that there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $F(x_1) = F(x_2)$. Define $e := x_1 - x_2$ and put $\psi : X \rightarrow \mathbb{R}$ by formula

$$\psi(x) := \frac{1}{2} \|F(x + x_2) - F(x_1)\|^2 = \varphi_{F(x_1)}(x + x_2).$$

Then ψ is of class C^1 and $\psi(\mathbf{0}_X) = \psi(e) = 0$. Moreover, $\mathbf{0}_X$ is a strict local minimum of ψ , since otherwise, in any neighbourhood of $\mathbf{0}_X$ we would have a nonzero x with $F(x + x_2) - F(x_1) = \mathbf{0}_H$ and this would contradict the fact that F defines a local diffeomorphism. Therefore we can apply Theorem 3 and, consequently, there exists $\bar{x} \in X$ such that $\psi(\bar{x}) > 0$ and $\psi'(\bar{x}) = \mathbf{0}_{X^*}$. Hence

$$\psi'(\bar{x})h = \langle F(\bar{x} + x_2) - F(x_1) \mid F'(\bar{x} + x_2)h \rangle = 0$$

for all $h \in X$. Again, by surjectivity of $F'(\bar{x} + x_2)$, we have $F(\bar{x} + x_2) - F(x_1) = 0$ and so $\psi(\bar{x}) = 0$, which contradicts $\psi(\bar{x}) > 0$. The obtained contradiction finishes the proof. $\square$

We would like to note that our arguments comply with the proof of the well known Hadamard’s Theorem, see Theorem 5.4 from [11], see also [16].

3. Proof of the Main Abstract Tool

Now, we can present the proof of the main Theorem.

Proof of Theorem 1. By (A2) we have $\|Au\| \geq \alpha \|u\|$ for $u \in D(A)$ and so

$$\|Au\| \geq \frac{\alpha}{1+\alpha} (\|Au\| + \|u\|) \geq \frac{\alpha}{1+\alpha} \|u\|_A. \quad (3)$$

Let $X := (D(A), \|\cdot\|_A)$ and let the operator $\tilde{N} : X \rightarrow H$ be defined by $\tilde{N} = N \circ i$, where $i : X \hookrightarrow (B, \|\cdot\|_B)$ is a compact embedding given by (A1). Then $\tilde{N} \in C^1(X, H)$ and the operator $\tilde{N}'(u)$ is symmetric, compact and linear for all $u \in X$, by (N1). Since $i(u) = u$, any solution of equation

$$Au = \tilde{N}(u)$$

is also a solution of Eq. (1).

Let us define $F : X \rightarrow H$ by

$$F(u) := Au - \tilde{N}(u). \quad (4)$$

Fix $y \in H$ and consider the mapping $\varphi_y : X \rightarrow \mathbb{R}$ given by

$$\varphi_y(u) := \frac{1}{2} \|F(u) - y\|^2. \quad (5)$$

Then $\varphi_y \in C^1(X, \mathbb{R})$ and $F \in C^1(X, H)$. Moreover, their derivatives are given, respectively, by the following formulas

$$\varphi'_y(u)h = \left\langle Au - \tilde{N}(u) - y \mid Ah - \tilde{N}'(u)h \right\rangle$$

and

$$F'(u)h = Ah - \tilde{N}'(u)h$$

for every $u, h \in X$.

In order to be able to use Theorem 2 we must show that φ_y satisfies the Palais–Smale condition and $F'(u)$ is bijective for all $u \in X$.

By applying (N2) we see that

$$\|F(u) - y\| = \|Au - N(u) - y\| \geq \|Au\| - \beta \|Au\| - \delta - \|y\| \geq \frac{(1-\beta)\alpha}{1+\alpha} \|u\|_X - \delta - \|y\|$$

for every $u \in X$. This implies that φ_y is coercive. Thus any Palais–Smale sequence can be assumed to be weakly convergent.

Now we show that the functional φ_y satisfies the Palais–Smale condition on X . Assume that $(u_n)_{n \in \mathbb{N}} \subset X$ is such that:

(PS1) $(\varphi_y(u_n))_{n \in \mathbb{N}}$ is bounded;

(PS2) $\varphi'_y(u_n) \rightarrow \mathbf{0}_{X^*}$ if $n \rightarrow \infty$.

Since φ_y is coercive, (PS1) shows that $(u_n)_{n \in \mathbb{N}}$ is bounded in X , and then possibly up to a subsequence, it is weakly convergent to some $u_0 \in X$. From (A1) there exists another subsequence, denoted by $(u_n)_{n \in \mathbb{N}}$, which is convergent in $(B, \|\cdot\|_B)$. So, by our assumptions we have

- $u_n \rightarrow u_0$ in B ;
- $\tilde{N}(u_n) \rightarrow \tilde{N}(u_0)$ in H ;
- $\tilde{N}'(u_n) \rightarrow \tilde{N}'(u_0)$ in $\mathcal{L}(X, H)$;
- $(A(u_n))_{n \in \mathbb{N}}$ is bounded in H .

Now, a direct calculation yields

$$\varphi'_y(u_n)(u_n - u_0) - \varphi'_y(u_0)(u_n - u_0) = \|Au_n - Au_0\|^2 + \sum_{k=1}^4 \psi_k(u_n), \quad (6)$$

where

$$\begin{aligned} \psi_1(u_n) &= \left\langle Au_0 - \tilde{N}(u_0) \mid \tilde{N}'(u_0)(u_n - u_0) \right\rangle, \\ \psi_2(u_n) &= \left\langle \tilde{N}(u_0) - \tilde{N}(u_n) \mid Au_n - Au_0 \right\rangle, \\ \psi_3(u_n) &= \left\langle \tilde{N}(u_n) - Au_n \mid \tilde{N}'(u_n)(u_n - u_0) \right\rangle, \\ \psi_4(u_n) &= \left\langle y \mid (\tilde{N}'(u_0) - \tilde{N}'(u_n))(u_n - u_0) \right\rangle. \end{aligned}$$

Then, using observations made above, we obtain

$$\begin{aligned} |\psi_1(u_n)| &\leq \|Au_0 - \tilde{N}(u_0)\| \|\tilde{N}'(u_0)(u_n - u_0)\| \rightarrow 0, \\ |\psi_2(u_n)| &\leq \|Au_n - Au_0\| \|\tilde{N}(u_n) - \tilde{N}(u_0)\| \rightarrow 0, \\ |\psi_3(u_n)| &\leq \|\tilde{N}(u_n) - Au_n\| \|\tilde{N}'(u_n)(u_n - u_0)\| \rightarrow 0, \\ |\psi_4(u_n)| &\leq \|y\| \|(\tilde{N}'(u_0) - \tilde{N}'(u_n))(u_n - u_0)\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. The convergence of $\tilde{N}'(u_n)(u_n - u_0)$ follows from inequality

$$\begin{aligned} \|\tilde{N}'(u_n)(u_n - u_0)\| &= \|(\tilde{N}'(u_n) - \tilde{N}'(u_0))(u_n - u_0) - \tilde{N}'(u_0)(u_n - u_0)\| \\ &\leq \|\tilde{N}'(u_n) - \tilde{N}'(u_0)\|_{\mathcal{L}(B, H)} \|u_n - u_0\| + \|\tilde{N}'(u_0)(u_n - u_0)\|. \end{aligned}$$

On the other hand, by (PS2) and the weak convergence of $(u_n)_{n \in \mathbb{N}}$ to u_0 in X , we have

$$|\varphi'_y(u_n)(u_n - u_0)| \leq \|\varphi'_y(u_n)\|_{X^*} \|u_n - u_0\|_X \rightarrow 0$$

and

$$|\varphi'_y(u_0)(u_n - u_0)| \rightarrow 0$$

as $n \rightarrow \infty$. Combining the above observations, we can now show that equality (6) implies

$$\|Au_n - Au_0\| \rightarrow 0$$

as $n \rightarrow \infty$ which means, by (3), that $(u_n)_{n \in \mathbb{N}}$ converges strongly to u_0 in X . This shows that φ_y satisfies the Palais-Smale condition.

Now, we show that $F'(u)$ is bijective for any $u \in X$. Fix $u \in X$. Since A is a self-adjoint operator and since $\tilde{N}'(u)$ is a symmetric, compact and linear operator, it follows by the RKNG Theorem in real Hilbert space, see [7], that $F'(u)$ is self-adjoint operator. Using (A2) and (N2) we get

$$\|Ah - \tilde{N}'(u)h\| \|h\| \geq \langle Ah - \tilde{N}'(u)h | h \rangle = \langle Ah | h \rangle - \langle N'(u)h | h \rangle \geq \alpha \|h\|^2 - \gamma \|h\|^2.$$

Therefore we have

$$\|F'(u)h\| = \|Ah - \tilde{N}'(u)h\| \geq (\alpha - \gamma) \|h\| \quad (7)$$

for all $h \in H$. Then, as $F'(u)$ is linear, it is injective. By Proposition 3.10 from [20] it follows that $F'(u)$ is also surjective, and this is why it is bijective.

Now we can apply Theorem 2 in order to obtain a unique $u^* \in X$ such that $0 = F(u^*) = Au^* - \tilde{N}(u^*)$. Moreover, using the same Theorem we see that F is a diffeomorphism. $\square$

4. Applications

As an application of Theorem 1 we discuss the solvability of problem (2). Firstly we show how to construct a suitable abstract framework. We take $H = L^2(\Omega)$ and as A , the operator $Au = -\Delta u$ with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, see [20, Proposition 10.19]. By the Poincaré inequality

$$c_\Omega \int_\Omega |u(x)|^2 dx \leq \sum_{k=1}^m \int_\Omega |\partial_k u(x)|^2 dx$$

and by the Green's formula we have

$$\langle Au | u \rangle \geq c_\Omega^2 \|u\|^2, \quad u \in D(A),$$

where c_Ω is a constant in Poincaré inequality and where $\langle \cdot | \cdot \rangle$ and $\|\cdot\|$ denote the scalar product and the norm in H , respectively. We note that on space $D(A)$ the graph norm of A and norm $\|\cdot\|_{H^2(\Omega)}$ are equivalent, see [20, p. 240]. Therefore, if we put

$$B_m(\Omega) := \begin{cases} C(\overline{\Omega}) & \text{if } m \leq 3, \\ L^{p_m}(\Omega) & \text{if } m \geq 4, \end{cases}$$

where $p_m > 2$ for $m = 4$ and $p_m \in \left(2, \frac{2m}{m-4}\right)$ for $m > 4$, we obtain the compact embedding

$$(D(A), \langle \cdot | \cdot \rangle_A) \hookrightarrow (B_m(\Omega), \|\cdot\|_{B_m}),$$

see [15, Theorem 1.51]. For $m \leq 3$ let $c_m > 0$ be such that $\|u\|_\infty \leq c_m \|Au\|$ for all $u \in D(A)$, where $\|\cdot\|_\infty$ denotes the supremum norm.

We will need the following assumptions on f :

(P1m)

(if $m \leq 3$) there exist $a_1, b_1 \in L^2(\Omega)$, $\|b_1\| < c_m^{-1}$ such that $|f(x, u)| \leq a_1(x) + b_1(x)|u|$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$;

(if $m \geq 4$) there exist $a_1 \in L^2(\Omega)$ and $b_1 \in (0, c_\Omega)$ such that $|f(x, u)| \leq a_1(x) + b_1|u|$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$;

(P2m)

- (if $m \leq 3$) there exist $a_2 \in L^2(\Omega)$ and $g \in C(\mathbb{R}, \mathbb{R})$ such that $|f'_u(x, u)| \leq a_2(x)g(u)$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$;
- (if $m \geq 4$) there exist $a_2 \in L^q(\Omega)$ and $b_2 > 0$ such that $|f'_u(x, u)| \leq a_2(x) + b_2|u|^r$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$, where $r = \frac{p_m-2}{2}$ and $q = \frac{2p_m}{p_m-2}$;
- (P3) there exists $b_3 \in (0, c_\Omega^2)$ such that $f'_u(x, u) < b_3$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$.

Under the above assumptions the operator $N_f : B_m(\Omega) \rightarrow L^2(\Omega)$ given by the formula $N_f(u)(x) = f(x, u(x))$ for $x \in \Omega$ is of class C^1 with $N'_f(u)(h) = N_{f'_u}(u)(h)$ for all $u, h \in B_m(\Omega)$. For the case $m \geq 4$ see [15, Proposition 2.78] and for the case $m \leq 3$ see “Appendix”. Clearly, $N'_f(u)$ is a symmetric operator for all $u \in B_m(\Omega)$.

In order to check (N2), take some $u \in B_m(\Omega)$. Using (P1m) we have for $m \leq 3$

$$\begin{aligned} \|N_f(u)\| &= \left(\int_{\Omega} |f(x, u(x))|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_{\Omega} |a_1(x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} b_1(x) |u(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \|a_1\| + \|b_1\| \|u\|_{\infty} \leq \|a_1\| + c_m \|b_1\| \|Au\|. \end{aligned}$$

and for $m \geq 4$

$$\begin{aligned} \|N_f(u)\| &= \left(\int_{\Omega} |f(x, u(x))|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_{\Omega} |a_1(x)|^2 dx \right)^{\frac{1}{2}} + b_1 \left(\int_{\Omega} |u(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \|a_1\| + b_1 \|u\| \leq \frac{b_1}{c_\Omega} \|Au\| + \|a_1\|. \end{aligned}$$

Assumption (P3) provides that for every $u, h \in B_m(\Omega)$ we have

$$\langle N'_f(u)h | h \rangle = \int_{\Omega} f'_u(x, u(x))h(x)h(x) dx \leq b_3 \int_{\Omega} |h(x)|^2 dx = b_3 \|h\|^2.$$

As a conclusion, we obtain

Theorem 4. Assume that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -Caratheodory function such that (P1m), (P2m) and (P3) hold. Then problem (2) has an unique solution in $H_0^1(\Omega) \cap H^2(\Omega)$.

As an example the following problems

$$\begin{cases} -\Delta u = C_1 \left(1 - \frac{1}{|x|}\right) (u - 1), \\ u|_{\partial\Omega} = 0, \\ -\Delta u = C_2 (-\arctan(u)x_2^2 + \exp(-u^2) + \phi(x_1, x_2, x_3)), \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is an open bounded set of class C^2 , constants C_1 and C_2 only depend on Ω , $\phi \in L^2(\Omega)$ and $|\cdot|$ denotes the Euclidean norm, have unique solutions in $H_0^1(\Omega) \cap H^2(\Omega)$.

5. Appendix

Let $m \leq 3$. Assume that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -Caratheodory function such that (P1m), (P2m) hold. We show that an operator $N_f : C(\overline{\Omega}) \rightarrow L^2(\Omega)$ given by the formula $N_f(u)(x) = f(x, u(x))$ for $x \in \Omega$ is of class C^1 with $N'_f(u)(h) = N_{f'_u}(u)(h)$ for all $u, h \in C(\overline{\Omega})$. By Theorem B in [3], it follows that if $|f(x, u)| \leq a(x)g(u)$ for all $x \in \Omega$ and $u \in \mathbb{R}$ with $a \in L^2(\Omega)$ and $g \in C(\mathbb{R}, \mathbb{R})$, then N_f is continuous from $C(\overline{\Omega})$ into $L^2(\Omega)$.

First, we show that for all $u, h \in C(\overline{\Omega})$, $N_{f'_u}(u)(h) \in L^2(\Omega)$. Indeed, we have

$$\begin{aligned} \|N_{f'_u}(u)(h)\| &= \left(\int_{\Omega} |f'_u(x, u(x))h(x)|^2 dx \right)^{\frac{1}{2}} \leq \|h\|_{\infty} \left(\int_{\Omega} |f'_u(x, u(x))|^2 dx \right)^{\frac{1}{2}} \\ &\leq \|h\|_{\infty} \left(\int_{\Omega} |a_2(x)g(u(x))|^2 dx \right)^{\frac{1}{2}} \leq \|h\|_{\infty} \|a\| \sup_{x \in \Omega} |g(u(x))| < \infty. \end{aligned}$$

Now, fix $u \in C(\overline{\Omega})$ and let

$$w(h) = N_f(u + h) - N_f(u) - N_{f'_u}(u)(h)$$

for all $h \in C(\overline{\Omega})$. Let $h \in C(\overline{\Omega})$ be now fixed. We have

$$f(x, u(x) + h(x)) - f(x, u(x)) = \int_0^1 f'_u(x, u(x) + \tau h(x))h(x) d\tau$$

Hence, using Fubini's theorem, we obtain

$$\begin{aligned} \|w(h)\| &= \left(\int_{\Omega} |f(x, u(x) + h(x)) - f(x, u(x)) - f'_u(x, u(x))h(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \|h\|_{\infty} \left(\int_{\Omega} \left| \int_0^1 f'_u(x, u(x) + \tau h(x)) - f'_u(x, u(x)) d\tau \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \|h\|_{\infty} \left(\int_0^1 \int_{\Omega} |f'_u(x, u(x) + \tau h(x)) - f'_u(x, u(x))|^2 dx d\tau \right)^{\frac{1}{2}} \end{aligned}$$

Since $N_{f'_u} : C(\overline{\Omega}) \rightarrow L^2(\Omega)$ is continuous, the above implies that

$$\frac{\|w(h)\|}{\|h\|_{\infty}} \rightarrow 0$$

as $\|h\|_\infty \rightarrow 0$. The continuity of the map $N'_f : C(\overline{\Omega}) \rightarrow \mathcal{L}(C(\overline{\Omega}), L^2(\Omega))$ follows now from the continuity of $N_{f'_u} : C(\overline{\Omega}) \rightarrow L^2(\Omega)$. Indeed, suppose that $u_n \rightarrow u_0$ in $C(\overline{\Omega})$. Then

$$\begin{aligned} \|N'_f(u_n) - N'_f(u_0)\|_{\mathcal{L}(C(\overline{\Omega}), L^2(\Omega))} &= \sup_{\|h\|_\infty=1} \left(\int_{\Omega} |f'_u(x, u_n(x))h(x) \right. \\ &\quad \left. - f'_u(x, u_0(x))h(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\Omega} |f'_u(x, u_n(x)) - f'_u(x, u_0(x))|^2 dx \right)^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow 0$.

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