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Extremal Polynomials and Entire Functions of Exponential Type

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Abstract. In this paper, we discuss asymptotic relations for the approximation of $|x|^{\alpha}, \alpha > 0$ in $L_{\infty}[-1, 1]$ by Lagrange interpolation polynomials based on the zeros of the Chebyshev polynomials of first kind. The limiting process reveals an entire function of exponential type for which we can present an explicit formula.

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1. Polynomial Interpolation and the Bernstein Constant

Let $\alpha > 0$ be not an even integer. Starting in year 1913 for the case $\alpha = 1$, and later in 1938 for the general case $\alpha > 0$, S.N. Bernstein [1,3] established the existence of the limit

$$\Delta_{\infty,\alpha} = \lim_{n\to\infty} n^{\alpha} E_n\left(|x|^{\alpha}, L_{\infty}[-1,1]\right),$$

where

$$E_n\left(f, L_p[a,b]\right) = \inf\left\{\|f-p\|_{L_p[a,b]} : \deg(p) \le n\right\}$$

denotes the error in best L_p approximation of a function f on the interval $[a, b]$ by polynomials of degree less or equal n. The proofs in [1,3] are highly difficult and long, missing many non-trivial technical details. In his 1938 paper, Bernstein made essential use of the homogeneity property of $|x|^{\alpha}$, namely that for $c > 0$ one has $|cx|^{\alpha} = c^{\alpha}|x|^{\alpha}$. Using this property, one gets for $a, b > 0$ and all $1 \le p \le \infty$ the relation (see [9], Lemma 8.2)

$$E_n\left(|x|^{\alpha}, L_p[-b,b]\right) = \left(\frac{b}{a}\right)^{\alpha+\frac{1}{p}} E_n\left(|x|^{\alpha}, L_p[-a,a]\right). \tag{1.1}$$

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This enabled Bernstein to relate the uniform best approximating error on $[-1,1]$ to that on $[-n,n]$. A routine argument shows that identity (1.1) sends the best approximating polynomials P_n^* of order n with respect to $[-1,1]$ into a sequence $\left\{n^\alpha P_n^*\left(\frac{\cdot}{n}\right) : n = 1,2,\ldots\right\}$ of scaled polynomials in $[-n,n]$. Bernstein also established a formulation of the limit as the error in approximation on the real line by entire functions of exponential type, namely,

$$\begin{aligned}\Delta_{\infty,\alpha} &= \lim_{n\to\infty} n^\alpha E_n\left(|x|^\alpha, L_\infty[-1,1]\right)\\ &= \lim_{n\to\infty} E_n\left(|x|^\alpha, L_\infty[-n,n]\right)\\ &= \lim_{n\to\infty}\left\| |x|^\alpha - n^\alpha P_n^*\left(\frac{\cdot}{n}\right)\right\|_{L_\infty[-n,n]}\\ &= \inf\left\{\left\||x|^\alpha - H\right\|_{L_\infty(\mathbb{R})} : H \text{ is entire of exponential type} \le 1\right\}.\end{aligned}$$

Recall that an entire function f is of exponential type $A \ge 0$ means that for each $\varepsilon > 0$ there is $z_0 = z_0(\varepsilon)$, such that

$$|f(z)| \le \exp\left(|z|(A+\varepsilon)\right), \quad \forall z \in \mathbb{C} : |z| \ge |z_0|. \tag{1.2}$$

Moreover, A is taken to be the infimum over all possible numbers for which (1.2) holds. The elegant formulation which introduces now functions of exponential type extends to spaces other than L_∞. Ganzburg [5] and Lubinsky [9] have shown that for all $1 \le p \le \infty$ positive constants $\Delta_{p,\alpha}$ exists, where $\Delta_{p,\alpha}$ is defined by

$$\begin{aligned}\Delta_{p,\alpha} &= \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} E_n\left(|x|^\alpha, L_p[-1,1]\right)\\ &= \inf\left\{\left\||x|^\alpha - H\right\|_{L_p(\mathbb{R})} : H \text{ is entire of exponential type } \le 1\right\}.\end{aligned} \tag{1.3}$$

From now on $\Delta_{p,\alpha}$ are called the Bernstein constants.

Only for $p = 1,2$ are the values $\Delta_{p,\alpha}$ known. In 1947, Nikolskii [11] proved that

$$\Delta_{1,\alpha} = \frac{\left|\sin\frac{\alpha\pi}{2}\right|}{\pi} 8\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^{\alpha+2}}, \quad \alpha > -1,$$

and in 1969, Raitsin [13] established

$$\Delta_{2,\alpha} = \frac{\left|\sin\frac{\alpha\pi}{2}\right|}{\pi} 2\Gamma(\alpha+1)\sqrt{\frac{\pi}{2\alpha+1}}, \quad \alpha > -\frac{1}{2}.$$

In contrast to the case of the L_∞ norm, no single value of $\Delta_{\infty,\alpha}$ is known. Bernstein speculated that

$$\Delta_{\infty,1} = \lim_{n\to\infty} nE_n\left(|x|, L_\infty[-1,1]\right) = \frac{1}{2\sqrt{\pi}} = 0.28209\ 47917\ldots$$

Over the years the speculation became known as the Bernstein conjecture in approximation theory. Some 70 years later Varga and Carpenter [17], using sophisticated high precision scientific computational methods, calculated the quantity numerically to

$$\Delta_{\infty,1} = 0.28016\ 94990\ 23869\ldots$$

Further extensive numerical explorations for the computation of $\Delta_{\infty,\alpha}$ have been provided later by Varga and Carpenter [18]. Their numerical work gave an enormous impact into the analytical investigation of approximation problems, not only restricted to the Bernstein constants. We would also like to mention the numerical work of Pachón and Trefethen ([12], Figure 4.4) from 2008, when they recomputed $\left\{nE_n\left(|x|, L_\infty[-1,1]\right) : n = 1,\ldots,10^4\right\}$ again and provided an graphical illustration indicating a monotonic growth behavior. As the story continued, the approximation of entire functions of exponential type became a much studied topic in function approximation, see [4,16], but also in connection to problems in number theory, see for instance [19]. As an further application in number theory, we would like to mention a recent paper of Ganzburg [7], where he discusses new asymptotic relations between Zeta-, Dirichlet- and Catalan functions in connection with the asymptotics of Lagrange–Hermite interpolation for $|x|^\alpha$.

Turning back to the Bernstein constants $\Delta_{p,\alpha}$, intensive emphasis has been placed on the structure of those entire functions of exponential type which minimize (1.3). For $p = 1$, the (unique) minimizing entire function of exponential type 1 may be expressed as an interpolation series at the nodes $\left\{\left(j - \frac{1}{2}\right)\pi : j = 1,2,\ldots\right\}$, see ([5], p. 197) or ([10], Formula 1.8). For $p = \infty$ an analogous interpolation series at unknown interpolation nodes was derived by Lubinsky in ([10], Theorem 1.1). In ([9], Theorem 1.1) he proved the following result.

Denote by P_n^* the best approximating polynomial of order n to $|x|^\alpha$ in the L_p norm. Then, for all $1 \le p \le \infty$, $\alpha > -\frac{1}{p}$ not an even integer, one has

$$\begin{aligned}\Delta_{p,\alpha} &= \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} \left\||x|^\alpha - P_n^*\right\|_{L_p[-1,1]} \\ &= \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} E_n\left(|x|^\alpha, L_p[-1,1]\right) \\ &= \lim_{n\to\infty} E_n\left(|x|^\alpha, L_p[-n,n]\right) \\ &= \lim_{n\to\infty} \left\||x|^\alpha - n^\alpha P_n^*\left(\frac{\cdot}{n}\right)\right\|_{L_p[-n,n]} \\ &= \left\||x|^\alpha - H_\alpha^*\right\|_{L_p(\mathbb{R})} \\ &= \inf\left\{\left\||x|^\alpha - H\right\|_{L_p(\mathbb{R})} : H \text{ is entire of exponential type} \le 1\right\}. \end{aligned} \tag{1.4}$$

Moreover, uniformly on compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} n^\alpha P_n^*\left(\frac{z}{n}\right) = H_\alpha^*(z),$$

and there is exactly one entire function H of exponential type ≤ 1 which minimizes (1.4). While various versions of this equality and relations (1.4) have been discussed by Bernstein, Raitsin and Ganzburg, the uniqueness of H_α^* proved in [9] is a highly nontrivial result.

From the Chebyshev alternation theorem it follows that for each integer n the best approximating polynomial P_n^* of order n to $|x|^\alpha$ in the in L_∞ norm can be represented as an interpolating polynomial with unknown consecutive nodes in $[-1,1]$. Thus, if one can find something about the nature of those best approximating interpolation nodes in $[-1,1]$, then we would successfully find an approach for a constructive analytical approximation towards some representations for the Bernstein constants $\Delta_{\infty,\alpha}$. One may not expect that a specific choice for such a node system would lead us into an instant range close to the Bernstein constants. But we can find out what type of formulas will be generated by the interpolation process itself for these node systems. It appears not to be out of range that these formulas may turn out to be part of a closed form expression for the Bernstein constants.

Since $|x|^\alpha$ is an even function a standard argument allows us to restrict ourselves to interpolation polynomials of even order $n=2m$. It is not surprising that Bernstein [2] himself, in 1937, studied the interpolation process to $|x|^\alpha$ by using the modified Chebyshev system

$$\begin{aligned} x_0^{(2n)} &= 0, \\ x_j^{(2n)} &= \cos\frac{(j-1/2)\,\pi}{2n}, \quad j=1,2,\dots 2n, \end{aligned}$$

where the $x_j^{(2n)}$ are the zeros of the Chebyshev polynomial T_{2n} of first kind, defined by $T_n(x)=\cos(n\arccos x)$. However, $x_0^{(2n)}$ is an additional choice, but not a zero of T_{2n}, in order to obtain the corresponding interpolation polynomial $P_{2n}^{(1)}$ of order $2n$ for $|x|^\alpha$. The final answer for its limit relation was given not before 2002 by Ganzburg ([5], Formula 2.7). For $\alpha>0$ one has

$$\lim_{n\to\infty}(2n)^\alpha\left\|\,|x|^\alpha - P_{2n}^{(1)}\right\|_{L_\infty[-1,1]} = \frac{2}{\pi}\left|\sin\frac{\pi\alpha}{2}\right|\int_0^\infty \frac{t^{\alpha-1}}{\cosh(t)}dt. \tag{1.5}$$

Let us give some remarks on equation (1.5). Firstly, we mention that in [2] Bernstein himself established a slightly weaker solution compared to formula (1.5). Secondly, an extension of limit relation (1.5) to complex values for α was obtained recently in [6].

It is remarkable that, since the beginning with Bernstein, no one has studied in detail the interpolation process by using the node system consisting of the $2n+1$ zeros of T_{2n+1}, since this node system automatically includes $x=0$ as a node and apparently it seems to be the more natural choice. To go into detail, let

$$x_j^{(2n+1)} = \cos\frac{(j-1/2)\,\pi}{2n+1}, \quad j=1,2,\dots 2n+1,$$

to be the zeros of T_{2n+1} and let us denote by $P_{2n}^{(2)}$ the corresponding interpolation polynomial of order $2n$ for $|x|^{\alpha}$. There is one paper [20], dealing with this node system and presenting the result that the approximation order $\||x|^{\alpha} - P_{2n}^{(2)}\|_{L_\infty[-1,1]} = O(1)/n^{\alpha}$ when $\alpha \in (0,1)$. In other words, the interpolation process attains the Jackson order. We also would like to mention a recent monograph by Ganzburg ([8], Theorem 4.2.3, Corollary 4.3.2 and Remark 4.3.3) for a more general approach to pointwise asymptotic relations within this topic.

In 2013, the author [14] established a strong asymptotic formula, valid for all $\alpha > 0$, from which he established an upper estimate for the error term, see ([14], Corollary 2), by showing that

$$\overline{\lim_{n\to\infty}} (2n)^{\alpha} \left\| |x|^{\alpha} - P_{2n}^{(2)} \right\|_{L_\infty[-1,1]} \leq \frac{2}{\pi} \left| \sin \frac{\pi\alpha}{2} \right| \int_0^\infty \frac{t^{\alpha}}{\sinh(t)} dt, \tag{1.6}$$

introducing an integral of similar nature to that in formula (1.5). In this paper we continue the investigation into the precise limiting quantity of $(2n)^{\alpha}$ $\||x|^{\alpha} - P_{2n}^{(2)}\|_{L_\infty[-1,1]}$ for all $\alpha > 0$.

The paper is organized as follows.

In Sect. 2 we collect some definitions for several constants and functions together with some standard results for later use.

In Sect. 3 we establish the precise limit relation (Theorem 3.1) and we show that the scaled polynomials $n^{\alpha} P_n^{(2)}\left(\frac{\cdot}{n}\right)$ uniformly converge on compact subsets of the real line to an entire function H_{α} of exponential type 1 (Theorems 3.2 and 3.3). We may also present an explicit expansion for H_{α} as an interpolating series for $|x|^{\alpha}$ (Theorem 3.3). As it can be seen later from the representation for the explicit limiting error term, i.e. from

$$\lim_{n\to\infty} (2n)^{\alpha} \left\| |x|^{\alpha} - P_{2n}^{(2)} \right\|_{L_\infty[-1,1]} = \|H(\alpha, \cdot)\|_{L_\infty[0,\infty)}, \tag{1.7}$$

the exact determination of the quantity on the right-hand side in (1.7) for individual values for α appears to be a rather difficult challenge.

2. Notation

In this section we record the following constants and functions, together with properties which are used later in the paper. The Chebyshev polynomials of first kind are denoted by T_n, where $T_n(x) = \cos(n \arccos x)$. For $x \in \mathbb{R}$, let $[x]$ to be the floor function, namely $[x] = \max\{m \in \mathbb{Z} : m \leq x\}$. Obviously, then $x - 1 < [x] \leq x$. We define the following constant, see also ([14], Remark 4).

$$C(\alpha) = \int_0^\infty \frac{t^{\alpha}}{\sinh(t)} dt, \quad \alpha > 0.$$

Next, we define the following functions.

$$H(\alpha,x)=\int_0^\infty \frac{t^\alpha}{\sinh(t)}\frac{x\sin(x)}{x^2+t^2}dt,\qquad \alpha>0,x>0,$$
$$H_1(\alpha,x)=\int_0^\infty \frac{t^\alpha}{\sinh(t)}\frac{x}{x^2+t^2}dt,\qquad \alpha>0,x>0,$$
$$H_2(\alpha,x)=\int_0^\infty \frac{t^\alpha}{\sinh(t)}\frac{x^2}{x^2+t^2}dt,\qquad \alpha>0,x>0.$$

Note that $H(\alpha,\cdot)$ should not be mixed up with the subsequent following definition of H_α. We collect the following easy to establish properties.

$$\begin{array}{ll}\text{(a) } 0\le H_2(\alpha,x)\le C(\alpha), & \alpha>0,x>0,\\ \text{(b) } |H(\alpha,x)|\le H_2(\alpha,x), & \alpha>0,x>0.\end{array}\tag{2.1}$$

For $\alpha>0$ the functions $H(\alpha,\cdot)$ and $H_2(\alpha,\cdot)$ can be extended for $x=0$ by interpreting the original definitions to be their limits $\lim_{x\to 0^+}$. The same can be done for $H_1(\alpha,\cdot)$ for values $\alpha\ge 1$. Some standard arguments then reveal

$$H_1(\alpha,0)=\begin{cases}\frac{\pi}{2}, & \alpha=1,\\ 0, & \alpha>1,\end{cases}$$
$$H(\alpha,0)=H_2(\alpha,0)=0,\quad \alpha>0.\tag{2.2}$$

Then, using (2.2), we define

$$H_\alpha(x)=|x|^\alpha-\frac{2}{\pi}\sin\frac{\pi\alpha}{2}H(\alpha,x),\quad \alpha>0,x\ge 0.\tag{2.3}$$

Finally, we apologize for the repulsive notation $\|f(x)\|$ instead of $\|f\|$ that we occasionally use in this paper.

3. The Limiting Error Term

Let $\alpha>0$ and $n\in\mathbb{N}$. We recall the definition of the nodes $x_j^{(2n+1)}=\cos\frac{(j-1/2)\pi}{2n+1}$ for $j=1,2,\dots 2n+1$ to be the zeros of the Chebyshev polynomial T_{2n+1}. Further denote by $P_{2n}^{(2)}$ the unique Lagrange interpolation polynomial for $|x|^\alpha$ in the interval $[-1,1]$.

Then, for $2n>\alpha>0$ and all $x\in[-1,1]$, we simply derive from ([14], Theorem 1) the asymptotic formula

$$(2n)^\alpha\left(|x|^\alpha-P_{2n}^{(2)}(x)\right)=(-1)^n\frac{2}{\pi}\sin\frac{\pi\alpha}{2}\left(1-\frac{1}{2n+1}\right)$$
$$\cdot T_{2n+1}(x)\int_0^\infty\frac{t^\alpha}{\sinh(t)}\frac{2nx}{(2nx)^2+t^2}dt+o(1),\quad n\to\infty,\tag{3.1}$$

where o(1) is independent of x.

The objective now is to find its limiting error term in the L_∞ norm. Since the error term is symmetric in $[-1,1]$ we prove the following

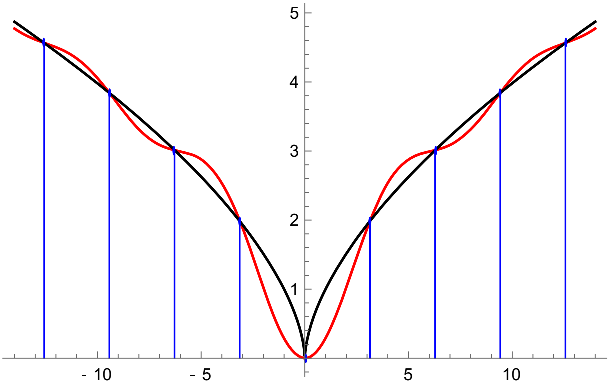

FIGURE 1. Interpolating entire function H_α of exponential type 1 from (3.3)

Theorem 3.1. *Let $\alpha > 0$. Then we have*

$$\lim_{n\to\infty} (2n)^\alpha \left\| |x|^\alpha - P_{2n}^{(2)} \right\|_{L_\infty[0,1]} = \frac{2}{\pi} \left| \sin \frac{\pi\alpha}{2} \right| \| H(\alpha,\cdot) \|_{L_\infty[0,\infty)}$$
$$= \frac{2}{\pi} \left| \sin \frac{\pi\alpha}{2} \right| \sup_{x\in[0,\infty)} \int_0^\infty \frac{t^\alpha}{\sinh(t)} \frac{x|\sin x|}{x^2+t^2} dt.$$

Theorem 3.2. *Let $\alpha > 0$. Then, uniformly on compact subsets in $[0,\infty)$,*

$$\lim_{n\to\infty} (2n)^\alpha P_{2n}^{(2)} \left(\frac{x}{2n} \right) = H_\alpha(x).$$

Theorem 3.3. *Let $\alpha > 0$ be not an even integer. Then H_α (interpreted as its extension into the complex domain) is an entire function of exponential type 1, interpolating $|x|^\alpha$ at the interpolation points $\{k\pi : k = 0,1,2,\ldots\}$ and H_α admits a representation as an interpolating series of the following form. Denote by $N = [\alpha/2]$. Then, for all $x \in \mathbb{R}$, we have*

$$H_\alpha(x) = \sin x \left(\frac{2}{\pi} \sum_{n=0}^{N-1} \sin\left(\frac{\pi(\alpha-2n-2)}{2} \right) C(\alpha-2n-2) x^{2n+1} \right.$$
$$\left. + 2x^{2N+1} \sum_{k=1}^\infty (-1)^k \frac{(k\pi)^{\alpha-2N}}{x^2-(k\pi)^2} \right). \tag{3.2}$$

For the special case $0 < \alpha < 2$ the expansion (see Fig. 1) is then represented by

$$H_\alpha(x) = 2x\sin x \sum_{k=1}^\infty (-1)^k \frac{(k\pi)^\alpha}{x^2-(k\pi)^2}. \tag{3.3}$$

We start with the proof for Theorem 3.1 by splitting it in several Lemmas. First, we present without a proof the following three Lemmas.

Lemma 3.1. *Let $x \in \left[0, \frac{1}{2}\right]$. Then* $0 \le \arcsin x - x \le x^2$.

Lemma 3.2. *For $n \in \mathbb{N}$ and $x \in [-1, 1]$ we have*

$$T_{2n+1}(x) = (-1)^n \sin\left((2n+1)\arcsin x\right).$$

Lemma 3.3. *Let $n \in \mathbb{N}$ and $x \in [-2n, 2n]$. Then we have*

$$\left|\frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x}\right| \le 1 + \frac{1}{2n}.$$

To carry the discussion further we proceed with

Lemma 3.4. *Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \max\left(C, \frac{C}{\varepsilon}\right)$. Then*

$$\left\|\frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} - \frac{(-1)^n \sin\left((2n+1)\frac{x}{2n}\right)}{x}\right\|_{L_\infty[0,C]} < \varepsilon.$$

Proof. For $x \in [0, C]$ we get $0 \le \frac{x}{2n} \le \frac{C}{2n} < \frac{C}{2C} = \frac{1}{2}$. Then, using Lemmas 3.1 and 3.2, we estimate

$$\begin{aligned}
&\left|\frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} - \frac{(-1)^n \sin\left((2n+1)\frac{x}{2n}\right)}{x}\right| \\
&\quad = \frac{1}{x}\left|\sin\left((2n+1)\arcsin\frac{x}{2n}\right) - \sin\left((2n+1)\frac{x}{2n}\right)\right| \\
&\quad \le \frac{2n+1}{x}\left|\arcsin\frac{x}{2n} - \frac{x}{2n}\right| \le \frac{2n+1}{x}\left(\frac{x}{2n}\right)^2 \le \frac{C}{n} < \varepsilon.
\end{aligned}$$

□

Lemma 3.5. *Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \frac{1}{2\varepsilon}$. Then*

$$\left\|\frac{\sin\left((2n+1)\frac{x}{2n}\right)}{x} - \frac{\sin x}{x}\right\|_{L_\infty[0,C]} < \varepsilon.$$

Proof. Let $x \in [0, C]$. Then by a standard argument we arrive at

$$\left|\frac{\sin\left((2n+1)\frac{x}{2n}\right)}{x} - \frac{\sin x}{x}\right| \le \frac{1}{x}\left|(2n+1)\frac{x}{2n} - x\right| = \frac{1}{2n} < \varepsilon.$$

□

Lemma 3.6. *Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \max\left(C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon}\right)$. Then*

$$\left\|\frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} - (-1)^n \frac{\sin x}{x}\right\|_{L_\infty[0,C]} < 2\varepsilon.$$

Proof. This follows directly by applying the triangle inequality combined together with Lemmas 3.4 and 3.5. □

Lemma 3.7. *Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \max\left(C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon}\right)$. Then, for $\alpha > 0$, we have*

$$\left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,C]} \leq \|H(\alpha, x)\|_{L_\infty[0,\infty)} + 2\varepsilon \cdot C(\alpha).$$

Proof. First, we remark that for $\alpha > 0$ the left-hand side in Lemma 3.7 is well defined by applying (2.2) together with Lemma 3.3. Using again the triangle inequality together with Lemma 3.6 and formula (2.1a), we arrive at

$$\begin{aligned} \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,C]} &= \left\| \frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} H_2(\alpha, x) \right\|_{L_\infty[0,C]} \\ \leq \left\| \frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} - (-1)^n \frac{\sin x}{x} \right\|_{L_\infty[0,C]} & \|H_2(\alpha, x)\|_{L_\infty[0,C]} \\ + \|\sin x \cdot H_1(\alpha, x)\|_{L_\infty[0,C]} &\leq 2\varepsilon C(\alpha) + \|H(\alpha, x)\|_{L_\infty[0,\infty)}. \end{aligned}$$

□

Our first substantial result is now the following

Lemma 3.8. *Let $\alpha > 0$. Then*

$$\overline{\lim}_{n\to\infty} \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \leq \|H(\alpha, x)\|_{L_\infty[0,\infty)}.$$

Proof. Let $\varepsilon > 0, C > \frac{C(\alpha)}{\varepsilon}$ and $n > \max\left(C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon}\right)$. Then

$$\begin{aligned} &\left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \\ &\leq \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,C]} + \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[C,2n]}. \end{aligned}$$

Using (2.1a), the latter part can be estimated to

$$\begin{aligned} \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[C,2n]} &= \left\| \frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} H_2(\alpha, x) \right\|_{L_\infty[C,2n]} \\ &\leq \frac{1}{C} \cdot C(\alpha) < \varepsilon. \end{aligned}$$

Combined together with the previous estimate and Lemma 3.7, we finally get

$$\left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \leq \|H(\alpha, x)\|_{L_\infty[0,\infty)} + 2\varepsilon \cdot C(\alpha) + \varepsilon.$$

By taking the $\overline{\lim}$ the result follows. □

Now, we are turning to the $\underline{\lim}$ case.

Lemma 3.9. *Let $\alpha > 0$ and $C > 0$ be fixed. Then*

$$\underline{\lim}_{n\to\infty} \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \geq \|H(\alpha, x)\|_{L_\infty[0,C]}.$$

Proof. Let $C > 0$, $\varepsilon > 0$ and $n > \max\left(C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon}\right)$. Then, by applying again the triangle inequality and combining together with Lemma 3.6 and (2.1a), we estimate

$$\begin{aligned}
&\left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \\
&\quad \geq \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,C]} \\
&\quad \geq \|H(\alpha, x)\|_{L_\infty[0,C]} - \left\| \left(\frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} - \frac{(-1)^n \sin x}{x} \right) H_2(\alpha, x) \right\|_{L_\infty[0,C]} \\
&\quad \geq \|H(\alpha, x)\|_{L_\infty[0,C]} - 2\varepsilon \|H_2(\alpha, x)\|_{L_\infty[0,C]} \\
&\quad \geq \|H(\alpha, x)\|_{L_\infty[0,C]} - 2\varepsilon \cdot C(\alpha).
\end{aligned}$$

Now, by taking $\underline{\lim}$ we establish the result. □

Our second substantial result is the following

Lemma 3.10. *Let* $\alpha > 0$. *Then*

$$\underline{\lim}_{n\to\infty} \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]} \geq \|H(\alpha, x)\|_{L_\infty[0,\infty)}.$$

Proof. Let $\varepsilon > 0$ and $C > \frac{C(\alpha)}{\varepsilon}$. Then, starting with the right-hand side in Lemma 3.10, we estimate

$$\|H(\alpha, x)\|_{L_\infty[0,\infty)} \leq \|H(\alpha, x)\|_{L_\infty[0,C]} + \|H(\alpha, x)\|_{L_\infty[C,\infty)}.$$

Using again (2.1a), the latter part can be estimated to

$$\begin{aligned}
\|H(\alpha, x)\|_{L_\infty[C,\infty)} &= \left\| \frac{\sin x}{x} H_2(\alpha, x) \right\|_{L_\infty[C,\infty)} \\
&\leq \frac{1}{C} \cdot C(\alpha) < \varepsilon.
\end{aligned}$$

Combined together with Lemma 3.9 and the previous estimate, we arrive at

$$\begin{aligned}
\|H(\alpha, x)\|_{L_\infty[0,\infty)} - \varepsilon &\leq \|H(\alpha, x)\|_{L_\infty[0,C]} \\
&\leq \underline{\lim}_{n\to\infty} \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]}.
\end{aligned}$$

Since the last expression holds for every $\varepsilon > 0$ we establish the result. □

Proof of Theorem 3.1. Let $\alpha > 0$. Then

$$\begin{aligned}
&\left\| T_{2n+1}(x) \int_0^\infty \frac{t^\alpha}{\sinh(t)} \frac{2nx}{(2nx)^2 + t^2} dt \right\|_{L_\infty[0,1]} \\
&\quad = \left\| T_{2n+1}\left(\frac{x}{2n}\right) \int_0^\infty \frac{t^\alpha}{\sinh(t)} \frac{x}{x^2 + t^2} dt \right\|_{L_\infty[0,2n]} \\
&\quad = \left\| T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,2n]}.
\end{aligned}$$

Combining now Lemmas 3.8 and 3.10 together with (3.1), gives the result and we are finished. □

Proof of Theorem 3.2. Let $\alpha > 0$. From (3.1) it follows that for every $\varepsilon > 0$ we can find some $n_0 = n_0(\varepsilon)$, such that for all $n > n_0$

$$\left\| (2n)^{\alpha} \left(|x|^{\alpha} - P_{2n}^{(2)}(x) \right) - (-1)^n \frac{2}{\pi} \sin \frac{\pi\alpha}{2} \left(1 - \frac{1}{2n+1} \right) \right.$$
$$\left. \cdot\, T_{2n+1}(x) \int_0^{\infty} \frac{t^{\alpha}}{\sinh t} \frac{2nx}{(2nx)^2 + t^2} dt \right\|_{L_\infty[0,1]} < \varepsilon.$$

Let $C > 0$ be fixed, $\varepsilon > 0$ and $n > \max\left(C, \frac{C}{\varepsilon}, \frac{1}{2\varepsilon}, \frac{\alpha}{2}, n_0\right)$. Then

$$\begin{aligned} &\left\| (2n)^{\alpha} P_{2n}^{(2)}\left(\frac{x}{2n}\right) - H_{\alpha}(x) \right\|_{L_\infty[0,C]} \\ &\quad = \left\| \frac{2}{\pi} \sin \frac{\pi\alpha}{2} H(\alpha, 2nx) - (2n)^{\alpha} \left(|x|^{\alpha} - P_{2n}^{(2)}(x) \right) \right\|_{L_\infty\left[0, \frac{C}{2n}\right]} \\ &\quad \le \frac{2}{\pi} \left| \sin \frac{\pi\alpha}{2} \right| \left\| H(\alpha, x) - (-1)^n \frac{2n}{2n+1} T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,C]} + \varepsilon. \end{aligned} \tag{3.4}$$

We proceed further by use of (2.1a), Lemmas 3.3 and 3.6.

$$\begin{aligned} &\left\| H(\alpha, x) - (-1)^n \frac{2n}{2n+1} T_{2n+1}\left(\frac{x}{2n}\right) H_1(\alpha, x) \right\|_{L_\infty[0,C]} \\ &\quad = \left\| H_1(\alpha, x) \left(\sin x - (-1)^n \frac{2n}{2n+1} T_{2n+1}\left(\frac{x}{2n}\right) \right) \right\|_{L_\infty[0,C]} \\ &\quad \le C(\alpha) \left(\left\| \frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} - (-1)^n \frac{\sin x}{x} \right\|_{L_\infty[0,C]} + \frac{1}{2n+1} \left\| \frac{T_{2n+1}\left(\frac{x}{2n}\right)}{x} \right\|_{L_\infty[0,C]} \right) \\ &\quad \le C(\alpha) \left(2\varepsilon + \frac{1}{2n} \right) \\ &\quad \le C(\alpha)\, 3\varepsilon. \end{aligned}$$

Combining together with (3.4), we obtain for every $\varepsilon > 0$ and n sufficiently large,

$$\left\| (2n)^{\alpha} P_{2n}^{(2)}\left(\frac{\cdot}{2n}\right) - H_{\alpha} \right\|_{L_\infty[0,C]} \le \frac{2}{\pi} \left| \sin \frac{\pi\alpha}{2} \right| C(\alpha)\, 3\varepsilon + \varepsilon.$$

Since any compact set K in $[0, \infty)$ can be included in some interval $[0, C]$ the result is established. □

Proof of Theorem 3.3. The expansion of H_{α} into the interpolating series (3.2) follows after some routine arguments from ([5], Formula 4.14). The special case (3.3) can be directly seen from ([5], Formula 4.16). The fact that H_{α} is an entire function of exponential type 1 can now be deduced from ([15], p. 183, Formula 15). The interpolation property is an easy consequence of (2.3). □

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References

[1] Bernstein, S.N.: Sur la meilleure approximation de $|x|$ par des polynômes des degrés donnés. Acta Math. **37**, 1–57 (1913)

[2] Bernstein, S.N.: Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of a Single Real Variable. State United Scientific and Technical Publishing House, Moscow (1937). (in Russian)

[3] Bernstein, S.N.: Sur la meilleure approximation de $|x|^p$ par des polynômes des degrés trés élevés. Bull. Acad. Sci. USSR Sér. Math. **2**, 181–190 (1938)

[4] Dryanov, D.P., Qazi, M.A., Rahman, Q.I.: Entire functions of exponential type in approximation theory. In: Bojanov, B.D. (ed.) Constructive Theory of Functions, Proceedings of the international conference, Varna, Bulgaria, June 19–23, 2002. DARBA, Sofia, pp. 86–135 (2003)

[5] Ganzburg, M.I.: The Bernstein constant and polynomial interpolation at the Chebyshev nodes. J. Approx. Theory **119**, 193–213 (2002)

[6] Ganzburg, M.I.: Polynomial interpolation, an L-function, and pointwise approximation of continuous functions. J. Approx. Theory **153**, 1–18 (2008)

[7] Ganzburg, M.I.: Lagrange interpolation and new asymptotic formulae for the Riemann zeta function. In: Approximation Theory XIII, San Antonio 2010, Springer Proceedings in Mathematics 13, pp. 79–93. Springer (2012)

[8] Ganzburg, M.I.: Polynomial interpolation and asymptotic representations for zeta functions. Dissertationes Mathematicae, vol. 496, Warszawa (2013)

[9] Lubinsky, D.S.: On the Bernstein constants of polynomial approximation. Constr. Approx. **25**(3), 303–366 (2007)

[10] Lubinsky, D.S.: Series representations for best approximating entire functions of exponential type. In: Chen, G., Lai, M. (ed.) Modern Methods in Mathematics, Athens, GA, USA, May 16–19, 2005, pp. 356–364. Nashboro Press , Brentwood (2006)

[11] Nikolskii, S.M.: On the best mean approximation by polynomials of the functions $|x-c|^s$. Izv. Akad. Nauk SSSR **11**, 139–180 (1947). (in Russian)

[12] Pachón, R., Trefethen, L.N.: Barycentric-Remez algorithms for best polynomial approximation in the chebfun system. BIT **49**(4), 721–741 (2009)

[13] Raitsin, R.A.: On the best approximation in the mean by polynomials and entire functions of finite degree of functions having an algebraic singularity. Izv. Vysch. Uchebn. Zaved. Mat. **13**, 59–61 (1969). (in Russian)

[14] Revers, M.: On the asymptotics of polynomial interpolation to $|x|^\alpha$ at the Chebyshev nodes. J. Approx. Theory **165**, 70–82 (2013)

[15] Timan, A.F.: Theory of Approximation of Functions of a Real Variable. Dover Publications, New York (1994)

[16] Trigub, R.M., Belinsky, E.S.: Fourier Analysis and Approximation of Functions. Kluwer, Dordrecht (2004)

[17] Varga, R.S., Carpenter, A.J.: On the Bernstein conjecture in approximation theory. Constr. Approx. **1**, 333–348 (1985)

[18] Varga, R.S., Carpenter, A.J.: Some numerical results on best uniform polynomial approximation of x^α on $[0, 1]$, Berlin, Springer 1993, Lect. Notes Math. vol. 1550, pp. 192–222 (1993)

[19] Vaaler, J.D.: Some extremal functions in fourier analysis. Bull. Am. Math. Soc. New Ser. **12**, 183–216 (1985)

[20] Zhu, L., Huang, Z.: On Lagrange interpolation for $|x|^\alpha$ $(0 < \alpha < 1)$. Anal. Theory Appl. **25**(1), 16–24 (2009)

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