Jumps of Milnor Numbers of Brieskorn–Pham Singularities in Non-degenerate Families

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Abstract. The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformation (f_s) . In the case f_s are non-degenerate singularities we call the jump non-degenerate. We give a formula (an inductive algorithm using diophantine equations) for the non-degenerate jump of f_0 in the case f_0 is a convenient singularity with only one $(n - 1)$ -dimensional face of its Newton diagram which equivalently (in our problem) can be replaced by the Brieskorn–Pham singularities.

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1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an (*isolated*) *singularity*, i.e. f_0 is the germ at 0 of a holomorphic function having an isolated critical point at $0 \in \mathbb{C}^n$, and $0 \in \mathbb{C}$ as the corresponding critical value. More specifically, there exists a representative $\hat{f}_0 : U \rightarrow \mathbb{C}$ of f_0 holomorphic in an open neighborhood U of the point $0 \in \mathbb{C}^n$ such that:

- $\hat{f}_0(0) = 0$,
- $\nabla \hat{f}_0(0) = 0$,
- $\nabla \hat{f}_0(z) \neq 0$ for $z \in U \setminus \{0\}$,

where for a holomorphic function f we put $\nabla f := (\partial f / \partial z_1, \dots, \partial f / \partial z_n)$.

The ring of germs of holomorphic functions of n variables will be denoted by $\mathcal{O}_n$.

A *deformation of the singularity* f_0 is the germ of a holomorphic function $f = f(s, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ such that:

- $f(0, z) = f_0(z)$,
- $f(s, 0) = 0$.

The deformation $f(s, z)$ of the singularity f_0 will also be treated as a family (f_s) of germs, putting $f_s(z) := f(s, z)$. Since f_0 is an isolated singularity, f_s has also isolated singularities near the origin, for sufficiently small s [8, Theorem 2.6 in Chapter I].

Remark 1. Notice that in the deformation (f_s) there can occur in particular smooth germs, that is germs satisfying $\nabla f_s(0) \neq 0$. In this context, the symbol ∇f_s will always denote $\nabla_z f_s(z)$.

By the above assumptions, for every sufficiently small s , one can define a (finite) number μ_s as the Milnor number of f_s , namely

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n / (\nabla f_s),$$

where (∇f_s) is the ideal in $\mathcal{O}_n$ generated by $\partial f_s / \partial z_1, \dots, \partial f_s / \partial z_n$.

Since the Milnor number is upper semi-continuous in the Zariski topology [8, Proposition 2.57 in Chapter II], there exists an open neighborhood S of the point $0 \in \mathbb{C}$ such that

- $\mu_s = \text{const.}$ for $s \in S \setminus \{0\}$,
- $\mu_0 \geq \mu_s$ for $s \in S$.

The (constant) difference $\mu_0 - \mu_s$ for $s \in S \setminus \{0\}$ will be called *the jump of the deformation* (f_s) and denoted by $\lambda((f_s))$. The smallest non-zero value among the jumps of all deformations of the singularity f_0 will be called *the jump (of the Milnor number) of the singularity* f_0 and denoted by $\lambda(f_0)$.

The first general result concerning the jump was Gusein-Zade's [9], who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$ and that for irreducible plane curve singularities it holds $\lambda(f_0) = 1$. In [4] the authors proved that $\lambda(f_0)$ is not a topological invariant of f_0 but it is an invariant of the stable equivalence of singularities. The computation of $\lambda(f_0)$ is not an easy task. It is related to the problem of adjacency of classes of singularities. Only for few classes of singularities we know the exact value of $\lambda(f_0)$. For plane curve singularities ($n = 2$) in the X_9 class (see [1] for terminology) that is singularities of the form $f_0^a(x, y) := x^4 + y^4 + ax^2y^2$, $a \in \mathbb{C}$, $a^2 \neq 4$, we have $\lambda(f_0^a) = 2$ (see [4]) and for specific homogeneous singularities $f_0^d(x, y) = x^d + y^d$, $d \geq 2$, we have $\lambda(f_0^d) = [d/2]$, where $[x]$ is the integer part of x (see [5]).

In the paper we consider a weaker problem: *compute the jump* $\lambda^{\text{nd}}(f_0)$ *of* f_0 *over all non-degenerate deformations of* f_0 (i.e. the f_s in deformations (f_s) of f_0 are non-degenerate singularities). Clearly, we always have $\lambda(f_0) \leq \lambda^{\text{nd}}(f_0)$.

Up to now, this problem has been studied only for plane curve singularities by Bodin [3], Brzostowski et al. [5] and Walewska [16, 17]. We give a formula (more precisely an algorithm) in n -dimensional case for $\lambda^{\text{nd}}(f_0)$, where f_0 is non-degenerate, convenient and has its Newton diagram reduced to one $(n-1)$ -dimensional face. Since for non-degenerate singularities f_s of a deformation (f_s) of f_0 the Milnor number $\mu(f_s)$ depends only on the Newton diagram of f_s and not on specific coefficients of f_s (see Sect. 2) we may restrict considerations to the class of the Brieskorn–Pham singularities

$$f_0(\mathbf{z}) = z_1^{p_1} + \cdots + z_n^{p_n}, \quad p_i \geq 2, \quad i = 1, \dots, n, \quad (1)$$

where $\mathbf{z} = (z_1, \dots, z_n)$.

Remark 2. Let us note that we compute non-degenerate jump of a singularity in a fixed system of coordinates. The reason is that non-degeneracy depends on coordinate system. For instance, for singularity $f_0(x, y) = x^3 + y^3$, $\lambda^{\text{nd}}(f_0) = 2$ in the original system of coordinates while for $L : x = x' - y'$, $y = y'$, we have $\lambda^{\text{nd}}(f_0 \circ L) = 1$.

The problem of finding the minimal non-degenerate jump of a singularity in arbitrary system of coordinates is open.

2. Non-degenerate Singularities

In this Section we recall the notion of non-degenerate singularities. Let $f_0(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}_0^n} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$ be a singularity and $\text{supp}(f_0) = \{\mathbf{i} \in \mathbb{N}_0^n : a_{\mathbf{i}} \neq 0\}$ the *support* of f_0 . The *Newton polyhedron* $\Gamma_+(f_0) \subset \mathbb{R}^n$ of f_0 is the convex hull of the set $\bigcup_{\mathbf{i} \in \text{supp}(f_0)} (\mathbf{i} + \mathbb{R}_+^n)$. The finite set of compact faces (of all dimensions) of the boundary of $\Gamma_+(f_0)$ constitutes the *Newton diagram* of f_0 and is denoted by $\Gamma(f_0)$. The singularity f_0 is *convenient* if $\Gamma_+(f_0)$ intersects all coordinate axes in $\mathbb{R}^n$. For each face $S \in \Gamma(f_0)$ we define a weighted homogeneous polynomial $(f_0)_S(\mathbf{z}) = \sum_{\mathbf{i} \in S} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$. We call the singularity f_0 *non-degenerate on* $S \in \Gamma(f_0)$ if the system of equations $\partial(f_0)_S / \partial z_i(\mathbf{z}) = 0$, $i = 1, \dots, n$ has no solutions in $(\mathbb{C}^*)^n$. The singularity f_0 is *non-degenerate* if f_0 is non-degenerate on every face $S \in \Gamma(f_0)$.

In the sequel we fix the following notations for coordinate hyperplanes in $\mathbb{R}^n$. Let $I = \{i_1, \dots, i_k\} \subset \{1, 2, \dots, n\}$. We denote

$$O_{x_{i_1} \dots x_{i_k}} := \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_j = 0, j \notin I\}$$

and

$$\{x_{i_1} = \dots = x_{i_k} = 0\} := \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_{i_1} = \dots = x_{i_k} = 0\}.$$

For a convenient singularity f_0 we define the *Newton number* $\nu(f_0)$ of f_0 by

$$\nu(f_0) := n!V - \sum_{i=1}^n (n-1)!V_i + \sum_{\substack{i,j=1 \\ i < j}}^n (n-2)!V_{ij} - \cdots + (-1)^n V_{12\dots n},$$

where V is the n -dimensional volume under $\Gamma_+(f_0)$, V_i is the $(n-1)$ -dimensional volume under $\Gamma_+(f_0)$ on the hyperplane $\{x_i = 0\}$, V_{ij} is the $(n-2)$ -dimensional volume under $\Gamma_+(f_0)$ on the hyperplane $\{x_i = x_j = 0\}$ and so on.

The importance of $\nu(f_0)$ has its source in the celebrated *Kouchnirenko theorem*.

Theorem 2.1 [11]. *If f_0 is a convenient singularity, then*

- (1) $\mu(f_0) \geq \nu(f_0)$,
- (2) *if f_0 is non-degenerate then $\mu(f_0) = \nu(f_0)$.*

Remark 3. For non-convenient singularities the Kouchnirenko Theorem also holds provided $\nu(f_0)$ is appropriately defined (see for instance [6]). It is interesting that both conditions of item 2. of the Theorem are equivalent in 2-dimensional case by Płoski [15]. For weaker non-degeneracy conditions equivalent to the equality $\mu(f_0) = \nu(f_0)$ in n -dimensional case see recent preprint by Mondal [12].

3. Non-degenerate Jump of Milnor Numbers of Singularities

Let $f_0 \in \mathcal{O}_n$ be a singularity. A deformation (f_s) of f_0 is called *non-degenerate* if f_s is non-degenerate for $s \neq 0$. The set of all non-degenerate deformations of the singularity f_0 will be denoted by $\mathcal{D}^{\text{nd}}(f_0)$. *Non-degenerate jump* $\lambda^{\text{nd}}(f_0)$ of the singularity f_0 is the minimal of non-zero jumps over all non-degenerate deformations of f_0 , which means

$$\lambda^{\text{nd}}(f_0) := \min_{(f_s) \in \mathcal{D}_0^{\text{nd}}(f_0)} \lambda((f_s)),$$

where by $\mathcal{D}_0^{\text{nd}}(f_0)$ we denote all non-degenerate deformations (f_s) of f_0 for which $\lambda((f_s)) \neq 0$.

Obviously

Proposition 3.1. *For each singularity f_0 we have the inequality*

$$\lambda(f_0) \leq \lambda^{\text{nd}}(f_0).$$

From the Kouchnirenko Theorem we get

Corollary 3.2. *If f_0 and $\tilde{f}_0$ are non-degenerate and convenient singularities and $\Gamma(f_0) = \Gamma(\tilde{f}_0)$ then $\lambda^{\text{nd}}(f_0) = \lambda^{\text{nd}}(\tilde{f}_0)$.*

Corollary 3.3. *If f_0 is a convenient and non-degenerate singularity such that the Newton diagram $\Gamma(f_0)$ has only one $(n-1)$ -dimensional face which intersects the axes $Ox_1, \dots, Ox_n$ in points $(p_1, 0, \dots, 0), \dots, (0, \dots, 0, p_n)$ respectively, then for the Brieskorn–Pham singularity $\bar{f}(\mathbf{z}) = z_1^{p_1} + \dots + z_n^{p_n}$ we have*

$$\lambda^{\text{nd}}(f_0) = \lambda^{\text{nd}}(\bar{f}).$$

In investigations concerning $\lambda^{\text{nd}}(f_0)$ we may restrict our attention to non-degenerate f_0 because the non-degenerate jump for degenerate singularities can be found using the proposition below (cf. [3, Lemma 5]). For a given f_0 let f_0^{nd} denote any non-degenerate singularity for which $\Gamma(f_0) = \Gamma(f_0^{\text{nd}})$. Such singularities always exist.

Proposition 3.4. *If f_0 is degenerate then*

$$\lambda^{\text{nd}}(f_0) = \begin{cases} \mu(f_0) - \mu(f_0^{\text{nd}}), & \text{if } \mu(f_0) - \mu(f_0^{\text{nd}}) > 0 \\ \lambda^{\text{nd}}(f_0^{\text{nd}}), & \text{if } \mu(f_0) - \mu(f_0^{\text{nd}}) = 0. \end{cases}$$

Proof. This follows from the fact that a generic small perturbation of coefficients of these monomials of f_0 which correspond to points belonging to $\bigcup \Gamma(f_0)$ (which are finite in number) give us non-degenerate singularities with the same Newton polyhedron as f_0 . $\square$

Remark 4. By the Płoski theorem ([14, Lemma 2.2], [15, Theorem 1.1]), for degenerate plane curve singularities ($n = 2$) the second possibility in Proposition 3.4 is excluded.

A crucial role in the search for the formula for $\lambda^{\text{nd}}(f_0)$ will be played by the monotonicity of the Newton number with respect to the Newton polyhedron. Namely, Gwoździewicz [10], Furuya [7] and Biviá-Ausina [2] proved.

Theorem 3.5. (Monotonicity Theorem) *Let $f_0, \tilde{f}_0 \in \mathcal{O}_n$ be two convenient singularities such that $\Gamma_+(f_0) \subset \Gamma_+(\tilde{f}_0)$. Then $\nu(f_0) \geq \nu(\tilde{f}_0)$.*

We need a stronger version of the above theorem in the particular case f_0 has only one $(n - 1)$ -dimensional face and $\tilde{f}_0$ differs from f_0 in one monomial.

Theorem 3.6. *If f_0 is a convenient and non-degenerate singularity such that the Newton diagram $\Gamma(f_0)$ has only one $(n - 1)$ -dimensional face and $f_0^{\mathbf{i}}(\mathbf{z}) := f_0(\mathbf{z}) + z^{\mathbf{i}}$, where the lattice point $\mathbf{i} \neq 0$ lies under $\Gamma_+(f_0)$ i.e. $0 \neq \mathbf{i} \in \mathbb{N}_0^n \setminus \Gamma_+(f_0)$, then $\nu(f_0) > \nu(f_0^{\mathbf{i}})$.*

Proof. By the Kouchnirenko Theorem we may consider only the Brieskorn–Pham singularities (1). We will use induction with respect to the number n of variables. For $n = 1$ we have $f_0(\mathbf{z}) = z^{p_1}$, $p_1 \geq 2$, and $f_0^{\mathbf{i}}(\mathbf{z}) = z^{p_1} + z^i$, $1 \leq i < p_1$. Hence $\nu(f_0) = p_1 - 1$ and $\nu(f_0^{\mathbf{i}}) = i - 1$. This gives $\nu(f_0) > \nu(f_0^{\mathbf{i}})$.

Suppose the theorem is true for any Brieskorn–Pham singularity in $(n - 1)$ variables. Let now $f_0(\mathbf{z}) = z_1^{p_1} + \dots + z_n^{p_n}$, $p_i \geq 2$, $i = 1, \dots, n$. Put $P_i := (0, \dots, p_i, \dots, 0)$.

If $\mathbf{i}$ does not lie on any coordinate hyperplane $\{x_i = 0\}$ then $\Gamma_+(f_0)$ and $\Gamma_+(f_0^{\mathbf{i}})$ are identical on all these hyperplanes (see Figure 1(a)) and hence

$$\nu(f_0) - \nu(f_0^{\mathbf{i}}) = n! \text{vol}(\Delta(\mathbf{i}, P_1, \dots, P_n)) > 0,$$

where $\Delta(\mathbf{i}, P_1, \dots, P_n)$ is the n -dimensional simplex with vertices $\mathbf{i}, P_1, \dots, P_n$.

Take now a lattice point $\mathbf{i} \neq 0$ under $\Gamma_+(f_0)$ which lies on a coordinate hyperplane. For simplicity we may assume $\mathbf{i} \in Ox_1 \dots x_{n-1}$ i.e. $\mathbf{i} =$

FIGURE 1. Possible cases in dimension three

$(i_1, \dots, i_{n-1}, 0)$. Two possible cases in dimension three are illustrated in Fig. 1b, c. By definition of the Newton number the difference $\nu(f_0) - \nu(f_0^i)$ is equal to the sum of volumes of differences of $\Gamma_+(f_0)$ and $\Gamma_+(f_0^i)$ on all hyperplanes $Ox_{i_1} \dots Ox_{i_l}$ multiplied by $(-1)^{n-l}l!$ ($i_1, \dots, i_l$ are different and $l = 1, \dots, n$). First we consider these differences in hyperplanes which do not contain the axis Ox_n . They are equal to the sum of volumes of differences for truncated singularities ${}^n f_0(\mathbf{z}) := z_1^{p_1} + \dots + z_{n-1}^{p_{n-1}}$ and ${}^n f_0^i(\mathbf{z}) := {}^n f_0(\mathbf{z}) + \mathbf{z}^i$ but taken with opposite signs. So, the sum of this differences is equal to $-(\nu({}^n f_0) - \nu({}^n f_0^i))$. Now we consider differences in coordinate hyperplanes which contain the axis Ox_n . Each such difference in a coordinate hyperplane $Ox_{i_1} \dots Ox_{i_k} x_n$ is a pyramid whose base is equal to the difference, say W , of $\Gamma_+(f_0)$ and $\Gamma_+(f_0^i)$ in $Ox_{i_1} \dots Ox_{i_k}$ and the vertex is equal to $P_n = (0, \dots, 0, p_n)$. Hence, in $\nu(f_0) - \nu(f_0^i)$ this pyramid will give the term

$$(-1)^{n-(k+1)}(k+1)! \frac{\text{vol}(W)p_n}{k+1} = (-1)^{n-1-k} k! \text{vol}(W)p_n.$$

But the latter expression is precisely the term in $(\nu({}^n f_0) - \nu({}^n f_0^i))p_n$ associated to the difference W . Together we get

$$\nu(f_0) - \nu(f_0^i) = (\nu({}^n f_0) - \nu({}^n f_0^i))(p_n - 1).$$

Since $p_n > 1$ and by induction hypothesis $\nu({}^n f_0) - \nu({}^n f_0^i)$ is positive, we get the assertion of the theorem. $\square$

Now we define specific non-degenerate deformations of a convenient and non-degenerate singularity $f_0 \in \mathcal{O}_n$ (in particular for Brieskorn–Pham singularities). Denote by J the set of lattice points (points with integer coordinates) $\mathbf{i} \neq 0$ lying under the Newton diagram of f_0 i.e. $0 \neq \mathbf{i} \in \mathbb{N}_0^n \setminus \Gamma_+(f_0)$. Obviously, J is a finite set. For $\mathbf{i} = (i_1, \dots, i_n) \in J$ we define the *monomial deformation* $(f_s^i)_{s \in \mathbb{C}}$ of f_0 by the formula

$$f_s^i(\mathbf{z}) := f_0(\mathbf{z}) + s\mathbf{z}^i.$$

Proposition 3.7. *For every $\mathbf{i} \in J$ the deformation (f_s^i) of f_0 is convenient and non-degenerate for all sufficiently small $|s|$.*

Proof. See [11] or [13, Appendix]. $\square$

Combining the Monotonicity Theorem with the above proposition we reach the conclusion that in order to find $\lambda^{\text{nd}}(f_0)$ it is enough to consider only the non-degenerate deformations of the type $(f_s^{\mathbf{i}})$.

Theorem 3.8. *If f_0 is a convenient and non-degenerate singularity such that the Newton diagram $\Gamma(f_0)$ has only one $(n-1)$ -dimensional face, then*

$$\lambda^{\text{nd}}(f_0) = \min_{\mathbf{i} \in J} \lambda((f_s^{\mathbf{i}})).$$

Proof. By the Kouchnirenko theorem it suffices to consider non-degenerate deformations of f_0 of the form

$$f_s(\mathbf{z}) = f_0(\mathbf{z}) + \sum_{\mathbf{i} \in J} a_{\mathbf{i}}(s) z^{\mathbf{i}}, \quad (2)$$

where $a_{\mathbf{i}}(s)$ are holomorphic at $0 \in \mathbb{C}$ and $a_{\mathbf{i}}(0) = 0$. Then by the Monotonicity Theorem and Theorem 3.6 we may restrict the scope of deformations (2) to deformations with only one term added i.e. to the deformations $(f_s^{\mathbf{i}})$ for $\mathbf{i} \in J$. $\square$

4. An Algorithm for the Non-degenerate Jump of Brieskorn–Pham Singularities

In this Section we give an algorithm for calculating $\lambda^{\text{nd}}(f_0)$ for a non-degenerate and convenient singularity with one $(n-1)$ -dimensional face of its Newton diagram. By Corollary 3.3 we may assume that f_0 is a Brieskorn–Pham singularity

$$f_0(\mathbf{z}) = z_1^{p_1} + \cdots + z_n^{p_n}, \quad p_i \geq 2. \quad (3)$$

We have $\mu(f_0) = \prod_{i=1}^n (p_i - 1)$. By Theorem 3.8 to find $\lambda^{\text{nd}}(f_0)$ we have to compare the jumps of all monomial deformations $(f_s^{\mathbf{i}})_{s \in \mathbb{C}}$ where $\mathbf{i} \neq 0$ is a lattice point lying under $\Gamma_+(f_0)$. First we prove a lemma relating the jumps of monomial deformations of the Brieskorn–Pham singularities to the jumps of their "truncations".

Lemma 4.1. *Let $f_0(\mathbf{z}) = z_1^{p_1} + \cdots + z_n^{p_n}$ be a Brieskorn–Pham singularity. Define ${}^k f_0(\mathbf{z}) := z_1^{p_1} + \cdots + z_{k-1}^{p_{k-1}} + z_{k+1}^{p_{k+1}} + \cdots + z_n^{p_n}$, $k = 1, \dots, n$. Let $\mathbf{i} \neq 0$ be a lattice point under $\Gamma_+(f_0)$ lying in the hyperplane $\{x_k = 0\}$. Then*

$$\lambda((f_s^{\mathbf{i}})) = \lambda(({}^k f_s^{\mathbf{i}}))(p_k - 1). \quad (4)$$

Proof. The proof follows the line of the proof of Theorem 3.6. For simplicity of notations we take $k = n$, i.e. $\mathbf{i} = (i_1, \dots, i_{n-1}, 0)$. Since $\lambda((f_s^{\mathbf{i}})) = \mathbf{v}(f_0) - \mathbf{v}(f_s^{\mathbf{i}})$, $s \neq 0$, by definition of the Newton number the difference $\mathbf{v}(f_0) - \mathbf{v}(f_s^{\mathbf{i}})$ is equal to the sum of volumes of differences of $\Gamma_+(f_0)$ and $\Gamma_+(f_s^{\mathbf{i}})$ on all hyperplanes $Ox_{i_1} \dots x_{i_l}$ multiplied by $(-1)^{n-l} l!$ ($i_1, \dots, i_l$ are different and $l = 1, \dots, n$). Similarly as in the proof of Theorem 3.6 we divide all differences in hyperplanes into two kinds: in those hyperplanes which do not contain the axis Ox_n

and those which do contain the axis Ox_n . The sum of the first ones is equal to the number $-\lambda((^n f_s^i))$ while the second ones $\lambda((^n f_s^i))p_n$. Together we get equality (4) for $k = n$. $\square$

Now we present an inductive algorithm for $\lambda^{\text{nd}}(f_0)$ where f_0 is any Brieskorn–Pham singularity

$$f_0(\mathbf{z}) = z_1^{p_1} + \cdots + z_n^{p_n}, \quad p_i \geq 2, \quad i = 1, \dots, n$$

with respect to the number of variables n . In each step we give the formula for $\lambda^{\text{nd}}(f_0)$ and simultaneously the point $\mathbf{i} \in J$ (precisely the deformation $(f_s^{\mathbf{i}})$) realizing this jump. For $n = 1$ we have $f_0(z_1) = z_1^{p_1}$, $p_1 \geq 2$ and $J = \{1, 2, \dots, p_1 - 1\}$. We have $\lambda^{\text{nd}}(f_0) = 1$ and the point $\mathbf{i} = (p_1 - 1)$ realizes this jump.

Assume that for all Brieskorn–Pham singularities f_0 in $(n - 1)$ -variables we can compute $\lambda^{\text{nd}}(f_0)$ and can indicate the point $\mathbf{i}$ realizing this jump.

Take the Brieskorn–Pham singularity $f_0(\mathbf{z}) = z_1^{p_1} + \cdots + z_n^{p_n}$, $p_i \geq 2$, $n \geq 2$, in n variables. By Theorem 3.8 we have to compare the jumps of all monomial deformations $(f_s^{\mathbf{i}})$, where $\mathbf{i} \neq 0$ is a lattice point lying under $\Gamma_+(f_0)$.

A. First we consider points $\mathbf{i} \neq 0$ lying in coordinate hyperplanes of dimension $(n - 1)$. By induction hypotheses on each coordinate hyperplane $Ox_1 \dots x_{k-1}x_{k+1} \dots x_n$ there exists a point $\mathbf{i}_k$ realizing non-degenerate jump of the truncated singularity ${}^k f_0 = z_1^{p_1} + \cdots + z_{k-1}^{p_{k-1}} + z_{k+1}^{p_{k+1}} + \cdots + z_n^{p_n}$ in $(n - 1)$ variables. Denote this jump by $\lambda_k := \lambda^{\text{nd}}({}^k f_0) = \lambda(({}^k f_s^{\mathbf{i}_k}))$. By the Lemma 4.1 the jump of the deformation $(f_s^{\mathbf{i}_k})$ is given by the formula

$$\lambda((f_s^{\mathbf{i}_k})) = \lambda_k(p_k - 1).$$

This formula implies that there is no "better" point in the hyperplane $Ox_1 \dots x_{k-1}x_{k+1} \dots x_n$ which could give a monomial deformation of f_0 with smaller jump. In fact, for any point $\mathbf{i} \in J \cap Ox_1 \dots x_{k-1}x_{k+1} \dots x_n$ we would have

$$\lambda((f_s^{\mathbf{i}})) \geq \lambda_k$$

and hence again by the Lemma 4.1

$$\lambda((f_s^{\mathbf{i}})) = \lambda(({}^k f_s^{\mathbf{i}}))(p_k - 1) \geq \lambda_k(p_k - 1).$$

Hence doing the same for all hyperplanes $Ox_1 \dots x_{k-1}x_{k+1} \dots x_n$, $k = 1, \dots, n$, we get that the minimum of the jumps of monomial deformations associated to points lying in these hyperplanes is equal to

$$\min_{1 \leq k \leq n} \lambda_k(p_k - 1).$$

Denote this number by λ^{hyp} .

B. Now we consider monomial deformations of f_0 associated to points $\mathbf{i} \neq 0$ under $\Gamma_+(f_0)$ not lying in coordinate hyperplanes i.e. lying in the interior of the simplex $\Delta(P_0, P_1, \dots, P_n)$, where $P_0 = (0, \dots, 0)$, $P_1 = (p_1, 0, \dots, 0)$, $\dots$, $P_n = (0, \dots, 0, p_n)$. Such a point $\mathbf{i} = (i_1, \dots, i_n)$ satisfies the conditions:

1. $0 < i_k < p_k$, $k = 1, \dots, n$,

$$2. \frac{i_1}{p_1} + \cdots + \frac{i_n}{p_n} < 1 \text{ (or equivalently } i_1 p_2 \cdots p_n + \cdots + i_n p_1 \cdots p_{n-1} < p_1 \cdots p_n \text{)}.$$

For such a point $\mathbf{i}$ we obviously have

$$\lambda((f_s^{\mathbf{i}})) = n! \operatorname{vol}(\Delta(\mathbf{i}, P_1, \dots, P_n)) = p_1 \cdots p_n - i_1 p_2 \cdots p_n - \cdots - i_n p_1 \cdots p_{n-1}.$$

This follows from the fact that $\operatorname{vol}(\Delta(P_0, \dots, P_n)) = \frac{p_1 \cdots p_n}{n!}$ and $\operatorname{vol}(\Delta(P_0, \dots, P_{k-1}, \mathbf{i}, P_{k+1}, \dots, P_n)) = \frac{i_k p_1 \cdots p_{k-1} p_{k+1} \cdots p_n}{n!}$, $k = 1, \dots, n$. So the problem is to find minimal l in the set $\{1, \dots, \lambda^{\text{hyp}} - 1\}$ for which there exist solutions $i_1, \dots, i_n$ of the diophantine equation

$$i_1 p_2 \cdots p_n + \cdots + i_n p_1 \cdots p_{n-1} = p_1 \cdots p_n - l \quad (5)$$

satisfying condition (1). Denote $p'_k := p_1 \cdots p_{k-1} p_{k+1} \cdots p_n$, $k = 1, \dots, n$, and their greatest common divisor by $d := \operatorname{GCD}(p'_1, \dots, p'_n)$.

Since $d \mid (p_1 \cdots p_n)$ equation (5) has integer solutions only for l such that $d \mid l$. Hence we reduce the problem to find minimal integer m , $1 \leq m \leq \frac{\lambda^{\text{hyp}} - 1}{d}$ for which there exist integer solutions $i_1, \dots, i_n$ of the equation

$$i_1 p'_1 + \cdots + i_n p'_n = p_1 \cdots p_n - md \quad (6)$$

satisfying

$$0 < i_k < p_k, \quad k = 1, \dots, n. \quad (7)$$

If such a minimal $m_0 \in \{1, \dots, [\frac{\lambda^{\text{hyp}} - 1}{d}]\}$ exists then $\lambda^{\text{nd}}(f_0) = m_0 d$. Moreover, the jump is realized by the monomial deformation $(f_s^{\mathbf{i}})$ where $\mathbf{i} = (i_1^0, \dots, i_n^0)$ and $i_1^0, \dots, i_n^0$ are solutions of (6) with restrictions (7) for $m = m_0$.

Remark 5. There is an interesting formula for $\operatorname{GCD}(p'_1, \dots, p'_n)$ in terms of p_i only. Namely using elementary properties of GCD and LCM one can prove: for given positive integers $p_1, \dots, p_n$

$$\operatorname{GCD}(p'_1, \dots, p'_n) = \prod_{k=2}^n \prod_{1 \leq i_1 < \cdots < i_k \leq n} \operatorname{GCD}(p_{i_1}, \dots, p_{i_k})^{(-1)^k}.$$

Since for integer solutions of (6) we impose restrictions (7) a simple computer search gives us the solution of our problem.

It is interesting that in particular case $\operatorname{GCD}(p_i, p_j) = 1$, $i, j = 1, \dots, n$, $i \neq j$, there is a particularly simple algorithm (using only Euclid's algorithm) to solve (6) with restrictions (7). Namely, assume $f_0(\mathbf{z}) = z_1^{p_1} + \cdots + z_n^{p_n}$, $p_i \geq 2$, $\operatorname{GCD}(p_i, p_j) = 1$ for $i \neq j$. Hence $d = \operatorname{GCD}(p'_1, \dots, p'_n) = 1$ (see Remark 5). Take $m \in \{1, \dots, \lambda^{\text{hyp}} - 1\}$. Using Euclid's algorithm we find integers $j_1, \dots, j_n \in \mathbb{Z}$ such that

$$j_1 p'_1 + \cdots + j_n p'_n = p_1 \cdots p_n - m.$$

By simple transformations we get another identity

$$i_1 p'_1 + \cdots + i_n p'_n = p_1 \cdots p_n - m, \quad (8)$$

where $0 < i_k < p_k$, $k = 1, \dots, n-1$ and $i_n \in \mathbb{Z}$.

If $0 < i_n < p_n$ then $\mathbf{i} = (i_1, \dots, i_n)$ is the solution (for this fixed m) you are looking for.

If i_n does not satisfy the inequality $0 < i_n < p_n$ then we claim there is no solution to our problem for this m . Assume to the contrary that $\tilde{i}_1, \dots, \tilde{i}_n$ is such a solution i.e.

$$\tilde{i}_1 p'_1 + \cdots + \tilde{i}_n p'_n = p_1 \cdots p_n - m, \quad 0 < \tilde{i}_k < p_k, \quad k = 1, \dots, n.$$

Hence and by (8) we obtain

$$(\tilde{i}_1 - i_1) p'_1 + \cdots + (\tilde{i}_n - i_n) p'_n = 0.$$

Since $p_1 \mid p'_k$, $k = 2, \dots, n$ then $p_1 \mid (\tilde{i}_1 - i_1) p'_1$. But p_1 is relatively prime to $p'_1 = p_2 \cdots p_n$. Hence $p_1 \mid (\tilde{i}_1 - i_1)$. But $|\tilde{i}_1 - i_1| < p_1$ which implies $\tilde{i}_1 = i_1$. Similarly $\tilde{i}_2 = i_2, \dots, \tilde{i}_{n-1} = i_{n-1}$. This implies $\tilde{i}_n = i_n$ which contradicts our supposition $0 < \tilde{i}_n < p_n$. So in this case there are no solutions (6) with restrictions (7) for this m .

The above considerations can be formulated in the following Lemma.

Lemma 4.2. *If $\text{GCD}(p_i, p_j) = 1$ for $i \neq j$ then for any $m \in \mathbb{Z}$ there exists unique solution of the diophantine equation*

$$i_1 p'_1 + \cdots + i_n p'_n = p_1 \cdots p_n - m$$

satisfying the inequality $0 < i_k < p_k$, $k = 1, \dots, n-1$.

We may sum up the above considerations in the following theorem

Theorem 4.3. *Let $f_0 \in \mathcal{O}^n$ be a convenient and non-degenerate singularity with only one $(n-1)$ -dimensional face of its Newton diagram. Assume the vertices (on axes) of this face are $(p_1, 0, \dots, 0), \dots, (0, \dots, p_n)$, $p_i \geq 2$. Then the non-degenerate jump $\lambda^{\text{nd}}(f_0)$ of f_0 can be inductively (with respect to n) calculated as follows. For $n = 1$, $\lambda^{\text{nd}}(f_0) = 1$. For $n \geq 2$ let λ_k be the non-degenerate jump of the restriction ${}^k f_0$ of f_0 to the hyperplane $\{z_k = 0\}$ and $\lambda^{\text{hyp}} := \min_{1 \leq k \leq n} (\lambda_k(p_k - 1))$. Then putting $p'_k := p_1 \cdots p_{k-1} p_{k+1} \cdots p_n$, $k = 1, \dots, n$, $d := \text{GCD}(p'_1, \dots, p'_n)$ and $M := \lceil \frac{\lambda^{\text{hyp}} - 1}{d} \rceil$ we have*

$$\lambda^{\text{nd}}(f_0) = \begin{cases} \min_{1 \leq m \leq M} (md) & \text{if there exist } i_1, \dots, i_n \in \mathbb{N} \text{ such that} \\ & i_1 p'_1 + \cdots + i_n p'_n = p_1 \cdots p_n - md, \\ & 0 < i_k < p_k, \quad k = 1, \dots, n, \\ \lambda^{\text{hyp}} & \text{otherwise.} \end{cases} \quad (9)$$

In the first case the jump is realized by the monomial deformation $(f_s^{\mathbf{i}})$ where $\mathbf{i} = (i_1, \dots, i_n)$ and in the second case by $(f_s^{\mathbf{i}})$ where $\mathbf{i}$ is the point in a coordinate hyperplane $\{x_{k_0} = 0\}$ for which $\lambda^{\text{hyp}} = \lambda_{k_0}(p_{k_0} - 1)$ and $\mathbf{i}$ realizes the jump for the restriction ${}^{k_0}f_0$ of f_0 to the hyperplane $\{z_{k_0} = 0\}$.

Moreover in the case $\text{GCD}(p_i, p_j) = 1$ for $i \neq j$ there is a simple algorithm using only Euclid's algorithm to decide if there are solutions in the first case.

Example 1 (Bodin [3], Walewska [16]). For $n = 2$ and $f_0(z) = z_1^{p_1} + z_2^{p_2}$ putting $d := \text{GCD}(p_1, p_2)$ we have (except the case $p_1 = p_2 = 2$) from the above theorem

$$\lambda^{\text{nd}}(f_0) = \begin{cases} d & \text{if } d < \min(p_1, p_2) \\ d - 1 & \text{if } d = \min(p_1, p_2) \end{cases}.$$

In fact, we have $\lambda_1 = \lambda_2 = 1$, $\lambda^{\text{hyp}} = \min(p_1 - 1, p_2 - 1)$ and we always have the solution i_1, i_2 of $i_1 p_2 + i_2 p_1 = p_1 p_2 - d$, $0 < i_1 < p_1$, $0 < i_2 < p_2$.

Example 2 Let $f_0(z) = z_1^{11} + z_2^6 + z_3^5$. We have $p_1 = 11$, $p_2 = 6$, $p_3 = 5$ and by Example 1 $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda^{\text{hyp}} = 4$. Since $\text{GCD}(p_i, p_j) = 1$ for $i \neq j$ by the algorithm given in the proof we easily find unique solutions (see Lemma (4.2)) of diophantine equations in the formula (9) for $m = 1, 2, 3$

$$\begin{aligned} 4p_2 p_3 + 5p_1 p_3 - 1p_1 p_2 &= p_1 p_2 p_3 - 1, \\ 8p_2 p_3 + 4p_1 p_3 - 2p_1 p_2 &= p_1 p_2 p_3 - 2, \\ 1p_2 p_3 + 3p_1 p_3 + 2p_1 p_2 &= p_1 p_2 p_3 - 3. \end{aligned}$$

Only coefficients in the last equation fulfills the condition in the formula. Hence

$$\lambda^{\text{nd}}(f_0) = 3$$

and the jump is realized by the monomial deformation

$$f_s^{(1,3,2)}(z) = z_1^{11} + z_2^6 + z_3^5 + s z_1 z_2^3 z_3^2.$$

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