Relations of the Spaces $A^p(\Omega)$ and $C^p(\partial\Omega)$

N. Georgakopoulos, V. Mastrantonis, and V. Nestoridis

Abstract. Let Ω be a Jordan domain in $\mathbb{C}$, J an open arc of $\partial\Omega$ and $\phi : D \rightarrow \Omega$ a Riemann map from the open unit disk D onto Ω . Under certain assumptions on ϕ we prove that if a holomorphic function $f \in H(\Omega)$ extends continuously on $\Omega \cup J$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$, then the following equivalence holds: the derivatives $f^{(l)}$, $1 \leq l \leq p$, $l \in \mathbb{N}$, extend continuously on $\Omega \cup J$ if and only if the function $f|_J$ has continuous derivatives on J with respect to the position of orders l , $1 \leq l \leq p$, $l \in \mathbb{N}$. Moreover, we show that for the relevant function spaces, the topology induced by the l -derivatives on Ω , $0 \leq l \leq p$, $l \in \mathbb{N}$, coincides with the topology induced by the same derivatives taken with respect to the position on J .

Mathematics Subject Classification. Primary 30E99; Secondary 30H50.

Keywords. Riemann map, Poisson Kernel, Jordan curve, smoothness on the boundary.

1. Introduction

In this paper we investigate the relationship between the continuous extendability of the derivatives of a function $f \in A(\Omega)$, for some Jordan domain Ω , and the differentiability of the map $t \mapsto f(\gamma(t))$ for some parametrization γ of $\partial\Omega$. Here, $A(\Omega)$ is the collection of all complex functions holomorphic on Ω and continuous on $\bar{\Omega}$. Specifically, it is well known that the first p derivatives of a function $f \in A(D)$, D being the unit disk in $\mathbb{C}$, continuously extend over $\bar{D}$ if and only if the map $t \mapsto f(e^{it})$ is p times continuously differentiable [6]. We generalize this for functions that are holomorphic on the unit disk but now continuously extend over an open arc of $\mathbb{T} = \partial D$ and prove an analogous equivalence for functions defined on Jordan domains that have sufficiently smooth Riemann maps.

The spaces $A^p(\Omega)$, $p \in \{0, 1, 2, \dots\} \cup \{\infty\}$, consist of all holomorphic functions f in Ω whose derivatives $f^{(l)}$, $l \in \{0, 1, \dots\}$, $l \leq p$, extend continuously on $\overline{\Omega}$. It is well known that for any f in the disc algebra $A(D)$, $f \in A^p(D)$ if and only if the map $t \mapsto f(e^{it})$ is C^p smooth. In other words, $A^p(D) = A(D) \cap C^p(\mathbb{T})$ both as sets and as topological spaces. Additionally, if $f \in A(D)$ and $g(t) = f(e^{it})$, $t \in \mathbb{R}$, the equation that relates the continuous extension of f' on $\mathbb{T}$ and the derivative of g is as expected, i.e. $\frac{dg}{dt}(t) = ie^{it}f'(e^{it})$. To prove this, one can use the Poisson representation, to recover the values of f in the disk from its boundary values, i.e. $f(re^{it}) = (g * P_r)(t)$ where P_r denotes the Poisson kernel, differentiate both sides with respect to t and let $r \rightarrow 1^-$. A detailed proof can be found in [6].

In this paper we prove analogous results for functions $f \in A(D)$ whose derivatives continuously extend on an open arc of the unit circle but not necessarily the entire circle. Moreover, using Riemann's mapping theorem, we can drop our initial assumption $f \in A(D)$ and instead assume that it only extends continuously over the specific arc we are interested in. Precisely, if $f : D \rightarrow \mathbb{C}$ is holomorphic and continuously extends over an open arc $J \subseteq \mathbb{T}$, then its first p derivatives continuously extend over that arc if and only if the map $t \mapsto f(e^{it})$ is in $C^p(I)$, where $I = (a, b)$ is an interval in $\mathbb{R}$ with $J = \{e^{it} : t \in I\}$. This motivates a more general definition of the spaces A^p .

In Sect. 3 we consider functions f holomorphic on a Jordan domain Ω and continuous on $\Omega \cup J$, for some open arc J of $\partial\Omega$. We prove that for any $p \in \{1, 2, \dots\} \cup \{\infty\}$, the derivatives $f^{(l)}$, $0 \leq l \leq p$, continuously extend over $\Omega \cup J$ if and only if the continuous extension of f on J is p times continuously differentiable on J with respect to the position [2]. To do this, we place a smoothness assumption for the Riemann map $\phi : D \rightarrow \Omega$ from the open unit disk D onto Ω . The condition is that $(\phi^{-1})'$ has a continuous extension on $\Omega \cup J$ and that $(\phi^{-1})'(z) \neq 0$ on $\Omega \cup J$.

Professor Paul Gauthier informed us that results in the same direction have already been proven in a paper of Steve Bell and Laszlo Lempert entitled "A C^∞ Schwarz Reflection principle in one and several complex variables" which appeared in J. Differential Geometry 32(1990)899–915. However, they do not cover our result Theorem 5. He also added that section 8 of a paper of Steven R. Bell and Steven G. Krantz entitled "Smoothness to the boundary of conformal maps", which appeared in Rocky Mountain Journal of Mathematics, vol. 17, no 1, winter 1987, shows that our result is the best possible in a certain sense.

2. Extendability Over an Open Arc of the Unit Circle

For $0 \leq p \leq +\infty$, $A^p(D)$ denotes the space of holomorphic functions on D whose derivatives of order $l \in \mathbb{N}$, $0 \leq l \leq p$, extend continuously over $\overline{D}$. It is topologized via the semi-norms:

$$|f|_l = \sup_{z \in \bar{D}} |f^{(l)}(z)| = \sup_{z \in \mathbb{T}} |f^{(l)}(z)|, 0 \leq l \leq p, l \in \mathbb{N}.$$

The following theorem is well known. A detailed proof can be found in [6],

Theorem 1. *For all $f \in A(D)$ the following equivalence holds: $f \in A^p(D)$ if and only if the map $g(t) = f(e^{it})$, $t \in \mathbb{R}$, is p times continuously differentiable. In that case:*

$$\frac{dg}{dt}(t) = ie^{it}f'(e^{it}). \quad (1)$$

We now generalize this on functions that are holomorphic on D and continuously extend over an open arc J of the unit circle. We prove that for any such function f and $p \in \{1, 2, \dots\} \cup \{\infty\}$ the first p derivatives of f continuously extend over $D \cup J$ if and only if the map $t \mapsto f(e^{it})$ is p times continuously differentiable in $I = \{t \in [a, a + 2\pi] : e^{it} \in J\}$ for a suitable $a \in \mathbb{R}$. Denote by $A^p(D, J)$ the space of holomorphic functions whose first p derivatives continuously extend over $D \cup J$ and let $C^p(J)$ be the class of functions $f : J \rightarrow \mathbb{C}$, such that the map $t \mapsto f(e^{it})$, $t \in I$, is p times continuously differentiable. The aim is to show the equality $A^p(D, J) = A(D, J) \cap C^p(J)$. For simplicity, we take $J = \{e^{it} : 0 < t < 1\}$ throughout this section. For any $z = re^{i\theta} \in \mathbb{C}$ we denote by $P_z(t)$ or $P_r(t)$ the Poisson kernel [1].

Proposition 1. *If $u : [0, 1] \rightarrow \mathbb{R}$ is a continuous function then:*

$$A(z) = \frac{1}{2\pi} \int_0^1 u(t) P_z(t) dt \quad (2)$$

is well defined in $\mathbb{C} \setminus \bar{J}$ and C^∞ harmonic.

Proof. To see that $A(z)$ is well defined in $\mathbb{C} \setminus \bar{J}$ observe that:

$$A(z) = \frac{1}{2\pi} \int_0^1 u(t) P_z(t) dt = \operatorname{Re} \left(\frac{1}{2\pi} \int_0^1 u(t) \frac{1 + e^{-it}z}{1 - e^{-it}z} dt \right) \quad (3)$$

and $1 - e^{-it}z = 0 \Leftrightarrow z = e^{i(t+2k\pi)}, k \in \mathbb{Z}$. For a fixed $z \in \mathbb{C} \setminus \{e^{it} : 0 \leq t \leq 1\}$ we have $\delta_z = \operatorname{dist}(1, \{ze^{-it} : 0 \leq t \leq 1\}) > 0$ and:

$$\left| u(t) \frac{1 + e^{-it}z}{1 - e^{-it}z} \right| \leq \sup_{t \in [0, 1]} |u(t)| \frac{1 + |z|}{\delta_z} < +\infty \quad (4)$$

for all $t \in [0, 2\pi]$. Since the quantity on the right hand side is integrable we deduce that $A(z)$ is indeed well defined in $\mathbb{C} \setminus \bar{J}$.

In order to prove that A is C^∞ harmonic it suffices to show that $g(z) = \frac{1}{2\pi} \int_0^1 u(t) \frac{1 + e^{-it}z}{1 - e^{-it}z} dt$ is holomorphic in $\mathbb{C} \setminus \bar{J}$, since A is the real part of g according to (3). Note that for $z \neq z_0$:

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{1}{2\pi} \int_0^1 u(t) \frac{2e^{-it}}{(1 - e^{-it}z)(1 - e^{-it}z_0)} dt \quad (5)$$

and hence for z sufficiently close to z_0 :

$$\left| \frac{g(z) - g(z_0)}{z - z_0} - \frac{1}{2\pi} \int_0^1 u(t) \frac{2e^{-it}}{(1 - e^{-it}z_0)^2} dt \right| \quad (6)$$

$$= \left| \frac{1}{2\pi} \int_0^1 u(t) \frac{2e^{-2it}(z - z_0)}{(1 - e^{-it}z)(1 - e^{-it}z_0)^2} dt \right| \quad (7)$$

$$\leq \frac{1}{2\pi} \sup_{t \in [0,1]} |u(t)| 2 \left(\frac{2}{\delta_{z_0}} \right)^3 |z - z_0| \xrightarrow{z \rightarrow z_0} 0. \quad (8)$$

Therefore g is holomorphic and the proof is complete. $\square$

Lemma 1. Let $u : [0, 1] \rightarrow \mathbb{R}$ be a continuous function and define $A(z)$ as in Proposition 1. For all $z = re^{i\theta} \in \mathbb{C} \setminus \bar{J}$:

$$\frac{dA}{d\theta}(re^{i\theta}) = \frac{1}{2\pi} \int_0^1 u(t) \frac{-2r(1 - r^2) \sin(\theta - t)}{(1 + r^2 - 2r \cos(\theta - t))^2} dt. \quad (9)$$

Thus, for $\theta \in (1, 2\pi)$ and $r = 1$:

$$\frac{dA}{d\theta}(e^{i\theta}) = 0. \quad (10)$$

Proof. Since $A(re^{i\theta}) = \frac{1}{2\pi} \int_0^1 u(t) P_r(\theta - t) dt = \frac{1}{2\pi} (f * P_r)$ and P_r is differentiable in respect to θ we have that $A(re^{i\theta})$ is differentiable in respect to θ ,

$$\frac{dA}{d\theta}(re^{i\theta}) = \frac{1}{2\pi} \frac{d(f * P_r)}{d\theta}(re^{i\theta}) = \frac{1}{2\pi} \left(f * \frac{dP_r}{d\theta} \right)(re^{i\theta}) \implies \quad (11)$$

$$\frac{dA}{d\theta}(re^{i\theta}) = \frac{1}{2\pi} \int_0^1 u(t) \frac{-2r(1 - r^2) \sin(\theta - t)}{(1 + r^2 - 2r \cos(\theta - t))^2} dt. \quad (12)$$

(10) is derived from (9) substituting $r = 1$ and $\theta \in (1, 2\pi)$. $\square$

Because every continuous function $u : [0, 1] \rightarrow \mathbb{C}$ can be considered a 2π -periodic function $u : \mathbb{R} \rightarrow \mathbb{C}$ such that $v(x) = u(x)$ for $x \in [0, 1] + 2\pi\mathbb{Z}$ and $v(x) = 0$ otherwise, one can expect that such a function when convolved with the Poisson kernel would retain the nice properties. More specifically, it will uniformly converge to 0 and to $u(x)$ on the compact subsets of the respective open arcs. We prove this in Propositions 2 and 3.

Proposition 2. If $A(z)$ is as before, then for all $\theta \in (1, 2\pi)$ and $l \in \mathbb{N}$:

$$\lim_{r \rightarrow 1^-} \frac{d^l A}{d\theta^l}(re^{i\theta}) = 0. \quad (13)$$

The convergence is uniform in the compact subsets of $(1, 2\pi)$.

Proof. We start with $l = 1$. It suffices to prove that for all $[\theta_1, \theta_2] \subset (1, 2\pi)$ the convergence is uniform. Observe that $0 < \theta_1 - t < \theta_2 - t < 2\pi$ for all $t \in [0, 1]$ and therefore $\cos(\theta - t) \leq \max\{\cos(\theta_1 - 1), \cos(\theta_2)\} = M < 1, \forall t \in [0, 1]$ and $\forall \theta \in [\theta_1, \theta_2]$. This implies that $1 + r^2 - 2r \cos(\theta - t) \geq 1 + r^2 - 2rM = (1 - r)^2 + 2r(1 - M)$ for all $t \in [0, 1]$ and $\theta \in [\theta_1, \theta_2]$. Thus, for all $\theta \in [\theta_1, \theta_2]$ and $0 < r < 1$:

$$\left| \frac{dA}{d\theta}(re^{i\theta}) \right| = \left| \frac{1}{2\pi} \int_0^1 u(t) \frac{-2r(1 - r^2) \sin(\theta - t)}{(1 + r^2 - 2r \cos(\theta - t))^2} dt \right| \quad (14)$$

$$\leq \frac{1}{2\pi} \|u\|_\infty \frac{2r(1 - r^2)}{((1 - r)^2 + 2r(1 - M))^2} \quad (15)$$

and hence $\sup_{\theta \in [\theta_1, \theta_2]} \left| \frac{dA}{d\theta}(re^{i\theta}) \right| \xrightarrow{r \rightarrow 1^-} 0$ since the right hand side of (15) converges to 0 as $r \rightarrow 1^-$.

Note that no matter how many times we differentiate P_r in respect to θ we will have a finite sum of fractions with numerator $c(1 - r^2)^k \cos(\theta)^l \sin(\theta)^m$ for $c \neq 0, k, l, m \in \mathbb{N}$ and denominator a power of $(1 + r^2 - 2r \cos(\theta - t))$. So for any $l \geq 2$ the same arguments apply. $\square$

Denote by $C^p([0, 1])$ the class of functions $u : [0, 1] \rightarrow \mathbb{C}$ that are p times continuously differentiable in $(0, 1)$ and $u^{(l)}$ continuously extend on $[0, 1]$ for all $0 \leq l \leq p, l \in \mathbb{N}$.

Proposition 3. Let $p \in \{0, 1, \dots\} \cup \{\infty\}$, $u : [0, 1] \rightarrow \mathbb{R}$ of class $C^p([0, 1])$ and define $A(z)$ as in Proposition 1. For all $\theta_0 \in (0, 1)$ and $0 \leq l \leq p, l \in \mathbb{N}$,

$$\lim_{\substack{z \rightarrow e^{i\theta_0} \\ |z| < 1}} \frac{d^l A}{d\theta^l}(z) = u^{(l)}(\theta_0). \quad (16)$$

The convergence is uniform in the compact subsets of $(0, 1)$.

Proof. By a theorem of Borel (see [5]), we can find a function $q : \mathbb{R} \rightarrow \mathbb{R}$ of class $C^\infty(\mathbb{R})$ such that $q^l(0) = u^l(0)$ and $q^l(1) = u^l(1)$ for $0 \leq l \leq p, l \in \mathbb{N}$. Thus, the function:

$$g(t) = \begin{cases} u(t), & t \in [0, 1] \\ q(t), & t \in (1, 2\pi) \end{cases}$$

is of class $C^p(\mathbb{T})$. Define:

$$G(z) = \frac{1}{2\pi} \int_0^{2\pi} g(t) P_z(t) dt. \quad (17)$$

It is well known that uniformly for all $\theta \in \mathbb{R}, 0 \leq l \leq p, l \in \mathbb{N}$:

$$\lim_{\substack{z \rightarrow e^{i\theta} \\ |z| < 1}} \frac{d^l G}{d\theta^l}(z) = g^{(l)}(\theta) \quad (18)$$

Let

$$A(z) = \frac{1}{2\pi} \int_0^1 u(t) P_z(t) dt \quad \text{and} \quad B(z) = \frac{1}{2\pi} \int_1^{2\pi} q(t) P_z(t) dt. \quad (19)$$

Note that $G(z) = A(z) + B(z)$ for all $|z| < 1$ and hence for $0 \leq l \leq p, l \in \mathbb{N}$:

$$\frac{d^l G}{d\theta^l}(z) = \frac{d^l A}{d\theta^l}(z) + \frac{d^l B}{d\theta^l}(z). \quad (20)$$

From (18) Proposition 2 we have that $\frac{d^l G}{d\theta^l}(re^{i\theta}) \rightarrow u^{(l)}(\theta)$ and $\frac{d^l B}{d\theta^l}(re^{i\theta}) \rightarrow 0$ as $r \rightarrow 1^-$, uniformly in the compact subsets of $(0, 1)$. As a result,

$$\lim_{\substack{z \rightarrow e^{i\theta_0} \\ |z| < 1}} \frac{d^l A}{d\theta^l}(z) = \lim_{\substack{z \rightarrow e^{i\theta_0} \\ |z| < 1}} \left(\frac{d^l G}{d\theta^l}(z) - \frac{d^l B}{d\theta^l}(z) \right) = u^{(l)}(e^{i\theta_0}) \quad (21)$$

while the convergence is uniform in the compact subsets of $(0, 1)$. $\square$

Remark 1. By linearity, Propositions 1, 2, 3 and Lemma 1 hold for complex functions $f = u + iv$ where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, since we can apply them to the real and imaginary part separately.

We now adapt the proof of Theorem 1 for functions whose derivatives only extend over an open arc of the unit circle.

Theorem 2. Let $p \in \{1, 2, \dots\} \cup \{\infty\}$, $f \in A(D)$ and $g(t) = f(e^{it})$, $t \in (0, 1)$. The following are equivalent: $f^{(l)}$ continuously extends on $D \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$ if and only if g is p times continuously differentiable in $(0, 1)$. In that case, for all $t \in (0, 1)$

$$\frac{dg}{dt}(t) = ie^{it} f'(e^{it}). \quad (22)$$

Proof. We prove it by induction on p . For $p = 1$, let $t_0 \in (0, 1)$ and $t_1 < t_0 < t_2$ such that $[t_1, t_2] \subset (0, 1)$. Assuming that $f^{(l)}$ continuously extends over $D \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$, let $f_r(t) = f(re^{it})$, $h(t) = ie^{it} f'(e^{it})$ and:

$$h_r(t) = \frac{df_r}{dt}(t) = ire^{it} f'(re^{it}) \quad (23)$$

for all $t \in (0, 1)$ and $0 < r < 1$. We have $h_r \rightarrow h$ as $r \rightarrow 1^-$ uniformly in $[t_1, t_2]$, since f' is continuous in $D \cup J$. Note that $f_r(t_0) \rightarrow f(e^{it_0})$ and hence from a well-known theorem $f_r \rightarrow \int h dt + c$, for a some $c \in \mathbb{C}$, as $r \rightarrow 1$, while the convergence is uniform in $[t_1, t_2]$. Additionally, $f_r \rightarrow g$, as $r \rightarrow 1$, uniformly in $[t_1, t_2]$ and therefore $g'(t) = h(t) = ie^{it} f'(e^{it})$, $t \in (t_1, t_2)$. However, t_0 was arbitrary thus, $g \in C^1((0, 1))$ and (22) holds.

For the converse let $g \in C^1((0, 1))$. We now use the Poisson representation:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) P_z(t) dt. \quad (24)$$

for $|z| < 1$. Define:

$$A(z) = \frac{1}{2\pi} \int_{t_1}^{t_2} g(t) P_z(t) dt \text{ and } B(z) = \frac{1}{2\pi} \int_{t_2}^{2\pi+t_1} f(e^{it}) P_z(t) dt. \quad (25)$$

Note that $f(z) = A(z) + B(z)$ for all $|z| < 1$. Consequently,

$$\frac{df}{dt}(re^{it}) = \frac{dA}{dt}(re^{it}) + \frac{dB}{dt}(re^{it}) \quad (26)$$

for all $0 < r < 1$ and $t \in \mathbb{R}$. Since $[t_1, t_2] \subset (0, 1)$, Propositions 2 and 3 imply that:

$$\lim_{r \rightarrow 1^-} \frac{dA}{dt}(re^{it}) = \frac{dg}{dt}(t) \quad (27)$$

and

$$\lim_{r \rightarrow 1^-} \frac{dB}{dt}(re^{it}) = 0 \quad (28)$$

uniformly for $t \in [t_1, t_2]$. Combining (26), (27) and (28) we get:

$$\lim_{r \rightarrow 1^-} f'(re^{it}) = \lim_{r \rightarrow 1^-} \frac{1}{ire^{it}} \frac{df}{dt}(re^{it}) = \frac{1}{ie^{it}} \frac{dg}{dt}(t) \quad (29)$$

uniformly for $t \in [t_1, t_2]$. Since t_0 was arbitrarily chosen in $(0, 1)$ we deduce that f' extends continuously on $D \cup J$. To complete the induction, let us assume that the theorem holds for some $p \geq 1$. If $f^{(l)}$ continuously extends on $D \cup J$ for all $0 \leq l \leq p+1, l \in \mathbb{N}$ it follows that $(f')^{(l)}$ continuously extends on $D \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$. By the induction hypothesis the map $t \mapsto f'(e^{it})$ belongs in the class $C^p((0, 1))$ and since, by the case of $p = 1$, $g'(t) = ie^{it} f'(e^{it})$ we have $g' \in C^p((0, 1))$ and hence $g \in C^{p+1}((0, 1))$. For the converse, if $g \in C^{p+1}((0, 1))$ it follows from (22) that $g'(t) = ie^{it} f'(e^{it}) \in C^p((0, 1))$ and therefore, the map $t \mapsto f'(e^{it})$ is of class $C^p((0, 1))$. By the induction hypothesis, $(f')^{(l)}$ continuously extends on $D \cup J$, for all $0 \leq l \leq p, l \in \mathbb{N}$ and hence $f^{(l)}$ continuously extends on $D \cup J$, for all $0 \leq l \leq p+1, l \in \mathbb{N}$. The case of $p = \infty$ follows easily. $\square$

Using Riemann's mapping theorem [1] we can drop the assumption of continuity over $\overline{D}$. Indeed, we can just have f continuously extend over the open arc we are dealing with, i.e. $f \in A(D, J)$.

Theorem 3. Let $p \in \{1, 2, \dots\} \cup \{\infty\}$, $f : D \cup J \rightarrow \mathbb{C}$ continuous on $D \cup J$ and holomorphic in D and $g(t) = f(e^{it}), t \in (0, 1)$. The following are equivalent: $f^{(l)}$ continuously extends over $D \cup J$, for all $0 \leq l \leq p, l \in \mathbb{N}$, if and only if g is p times continuously differentiable in $(0, 1)$. In that case:

$$\frac{dg}{dt}(t) = ie^{it} f'(e^{it}) \quad (30)$$

for all $t \in (0, 1)$.

Proof. We prove it by induction on p . For $p = 1$, the only if part is proven like Theorem 2; we also obtain (30). For the converse, let $t_0 \in (0, 1)$, $t_1 < t_0 < t_2$ such that $[t_1, t_2] \subseteq (0, 1)$ and $J' = \{e^{it} : t_1 < t < t_2\}$. Set $V = \{|z| < 1, z \neq 0 : \frac{z}{|z|} \in J'\}$. It is easily verified that V is simply connected and hence there is a conformal map $\phi : D \rightarrow V$ which extends to homeomorphism over the closures $\phi : \bar{D} \rightarrow \bar{V}$, by the Osgood–Carathéodory theorem [4]. Since $\phi(\mathbb{T}) = \{e^{it} : t_1 \leq t \leq t_2\} \cup \{re^{it_1} : 0 \leq r \leq 1\} \cup \{re^{it_2} : 0 \leq r \leq 1\}$ and $\{e^{it} : t_1 \leq t \leq t_2\}$ is connected we deduce that $\phi^{-1}(\{e^{it} : t_1 \leq t \leq t_2\})$ is closed and connected in $\mathbb{T}$. Without loss of generality, assume that $\phi^{-1}(\{e^{it} : t_1 \leq t \leq t_2\}) = \{e^{it} : 0 \leq t \leq 1\}$. Consequently, $f \circ \phi : \bar{D} \rightarrow \mathbb{C}$ is continuous on $\bar{D}$ and holomorphic in D that is, of class $A(D)$. Using the Reflection Principle, we deduce that ϕ conformally extends on an open $G \supset V \cup \{e^{it} : t_1 \leq t \leq t_2\}$ thus, $t \mapsto \phi(e^{it}), t \in (t_1, t_2)$ is of class $C^\infty((t_1, t_2))$ with non-vanishing derivative (see [1] pp. 233–235). Since $g \in C^1((t_1, t_2))$ and $t \mapsto \phi(e^{it})$ is in $C^\infty((0, 1))$ we have that the map $t \mapsto f(\phi(e^{it}))$ is in $C^1((0, 1))$. By Theorem 2 the derivative $(f \circ \phi)'$ continuously extends on $D \cup J$ and hence f' continuously extends in $D \cup J'$, because $f = (f \circ \phi) \circ \phi^{-1}$ and $f' = (f \circ \phi)' \circ \phi^{-1} \cdot (\phi^{-1})'$. Our choice of t_0 was arbitrary and therefore f' continuously extends on $D \cup J$. To complete the induction, we follow the proof of Theorem 2. The case of $p = \infty$ follows easily. $\square$

In other words, Theorem 3 states that for any open arc J of $\mathbb{T}$ and $p \in \{0, 1, 2, \dots\} \cup \{\infty\}$, $A^p(D, J) = A(D, J) \cap C^p(J)$ holds.

Remark 2. In all Propositions and Lemmas of this section, $[0, 1]$ can be replaced by any interval $[a, b]$ with $0 \leq a < b < a + 2\pi$.

3. Jordan Domains

Motivated by Theorem 3 we now give a more general definition of the spaces A^p . Let Ω be a Jordan domain and J an open arc of $\partial\Omega$. Denote by $A^p(\Omega, J)$ the collection of all functions f holomorphic on Ω such that $f^{(l)}$ continuously extends on $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$. Note that $A^p(\Omega, \partial\Omega) = A^p(\Omega)$. If $\gamma : I \rightarrow J$ is a parametrization of J , denote by $C_\gamma^p(J)$ the class of all functions $f : G \rightarrow \mathbb{C}$ defined on a varying set G , $J \subset G$, such that $f \circ \gamma : I \rightarrow \mathbb{C}$ is p times continuously differentiable. Moreover, if $\phi : \bar{D} \rightarrow \bar{\Omega}$ is a Riemann map, denote by $C_\phi^p(J)$ the class $C_\gamma^p(J)$ where $\gamma(t) = \phi(e^{it})$. Since any two Riemann maps differ by an automorphism of the unit disk it is easily verified that the spaces $C_\phi^p(J)$ do not depend on the chosen Riemann map. Next, we consider differentiability on J with respect to the position [2].

Definition 1. Let J be a Jordan arc and $f : J \rightarrow \mathbb{C}$. We define the derivative of f on $z_0 \in J$ by:

$$\frac{df}{dz}(z_0) = \lim_{z \rightarrow z_0, z \in J} \frac{f(z) - f(z_0)}{z - z_0} \quad (31)$$

if this limit exists and is a complex number.

In order to go one step further, we consider the derivative $\frac{df}{dz}$ on J of Definition 1 and we take its derivative on J with respect to the position.

Definition 2. A function $f : J \rightarrow \mathbb{C}$ belongs to the class $C^1(J)$ if $\frac{df}{dz}(z)$ exists and is continuous for $z \in J$. Inductively, suppose that $\frac{d^{p-1}f}{dz^{p-1}}$ is well defined on J for some $p = 2, \dots, +\infty$, we say that f is of class $C^p(J)$ if

$$\frac{d^p f}{dz^p}(z) = \frac{d(\frac{d^{p-1}f}{dz^{p-1}})}{dz}(z) \quad (32)$$

exists and is continuous on J .

Remark 3. In [2] the following fact is proven. If $\gamma : I \rightarrow J$ is a C^n regular (with non vanishing derivative) parametrization of a Jordan arc $J, n \in \{1, 2, \dots\} \cup \{\infty\}$, a function f is of class $C^p(J)$ if and only if $g(t) = (f \circ \gamma)(t), t \in I$, is of class $C^p(I), p \in \{1, 2, \dots\} \cup \{\infty\}, p \leq n$. Additionally,

$$\frac{dg}{dt}(t) = \frac{df}{dz}(\gamma(t)) \cdot \gamma'(t). \quad (33)$$

In this section we prove that given a Jordan domain Ω , $\phi : \overline{D} \rightarrow \overline{\Omega}$ a Riemann map such that $\phi^{-1} \in A^1(\Omega, J)$ and $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$, $f \in A(\Omega, J)$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$ the following are equivalent: $f^{(l)}$ continuously extend over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$ if and only if $f|_J$ is of class $C^p(J)$. That is $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$.

Let us first prove a straightforward fact concerning the parametrization of an arc J of $\partial\Omega$ induced by a Riemann map ϕ .

Theorem 4. Let Ω be a Jordan domain, $\phi : \overline{D} \rightarrow \overline{\Omega}$ a Riemann map and J an open arc of $\partial\Omega$ such that ϕ^{-1} is of class $A^n(\Omega, J)$ for some $n \in \{1, 2, \dots\} \cup \{\infty\}$ and $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$. The following holds for any $p \in \{1, 2, \dots\} \cup \{\infty\}, p \leq n$: $A^p(\Omega, J) = A(\Omega, J) \cap C^p_\phi(J)$. In that case, if $f \in A^p(\Omega, J)$ and $g(t) = (f \circ \phi)(e^{it})$:

$$\frac{dg}{dt}(t) = ie^{it} f'(\phi(e^{it})) \phi'(e^{it}) \quad (34)$$

for all $t \in \{s \in \mathbb{R} : \phi(e^{is}) \in J\}$.

On the right hand side of (34) f' denotes the continuous extension of $f'(z)$ from Ω to $\Omega \cup J$.

Before proceeding to the proof, let us note that the choice of the Riemann map is irrelevant. Suppose that Φ, Ψ are two Riemann maps and J an open arc of $\partial\Omega$ such that Φ^{-1} is of class $A^n(\Omega, J), n \in \{1, 2, \dots\} \cup \{\infty\}$, and $(\Phi^{-1})'(z) \neq 0$ for all $z \in \Omega \cup J$. Then, Ψ^{-1} is also of class $A^n(\Omega, J)$ and $(\Psi^{-1})'$ is non zero in $\Omega \cup J$. To see this, observe that $\Phi^{-1} \circ \Psi$ is an automorphism of the unit disk and hence there are $a \in D$ and $c \in \mathbb{T}$ such that $\Psi = \Phi \circ \phi_a$, where $\phi_a(z) = c \frac{z-a}{1-\overline{a}z}, z \in D$. Note that ϕ_a is holomorphic in $D(0, \frac{1}{|a|}) \supset D$ with non vanishing derivative

thus, $\Psi^{(l)}$ can be extended over $\overline{D}$ for all $0 \leq l \leq n, l \in \mathbb{N}$. Additionally, $(\Psi^{-1})'(z) = \frac{1-|a|^2}{(1+\overline{a}\Phi^{-1}(z))^2} \cdot (\Phi^{-1})'(z) \neq 0$ for all $z \in \Omega \cup J$. Moreover, Φ^{-1} being of class $A^n(\Omega, J)$ is equivalent to Φ being of class $A^n(D, I)$, $I = \phi^{-1}(J)$. This is easy to see given that $(\Phi^{-1})'(z) \neq 0$ for $z \in \Omega \cup J$. In other words, if a Riemann map induces a regular n times continuously differentiable parametrization of J then, the same holds for any other Riemann map.

Proof. (Theorem 4) Set $I = \phi^{-1}(J)$ and $\tilde{I} = \{s \in \mathbb{R} : \phi(e^{is}) \in J\}$. Given an $f \in A(\Omega, J)$ one can easily verify that f is of class $A^p(\Omega, J)$ if and only if $f \circ \phi$ is of class $A^p(D, I)$, since ϕ' is non zero in $D \cup I$ and $p \leq n$. By Theorem 3, $f \circ \phi$ is of class $A^p(D, I)$ if and only if the map $t \mapsto (f \circ \phi)(e^{it}), t \in \tilde{I}$, is p times continuously differentiable which by definition is equivalent to f being of class $C^p_\phi(J)$. To see that (34) holds, set $g_r(t) = f(\phi(re^{it})), t \in \mathbb{R}, 0 < r < 1$. Note that $g_r \rightarrow g$ as $r \rightarrow 1^-$ uniformly in the compact subsets of $\tilde{I}$. Additionally, $\frac{dg_r}{dt}(t) = ire^{it}f'(\phi(re^{it}))\phi'(e^{it})$ which uniformly converges to $ie^{it}f'(\phi(e^{it}))\phi'(e^{it})$ as $r \rightarrow 1^-$ in the compact subsets of $\tilde{I}$. From a well known theorem of calculus (34) follows. $\square$

In the next theorem we prove that for any function $f \in A(\Omega, J)$ the first p derivatives of f continuously extend over $\Omega \cup J$ if and only if $f|_J$ is of class $C^p(J)$, given that J is a smooth enough open arc of the Jordan domain Ω . That is, if ϕ is a Riemann map and ϕ^{-1} is of class $A^1(\Omega, J)$ with $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$ then $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$ for all $p \in \{1, 2, \dots\} \cup \{\infty\}$.

Theorem 5. *Let Ω be a Jordan domain, $\phi : D \rightarrow \Omega$ a Riemann map and J an open arc of $\partial\Omega$ such that ϕ^{-1} is of class $A^1(\Omega, J)$ with $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$. Given an $f \in A(\Omega, J)$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$ we have that: $f^{(l)}$ continuously extend over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$ if and only if $f|_J$ is of class $C^p(J)$. That is, $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$. In that case:*

$$f^{(l)}(z) = \frac{d^l f|_J}{dz^l}(z) \quad (35)$$

for all $z \in J$ and $0 \leq l \leq p, l \in \mathbb{N}$.

On the left hand side of (35), $f^{(l)}(z)$ denotes the continuous extension of $f^{(l)}$ on $\Omega \cup J$, while on the right hand side the differentiation is with respect to the position.

Proof. Let $p = 1$ and $f \in A(\Omega, J)$. Denote by $\tilde{I} = \{s \in \mathbb{R} : \phi(e^{is}) \in J\}$. By Theorem 4 we have that $f \in A^1(\Omega, J)$ if and only if the map $g(t) = f(\phi(e^{it})), t \in \tilde{I}$ is continuously differentiable. Additionally, $\frac{dg}{dt}(t) = ie^{it}\phi'(e^{it})f'(\phi(e^{it}))$ and:

$$\frac{dg}{dt}(e^{it}) = \frac{d f|_J}{dz}(\phi(e^{it})) \cdot \frac{d \phi|_{\mathbb{T}}}{dz}(e^{it}) \cdot ie^{it} = ie^{it}\phi'(e^{it})\frac{d f|_J}{dz}(\phi(e^{it})) \quad (36)$$

Since $ie^{it}\phi'(e^{it})$ is non zero we have that g is continuously differentiable if and only if $f|_J$ is of class $C^1(J)$. Consequently, f is of class $A^1(\Omega, J)$ if and only if g is of class $C^1(\bar{I})$ if and only if $f|_J$ is of class $C^1(J)$. Equation (35) follows from Eqs. (34) and (36). For the induction step, assume that $p > 1$. We have that $f \in A^p(\Omega, J)$ if and only if $f' \in A^{p-1}(\Omega, J)$ which by the induction hypothesis is equivalent to $f'|_J \in C^{p-1}(J)$ and that is equivalent to $f|_J \in C^p(J)$, by (35) for $l = 1$. Moreover,

$$\frac{d^l f|_J}{dz^l}(z) = \frac{d^{(l-1)}}{dz^{(l-1)}} \left(\frac{d f|_J}{dz}(z) \right) = \frac{d^{(l-1)}}{dz^{(l-1)}} (f'(z)) = (f')^{(l-1)}(z) = f^{(l)}(z) \quad (37)$$

for all $z \in J$ and $1 \leq l \leq p, l \in \mathbb{N}$. $\square$

Remark 4. Note that if Ω is a Jordan domain, J an analytic arc of $\partial\Omega$ and $\phi : D \rightarrow \Omega$ a Riemann map we know from [1] (p. 235) that ϕ^{-1} has a conformal extension over an open set $G \supset \Omega \cup J$. Consequently, if J is an analytic arc we immediately have $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$ for all $p \in \{1, 2, \dots\} \cup \{\infty\}$.

In addition, if we have γ a C^n regular parametrization of $J, n \in \{1, 2, \dots\} \cup \{\infty\}$ we have a triple equivalence as derived from Remark 3 and Theorem 5.

Theorem 6. *Let Ω be a Jordan domain, ϕ a Riemann map and J an open arc of $\partial\Omega$ such that ϕ^{-1} is of class $A^1(\Omega, J)$ and $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$, γ a C^n regular parametrization of $J, n \in \{1, 2, \dots\} \cup \{\infty\}$, $f \in A(\Omega, J)$ and $p \in \{1, 2, \dots\} \cup \{\infty\}, p \leq n$. The following are equivalent:*

1. $f^{(l)}$ continuously extend over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$. That is, f is of class $A^p(\Omega, J)$.
2. $f|_J$ is of class $C^p(J)$.
3. $g(t) = (f \circ \gamma)(t), t \in I$ is p times continuously differentiable. That is f is of class $C^p_\gamma(J)$.

In other words, $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J) = A(\Omega, J) \cap C^p_\gamma(\Omega, J)$. In that case:

$$\frac{dg}{dt}(t) = \frac{d f|_J}{dz}(\gamma(t))\gamma'(t) = f'(\gamma(t))\gamma'(t) \quad (38)$$

for all $t \in I$.

Remark 5. We notice that the assumption on Theorem 5 that the Riemann map $\phi : D \rightarrow \Omega$ is such that $\phi^{-1} \in A(\Omega, J)$ with $(\phi^{-1})'(z) \neq 0$ for $z \in \Omega \cup J$ is in fact equivalent to say that the parametrization of J induced by ϕ is C^1 regular.

We topologize $A^p(\Omega)$ by the semi-norms

$$|f|_l = \sup_{z \in \bar{\Omega}} |f^{(l)}(z)| = \sup_{z \in \partial\Omega} |f^{(l)}(z)|, 0 \leq l \leq p, l \in \mathbb{N} \quad (39)$$

and $C^p(\partial\Omega)$ by the semi-norms

$$|g|_l = \sup_{z \in \partial\Omega} \left| \frac{d^l g}{dz^l}(z) \right|, 0 \leq l \leq p, l \in \mathbb{N} \quad (40)$$

see also [2]. By Eq. (35) it easily seen that the topology on $A^p(\Omega)$ and $A(\Omega) \cap C^p(\partial\Omega)$ induced by $C^p(\partial\Omega)$ coincide. Summarizing:

Corollary 1. *Let Ω be a Jordan domain and $\phi : D \rightarrow \Omega$ a Riemann map of class $A^1(D)$ such that $\phi'(z) \neq 0$ for all $z \in \overline{D}$. The following equivalence holds for all $f \in A(\Omega)$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$: $f \in A^p(\Omega)$ if and only if $f|_{\partial\Omega}$ is p times continuously differentiable with respect to the position. In this case we have:*

$$\frac{d^l f}{dz^l}(z) = f^{(l)}(z) \quad (41)$$

For all $z \in \partial\Omega$ and $0 \leq l \leq p, l \in \mathbb{N}$. On the right hand side of (41) $f^{(l)}(z)$ denotes the continuous extension of $f^{(l)}$ from Ω to $\Omega \cup J$. We also have, $A^p(\Omega) = A(\Omega) \cap C^p(\partial\Omega)$ as topological spaces.

Remark 6. There is a natural way to topologize the spaces $A^p(\Omega, J)$ for any Jordan domain Ω , open arc $J \subset \partial\Omega$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$. Then, the equality $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$ holds taking into account the topologies of the spaces as well. However, we will not deal with the topological properties of the spaces $A^p(\Omega, J)$ in this paper.

One can easily extend the results of this section for the complement of a Jordan domain Ω given that the functions considered vanish at infinity. That is, if $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$ is the class of all functions f with $\lim_{z \rightarrow \infty} f(z) = 0$ and holomorphic in $\mathbb{C} \setminus \overline{\Omega}$ such that their first p derivatives extend over $\mathbb{C} \setminus \Omega$ for all $0 \leq l \leq p, l \in \mathbb{N}$, then $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega}) = A_0(\hat{\mathbb{C}} \setminus \overline{\Omega}) \cap C^p(\partial\Omega)$ under similar assumptions to that of Theorem 5. For J an open arc of $\partial\Omega$ we define $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega}, J)$ similarly.

Remark 7. The previous results can be generalized to the case of finitely connected domains bounded by a finite set of disjoint Jordan curves. For this we use the Laurent Decomposition [3].

We will take a moment to sketch the proof. Let Ω be a bounded domain whose boundary consists of a finite number of disjoint Jordan curves. If $V_0, V_1, \dots, V_{n-1}$ are the connected components of $\hat{\mathbb{C}} \setminus \Omega$, $\infty \in V_0$ and $\Omega_0 = \hat{\mathbb{C}} \setminus V_0, \dots, \Omega_{n-1} = \hat{\mathbb{C}} \setminus V_{n-1}$, then for every f which is holomorphic in Ω we know that there exist functions $f_0, \dots, f_{n-1}$ which are holomorphic in $\Omega_0, \dots, \Omega_{n-1}$ respectively such that $f = f_0 + f_1 + \dots + f_{n-1}$ and $\lim_{z \rightarrow \infty} f_j(z) = 0, j \in \{1, \dots, n-1\}$ [3]. Let $\phi_j : \overline{D} \rightarrow \overline{\Omega}_j, j \in \{0, 1, \dots, n-1\}$, be the respective conformal maps. Without loss of generality, take J an open arc of $\partial\Omega_0$ such that ϕ_0^{-1} is of class $A^1(\Omega_0, J)$ with non vanishing derivative. Let also f be a holomorphic function in Ω and continuous in $\Omega \cup J$ such that $f = f_0 + f_1 + \dots + f_{n-1}$,

f_j holomorphic in Ω_j , is the extended Laurent decomposition of f . Since f_j is holomorphic in a neighborhood of $\partial\Omega_0$ for all $j \neq 0$, we have that f_0 continuously extends over $\Omega_0 \cup J$ since f continuously extends over $\Omega \cup J$. Similarly, $f^{(l)}$ continuously extends over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$, if and only if f_0 is of class $A^p(\Omega_0, J)$. By Theorem 5, f is of class $A^p(\Omega_0, J)$ if and only if $f_0|_J$ is of class $C^p(J)$, which happens if and only if $f|_J$ is of class $C^p(J)$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Ahlfors, L.V.: Complex Analysis, 3rd edn. Mc Graw-Hill Book Co, New York (1978)
- [2] Bolkas, E., Nestoridis, V., Panagiotis, C., Papadimitrakis, M.: One sided extendability and p -continuous analytic capacities, submitted. [arXiv:1606.05443](https://arxiv.org/abs/1606.05443)
- [3] Costakis, G., Nestoridis, V., Papadoperakis, I.: Universal Laurent series. Proc. Edinb. Math. Soc. (2) **48**(3), 571–583 (2005)
- [4] Koosis, P.: An Introduction of H_p Spaces. Cambridge University Press, Cambridge (1998)
- [5] Krantz, S., Parks, H.: A Primer of Real Analytic Functions, 2nd edn. Springer Birkhäuser, Boston (2002)
- [6] Mastrantonis, V.: Relations of the spaces $A^p(\Omega)$ and $C^p(\partial\Omega)$. [arXiv:1611.02791](https://arxiv.org/abs/1611.02791)

N. Georgakopoulos, V. Mastrantonis and V. Nestoridis
 Department of Mathematics
 National and Kapodistrian University of Athens
 Panepistimioupolis
 157 84 Athens
 Greece
 e-mail: nikolaosgmth@gmail.com

V. Mastrantonis
 e-mail: blasismas@hotmail.com

V. Nestoridis
 e-mail: vnestor@math.uoa.gr

Received: August 24, 2017.

Accepted: March 13, 2018.