On Stability and Hyperstability of an Equation Characterizing Multi-Cauchy–Jensen Mappings

Anna Bahyrycz and Jolanta Olko

Abstract. Recently, functions of several variables satisfying, with respect to each variable, some functional equation (among them Cauchy's, Jensen's, quadratic and other ones) have been studied. We give a new characterization of multi-Cauchy–Jensen mappings, which states that a function fulfilling some equation on a restricted domain is multi-Cauchy–Jensen. Next, using a fixed point theorem, it is proved that a function which approximately satisfies (on restricted domain) the equation characterizing such functions is close (in some sense) to the solution of the equation. This result is a tool for obtaining a generalized Hyers–Ulam stability or hyperstability of this equation for particular control functions, which is presented in several examples.

Mathematics Subject Classification. 39B52, 39B82, 39B72, 47H10.

Keywords. Multi-Cauchy–Jensen mapping, Hyers–Ulam stability, hyperstability, fixed point theorem.

1. Introduction

It is well-known that among functional equations the Cauchy equation

$$f(x + y) = f(x) + f(y) \quad (1)$$

and the Jensen equation

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2} \quad (2)$$

(which is closely connected with the notion of convex function) play a prominent role. A lot of information about them and their applications can be found for instance in [29, 30]. The first positive answer to celebrated Ulam's question concerning the problem of stability of functional equations was given by Hyers in the case of Eq. (1) in Banach spaces (see [23]). The history and recent results concerning the notion of Hyers–Ulam stability can be found in many papers (see e.g. [12, 13, 16, 21, 28, 29] and references included there).

The multi-Cauchy–Jensen mappings mentioned in the title are functions of several variables satisfying Cauchy's functional equation in each of some chosen variables and Jensen's functional equation in each of the remaining ones. Namely, if it holds for k and l variables, respectively, such a function is called k -Cauchy and l -Jensen (see [15]). Without loss of generality it can be assumed that such functions satisfy (1) for the first few variables, and (2) for the next ones.

Let us note that for $k = n$ the above definition leads to the so-called *multi-additive mappings* (some basic facts on such mappings can be found for instance in [30], where their application to the representation of polynomial functions is also presented); for $k = 0$ we obtain the notion of *multi-Jensen function* (which was introduced in 2005 by Prager and Schwaiger (see [33]) in the connection with generalized polynomials), and an 1-Cauchy and 1-Jensen mapping is just a *Cauchy–Jensen mapping* defined by Park and Bae [32].

In this paper, we give a new characterization of multi-Cauchy–Jensen mappings, which states that a function fulfilling some equation on a restricted domain is multi-Cauchy–Jensen on the whole space. Next it is proved that a function which approximately satisfies (on restricted domain) the equation characterizing such functions is close (in some sense) to the solution of the equation. This result is a tool for obtaining a generalized Hyers–Ulam stability or hyperstability of this equation for particular control functions, which is presented in several examples. Our results are significant counterparts of some classical outcomes from [1, 11, 22, 23, 35] and recent results from [2–4, 10, 17–20, 24–27, 31, 32, 34, 36].

In the proof of our stability result (Theorem 6) we use the fixed point method, which was used for the investigation of the Hyers–Ulam stability of functional equations for the first time by Baker [5]. For more information about this method we refer the reader to recent survey papers [13, 21].

To finish this introductory section let us finally mention that some results on the stability of Cauchy–Jensen mappings can be found in [6, 7, 24–27, 31, 32].

2. Characterizations of Multi-Cauchy–Jensen Mappings

Throughout this paper $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $[m] := \{1, \dots, m\}$ for $m \in \mathbb{N}$.

Let us recall that an abelian semigroup G is called *uniquely divisible by 2* provided for every $x \in G$ there exists a unique $y \in G$ (which is denoted in the sequel by $\frac{x}{2}$ or $\frac{1}{2}x$) such that $x = y + y$. The symbol my denotes $(m-1)y + y$ for $m \in \mathbb{N}$, $m \geq 2$. We will denote by G_0 the set $G \setminus \{0\}$, where 0 is an identity element. G is said to be torsion free, if the identity element is the only one of finite order. In this case in particular, $2x, 3x \in G_0$ for $x \in G_0$.

For a nonempty set X and $l, m \in \mathbb{N}$ we identify m -tuple $x = (x_1, \dots, x_m) \in X^m$ with $(y, z) \in X^l \times X^{m-l}$, where $y := (x_1, \dots, x_l)$ and $z := (x_{l+1}, \dots, x_m)$.

Given $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in X^m$ and a set $I \subset [m]$, we will denote by $[x, y]_I$ the m -tuple obtained from x by replacing coordinates x_i with y_i for all $i \notin I$, namely $[x, y]_I = (z_1, \dots, z_m)$, where

$$z_i = \begin{cases} x_i & \text{if } i \in I \\ y_i & \text{if } i \notin I \end{cases}, \quad i \in \{1, \dots, m\}$$

It is clear that $[x, x]_I = x$ for $x \in X^m$, $I \subset [m]$. Moreover for $l \in [m]$, $I \subset [m]$

$$[(y, u), (z, u)]_I = ([y, z]_{[l] \cap I}, u), \quad (y, u), (z, u) \in X^l \times X^{m-l}$$

and

$$[(u, y), (u, z)]_I = \begin{cases} (u, y) & \text{if } m \in I \\ (u, z) & \text{if } m \notin I \end{cases} \quad (u, y), (u, z) \in X^{m-1} \times X.$$

Moreover, we assume that V and W are linear spaces over the rationals, $V_0 := V \setminus \{0\}$, $v \in V_0$ and $n \in \mathbb{N}$, $k \in \{0, \dots, n\}$.

In [9], a characterization of multi-Cauchy–Jensen mappings was proved. Using our notations we can rephrase it in the following form.

Theorem 1. *Assume that G is a semigroup uniquely divisible by 2 and with an identity element, and W is a linear space over the rationals. Then a function $f : G^n \rightarrow W$ is k -Cauchy and $n - k$ -Jensen if and only if for any $x = (x^1, x^2), y = (y^1, y^2) \in G^k \times G^{n-k}$ we have*

$$2^{n-k} f\left(x^1 + y^1, \frac{x^2 + y^2}{2}\right) = \sum_{I \subset [n]} f([x, y]_I). \quad (3)$$

Substituting $k = n$ we have a characterization of multi-additive mappings. A counterpart of this theorem, for mappings defined on linear space over rationals which satisfy (1) on a restricted domain, was proved in [8]. The theorem is still true if we assume that the domain of f is a group satisfying some additional assumptions, which will be proven with the aid of the following characterization of multi-Jensen mappings.

Theorem 2. *Assume that G is a torsion free group uniquely divisible by 2, $G_0 \neq \emptyset$. A function $f : G^k \rightarrow W$ satisfies the equation*

$$2^k f\left(\frac{x + y}{2}\right) = \sum_{I \subset [k]} f([x, y]_I), \quad (4)$$

for all $x, y \in G_0^k$, if and only if f is a multi-Jensen mapping.

Proof. First observe that by Theorem 1, every multi-Jensen mapping of k variables satisfies (4) on G^k . The proof of the converse theorem is by induction on k . It is true for $k = 1$. Indeed, in this case (4) means that for $x \neq 0$ and $y \neq 0$

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad (5)$$

and it suffices to prove the above equality for $x \in G$ and $y = 0$ which is equivalent that for every $x \neq 0$

$$2f(x) = f(2x) + f(0). \quad (6)$$

Obviously, the above equality holds for $x = 0$. If $x \neq 0$, applying (5) for pairs of nonzero elements $2x$, x and next $2x$, $-x$ we get

$$2f\left(\frac{3x}{2}\right) = 2f\left(\frac{2x+x}{2}\right) = f(2x) + f(x),$$

and

$$2f\left(\frac{x}{2}\right) = 2f\left(\frac{2x-x}{2}\right) = f(2x) + f(-x).$$

Adding the above equalities and applying (5) for elements $\frac{3x}{2}$, $\frac{x}{2}$ and next for x , $-x$ we have

$$2f(x) = 2f\left(\frac{\frac{3x}{2} + \frac{x}{2}}{2}\right) = f\left(\frac{3x}{2}\right) + f\left(\frac{x}{2}\right) = f(2x) + f(0),$$

and the proof of the base case is complete.

Now assume that every function of k variables satisfying (4) is multi-Jensen and fix $v \in G_0$

Let $f : G^{k+1} \rightarrow W$ satisfies

$$2^{k+1}f\left(\frac{x+y}{2}\right) = \sum_{I \subset [k+1]} f([x, y]_I), \quad (7)$$

for all $x, y \in G_0^{k+1}$. Consequently for $x \in G_0^k$

$$\begin{aligned} 2^{k+1}f\left(\frac{(x, v) + (y, -v)}{2}\right) &= \sum_{I \subset [k+1]} f([(x, v), (y, -v)]_I) \\ &= \sum_{J \subset [k]} \left(f([x, x]_J, v) + f([x, x]_J, -v) \right) \\ &= 2^k \left(f(x, v) + f(x, -v) \right), \end{aligned}$$

and thus

$$2f(x, 0) = f(x, v) + f(x, -v). \quad (8)$$

Similarly, for $x \in G_0^k$ and $z \in G_0$

$$2f(x, 2z) = f(x, 3z) + f(x, z). \quad (9)$$

Fix $z \in G_0$ and define the function $g_z(x) := f(x, 2z)$ for $x \in G^k$. Applying (7) and (9) we have for all $x, y \in G_0^k$

$$\begin{aligned}
 2^{k+1}g_z\left(\frac{x+y}{2}\right) &= 2^{k+1}f\left(\frac{(x, 3z) + (y, z)}{2}\right) \\
 &= \sum_{I \subset [k+1]} f([(x, 3z), (y, z)]_I) \\
 &= \sum_{J \subset [k]} f([x, y]_J, 3z) + \sum_{J \subset [k]} f([x, y]_J, z) \\
 &= \sum_{J \subset [k]} \{f([x, y]_J, 3z) + f([x, y]_J, z)\} \\
 &= \sum_{J \subset [k]} 2f([x, y]_J, 2z) = 2 \sum_{J \subset [k]} g_z([x, y]_J),
 \end{aligned}$$

which with the inductive assumption implies that g_z is a multi-Jensen function.

Similarly, a function $g_0 : G^k \rightarrow W$ given by the formula $g_0(x) = f(x, 0)$ is multi-Jensen, since according to (7) and (8) it satisfies for $x, y \in G_0^k$

$$\begin{aligned}
 2^{k+1}g_0\left(\frac{x+y}{2}\right) &= 2^{k+1}f\left(\frac{(x, v) + (y, -v)}{2}\right) \\
 &= \sum_{I \subset [k+1]} f([(x, v), (y, -v)]_I) \\
 &= \sum_{J \subset [k]} f([x, y]_J, v) + \sum_{J \subset [k]} f([x, y]_J, -v) \\
 &= \sum_{J \subset [k]} (f([x, y]_J, v) + f([x, y]_J, -v)) \\
 &= \sum_{J \subset [k]} 2f([x, y]_J, 0) \\
 &= 2 \sum_{J \subset [k]} g_0([x, y]_J).
 \end{aligned}$$

It suffices to show that f is a Jensen function with respect to the last variable. Fix $x = (x_1, \dots, x_k) \in G^k$ and define $h_x : G \rightarrow W$ as follows $h_x(y) = f(x, y)$. Let

$$u_i = \begin{cases} v & x_i = 0 \\ \frac{3x_i}{2} & x_i \neq 0 \end{cases}, \quad w_i = \begin{cases} -v & x_i = 0 \\ \frac{x_i}{2} & x_i \neq 0 \end{cases}, \quad i \in \{1, \dots, k\}.$$

Thus $u = (u_1, \dots, u_k), w = (w_1, \dots, w_k) \in G_0^k$ and $x = \frac{u+w}{2}$. We will show that h_x fulfills (4) on G_0 . To this end take $y, z \in G_0$, then $(u, y), (w, z) \in G_0^{k+1}$ and the functions g_y, g_z are multi-Jensen. Therefore

$$\begin{aligned}
2^{k+1}h_x\left(\frac{y+z}{2}\right) &= 2^{k+1}f\left(\frac{(u,y)+(w,z)}{2}\right) \\
&= \sum_{I \subset [k+1]} f([(u,y), (w,z)]_I) \\
&= \sum_{J \subset [k]} f([u,w]_J, y) + \sum_{J \subset [k]} f([u,w]_J, z) \\
&= \sum_{J \subset [k]} g_y([u,w]_J) + \sum_{J \subset [k]} g_z([u,w]_J) \\
&= 2^k g_y\left(\frac{u+w}{2}\right) + 2^k g_z\left(\frac{u+w}{2}\right) \\
&= 2^k (g_y(x) + g_z(x)) \\
&= 2^k (h_x(y) + h_x(z)).
\end{aligned}$$

From what has already been proved in the base step, we conclude that h_x is Jensen, and finally induction completes the proof. $\square$

We are thus led to the following new proof of a refinement of a characterization for multi-additive mappings given in [8].

Proposition 3. *Assume that G is a torsion free group uniquely divisible by 2, $G_0 \neq \emptyset$. A function $f : G^k \rightarrow W$ satisfies the equation*

$$f(x+y) = \sum_{I \subset [k]} f([x, y]_I), \quad (10)$$

for all $x, y \in G_0^k$, if and only if f is a multi-additive mapping.

Proof. First observe that by Theorem 1, every multi-additive mapping of k variables satisfies (10) on G^k . Now assume that (10) is fulfilled for $x, y \in G_0^k$. According to Theorem 1, it suffices to show that it holds on G^k . We begin by proving that f is 2-homogeneous of degree k , namely

$$f(2x) = 2^k f(x) \quad \text{for } x \in G^k. \quad (11)$$

Indeed, if $x \in G_0^k$ we conclude from (10) that

$$f(2x) = f(x+x) = \sum_{I \subset [k]} f([x, x]_I) = 2^k f(x).$$

For any $x = (x_1, \dots, x_k) \in G^k \setminus G_0^k$, fix $v \in G_0$ and define

$$y_i = \begin{cases} v & x_i = 0 \\ x_i & x_i \neq 0 \end{cases}, \quad z_i = \begin{cases} -v & x_i = 0 \\ x_i & x_i \neq 0 \end{cases}, \quad i \in \{1, \dots, k\}.$$

Then $2x = y + z$ with $y = (y_1, \dots, y_k), z = (z_1, \dots, z_k) \in G_0^k$, and

$$\begin{aligned} f(2x) &= f(y + z) = \sum_{I \subset [k]} f([y, z]_I) = \sum_{I \subset [k]} f\left(2\left[\frac{y}{2}, \frac{z}{2}\right]_I\right) \\ &= \sum_{I \subset [k]} 2^k f\left(\left[\frac{y}{2}, \frac{z}{2}\right]_I\right) \\ &= 2^k f\left(\frac{y}{2} + \frac{z}{2}\right) = 2^k f(x). \end{aligned}$$

Since (10) holds on G_0^k and f is 2-homogeneous of degree k , for $x, y \in G_0^k$ we obtain

$$2^k f\left(\frac{x+y}{2}\right) = \sum_{I \subset [k]} 2^k f\left(\left[\frac{x}{2}, \frac{y}{2}\right]_I\right) = \sum_{I \subset [k]} f([x, y]_I).$$

Therefore f is k -Jensen and satisfies (4) on G^k , by Theorems 2 and 1. Finally, applying (4) and (11), for $x, y \in G^k$ we see that

$$f(x + y) = f\left(2\frac{x+y}{2}\right) = 2^k f\left(\frac{x+y}{2}\right) = \sum_{I \subset [k]} f([x, y]_I),$$

which completes the proof. $\square$

We are now in a position to show the second characterization of multi-Cauchy–Jensen mappings.

Theorem 4. Assume that G is a torsion free group uniquely divisible by 2, $G_0 \neq \emptyset$. A function $f : G^n \rightarrow W$ satisfies the Eq. (3) for all $x = (x^1, x^2), y = (y^1, y^2) \in G_0^k \times G_0^{n-k}$ if and only if f is a multi-Cauchy–Jensen mapping.

Proof. It suffices to prove that if a function $f : G^n \rightarrow W$ satisfies the Eq. (3) for all $x = (x^1, x^2), y = (y^1, y^2) \in G_0^k \times G_0^{n-k}$ then f is a multi-Cauchy–Jensen mapping.

By (3) for all $x, y \in G_0^k$ and $z \in G_0^{n-k}$ we have

$$f(x + y, z) = \sum_{I \subset [k]} f([x, y]_I, z). \quad (12)$$

Since for any $z \in G_0^{n-k}$ a mapping $g_z : G^k \rightarrow W$ given by

$$g_z(x) := f(x, z), \quad x \in G^k$$

satisfies the Eq. (10) for all $x, y \in G_0^k$, Lemma 3 shows that the function g_z is multi-additive, which means that

$$f(x + y, z) = \sum_{I \subset [k]} f([x, y]_I, z) \quad (13)$$

for $z \in G_0^{n-k}, x, y \in G^k$.

On the other hand, setting $y = \mathbf{0} := (0, \dots, 0) \in G^k$ in (13) we have

$$f\left(x + \mathbf{0}, \frac{z+w}{2}\right) = \sum_{I \subset [k]} f\left([x, \mathbf{0}]_I, \frac{z+w}{2}\right) = f\left(x, \frac{z+w}{2}\right), \quad (14)$$

for any $x \in G^k$ and $z, w \in G_0^{n-k}$, since $f([x, \mathbf{0}]_I, \frac{z+w}{2}) = 0$ if $I \neq [k]$. Therefore using (3) for all $x \in G^k$ and $z, w \in G_0^{n-k}$ we have

$$2^{n-k} f\left(x, \frac{z+w}{2}\right) = \sum_{I \subset [n-k]} f(x, [z, w]_I). \quad (15)$$

Thus for any $x \in G^k$ the function $h_x : G^{n-k} \rightarrow W$ given by

$$h_x(y) := f(x, y), \quad y \in G^{n-k}$$

satisfies the equation

$$2^{n-k} h_x\left(\frac{z+w}{2}\right) = \sum_{I \subset [n-k]} h_x([z, w]_I),$$

for all $z, w \in G_0^{n-k}$. Lemma 2 shows that the function h_x is multi-Jensen, which means (15) holds for all $x \in G^k$ and $z, w \in G^{n-k}$, and finishes the proof that f is a multi-Cauchy-Jensen mapping. $\square$

3. Stability of Multi-Cauchy-Jensen Mappings on Restricted Domain

In this section we prove stability of Eq. (3) on restricted domain. This result generalizes Theorem 3.2 from [9]. The proof is based on a fixed point result that can be derived from [14] (Theorem 1). To present it we need the following three hypothesis:

- (H1) E is a nonempty set, Y is a Banach space, $f_1, \dots, f_k : E \rightarrow E$ and $L_1, \dots, L_k : E \rightarrow \mathbb{R}_+$ are given.
- (H2) $\mathcal{T} : Y^E \rightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^j L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in Y^E, x \in E.$$

- (H3) $\Lambda : \mathbb{R}_+^E \rightarrow \mathbb{R}_+^E$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^j L_i(x) \delta(f_i(x)), \quad \delta \in \mathbb{R}_+^E, x \in E.$$

For the convenience of the reader, we recall the above mentioned fixed point theorem.

Theorem 5. *Let hypotheses (H1)–(H3) be valid and functions $\varepsilon: E \rightarrow \mathbb{R}_+$ and $\varphi: E \rightarrow Y$ fulfill the following two conditions*

$$\begin{aligned} \|\mathcal{T}\varphi(x) - \varphi(x)\| &\leq \varepsilon(x), & x \in E, \\ \varepsilon^*(x) &:= \sum_{l=0}^{\infty} \Lambda^l \varepsilon(x) < \infty, & x \in E. \end{aligned}$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in E.$$

Moreover

$$\psi(x) := \lim_{l \rightarrow \infty} \mathcal{T}^l \varphi(x), \quad x \in E.$$

In the sequel, we assume that W is a Banach space, $k \geq 1$ and D, E are the nonempty subsets of V such that $E \subset D$ and $x_1 + x_2, \frac{x_1 + x_2}{2} \in D$ for all $x_1, x_2 \in D$.

Write

$$(\Phi f)(x, y) := 2^{n-k} f\left(x^1 + y^1, \frac{x^2 + y^2}{2}\right) - \sum_{I \subset [n]} f([x, y]_I),$$

for $f: D^n \rightarrow W$ and $x = (x^1, x^2), y = (y^1, y^2) \in D^k \times D^{n-k}$.

Theorem 6. *Let $f: D^n \rightarrow W$ and $\theta: E^{2n} \rightarrow \mathbb{R}_+$ be mappings satisfying the inequality*

$$\|(\Phi f)(x, y)\| \leq \theta(x, y), \quad (16)$$

for $x, y \in E^n$. Assume also that there is an $s \in \{-1, 1\}$ such that

$$\varepsilon^*(x) := \frac{1}{2^{n+k} \left(\frac{s-1}{2}\right)} \sum_{l=0}^{\infty} \left(\frac{1}{2^{sk}}\right)^l \theta\left(2^{sl+\frac{s-1}{2}} x^1, x^2, 2^{sl+\frac{s-1}{2}} x^1, x^2\right) < \infty, \quad (17)$$

for $x = (x^1, x^2) \in E^k \times E^{n-k}$,

$$\lim_{l \rightarrow \infty} \left(\frac{1}{2^{sk}}\right)^l \theta(2^{sl} x^1, x^2, 2^{sl} y^1, y^2) = 0, \quad (18)$$

for $x^1, y^1 \in E^k, x^2, y^2 \in E^{n-k}$, and the set E fulfils a condition

$$2^s e \in E, \quad e \in E. \quad (19)$$

Then there exists a unique function $F: E^n \rightarrow W$ satisfying Eq. (3) for all $x = (x^1, x^2), y = (y^1, y^2) \in E^k \times E^{n-k}$ and such that

$$\|f(x) - F(x)\| \leq \varepsilon^*(x), \quad x \in E^n. \quad (20)$$

Proof. If $s = 1$ putting in (16) $x = y = (x^1, x^2) \in E^k \times E^{n-k}$ and then dividing by 2^n we get

$$\left\| \frac{1}{2^k} f(2x^1, x^2) - f(x) \right\| \leq \frac{1}{2^n} \theta(x, x). \quad (21)$$

If $s = -1$ putting in (16) $x = y = \left(\frac{1}{2}x^1, x^2\right) \in E^k \times E^{n-k}$ and then dividing by 2^{n-k} we have

$$\left\| 2^k f\left(\frac{1}{2}x^1, x^2\right) - f(x) \right\| \leq \frac{1}{2^{n-k}} \theta(x, x), \quad (22)$$

Let $s \in \{1, -1\}$. Write

$$\begin{aligned} \mathcal{T}\xi(x) &:= \frac{1}{2^{sk}} \xi(2^s x^1, x^2), \quad \xi \in W^{D^n}, \quad x = (x^1, x^2) \in D^k \times D^{n-k} \\ \varepsilon(x) &:= \begin{cases} \frac{1}{2^n} \theta(x, x) & \text{if } s = 1, \\ \frac{1}{2^{n-k}} \theta\left(\frac{1}{2}x^1, x^2, \frac{1}{2}x^1, x^2\right) & \text{if } s = -1, \end{cases} \quad x = (x^1, x^2) \in E^k \times E^{n-k}. \end{aligned}$$

Then (21) if $s = 1$, (22) if $s = -1$ takes the form

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \quad x \in E^n.$$

Define

$$\Lambda\eta(x) := \frac{1}{2^{sk}} \eta(2^s x^1, x^2), \quad \eta \in \mathbb{R}_+^{E^n}, \quad x = (x^1, x^2) \in E^k \times E^{n-k}.$$

Then it is easily seen that Λ has the form described in (H3) with $j = 1$ and $f_1(x) = (2^s x^1, x^2)$, $L_1(x) = \frac{1}{2^{sk}}$, for $x = (x^1, x^2) \in E^k \times E^{n-k}$. Moreover, for every $\xi, \mu \in W^{D^n}$, $x \in D^n$

$$\begin{aligned} \|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| &= \left\| \frac{1}{2^{sk}} \xi(2^s x^1, x^2) - \frac{1}{2^{sk}} \mu(2^s x^1, x^2) \right\| \\ &\leq L_1(x) \|\xi(f_1(x)) - \mu(f_1(x))\|, \end{aligned}$$

so (H2) is valid.

It is easy to check that for $x = (x^1, x^2) \in E^k \times E^{n-k}$

$$\begin{aligned} \Lambda^l \varepsilon(x) &= \left(\frac{1}{2^{sk}}\right)^l \varepsilon(2^{sl} x^1, x^2) \\ &= \frac{1}{2^{n+k\left(\frac{s-1}{2}\right)}} \left(\frac{1}{2^{sk}}\right)^l \theta\left(2^{sl+\frac{s-1}{2}} x^1, x^2, 2^{sl+\frac{s-1}{2}} x^1, x^2\right). \end{aligned}$$

Hence and from (17), according to Theorem 5 there exists a unique solution $F: E^n \rightarrow Y$

$$F(x) = \frac{1}{2^{sk}} F(2^s x^1, x^2), \quad x = (x^1, x^2) \in E^k \times E^{n-k},$$

such that (20) holds. Moreover,

$$F(x) := \lim_{l \rightarrow \infty} (\mathcal{T}^l f)(x), \quad x \in E^n.$$

One can now show, by induction, that

$$\|\Phi(T^l f)(x, y)\| \leq \left(\frac{1}{2^{sk}}\right)^l \theta(2^{sl} x^1, x^2, 2^{sl} y^1, y^2), \quad (23)$$

for every $l \in \mathbb{N}_0$, $x = (x^1, x^2), y = (y^1, y^2) \in E^k \times E^{n-k}$. Letting $l \rightarrow \infty$ in (23) and using (18) we obtain

$$\Phi(F)(x, y) = 0, \quad x, y \in E^n,$$

which means the function F satisfies Eq. (3) for all $x = (x^1, x^2), y = (y^1, y^2) \in E^k \times E^{n-k}$. Now, we assume that $F' : E^n \rightarrow W$ is another function satisfying the Eq. (3) for all $x = (x^1, x^2), y = (y^1, y^2) \in E^k \times E^{n-k}$ and the inequality (20). Then using (3) and (20), we have for $x = (x^1, x^2) \in E^k \times E^{n-k}$, $l \in \mathbb{N}$

$$\begin{aligned} \|F(x) - F'(x)\| &= \left\| \frac{1}{2^{skl}} F(2^{sl} x^1, x^2) - \frac{1}{2^{skl}} F'(2^{sl} x^1, x^2) \right\| \\ &\leq \frac{1}{2^{skl}} \left(\|F(2^{sl} x^1, x^2) - f(2^{sl} x^1, x^2)\| + \|F'(2^{sl} x^1, x^2) - f(2^{sl} x^1, x^2)\| \right) \\ &\leq \frac{2}{2^{skl}} \varepsilon^*(2^{sl} x^1, x^2) = \frac{2}{2^{n+k} \left(\frac{s-1}{2}\right)} \sum_{j=l}^{\infty} \left(\frac{1}{2^{sk}}\right)^j \theta(2^{sj+\frac{s-1}{2}} x^1, x^2, 2^{sj+\frac{s-1}{2}} x^1, x^2), \end{aligned}$$

whence letting $l \rightarrow \infty$ and using (17) we obtain $F(x) = F'(x)$ for $x \in E^n$, which finishes the proof. $\square$

4. Stability and Hyperstability Results

Applying the above Theorem 6 for specific functions θ and $D = V$, $E = V_0$ yields the following stability results. These results are significant generalizations of some outcomes from [8, 27].

Corollary 7. Assume that $f : V^n \rightarrow W$ is a mapping satisfying the inequality

$$\|(\Phi f)(x, y)\| \leq C \prod_{j=1}^n \|x_j\|^{p_j} \|y_j\|^{q_j}, \quad (24)$$

for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n$, with some $C \in (0, +\infty)$ and $p_j, q_j \in \mathbb{R}$ such that

$$d := \sum_{j=1}^k (p_j + q_j) - k \neq 0. \quad (25)$$

Then there exists a unique k -Cauchy and $n-k$ -Jensen mapping $F : V^n \rightarrow W$ such that

$$\|f(x) - F(x)\| \leq \frac{C}{2^n |1 - 2^d|} \prod_{j=1}^n \|x_j\|^{p_j + q_j}, \quad x = (x_1, \dots, x_n) \in V_0^n.$$

Proof. Put

$$\theta(x, y) := C \prod_{j=1}^n \|x_j\|^{p_j} \|y_j\|^{q_j}, \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n.$$

From (25) we get that $d < 0$ or $d > 0$. Then there exists $s \in \{1, -1\}$ such that $2^{sd} < 1$ ($s = 1$ if $d < 0$, $s = -1$ if $d > 0$), and

$$\sum_{l=0}^{\infty} (2^{sd})^l = \frac{1}{1 - 2^{sd}}.$$

Using Theorem 6, because

$$\varepsilon^*(x) = \frac{C}{2^n |1 - 2^d|} \prod_{j=1}^n \|x_j\|^{p_j + q_j} < +\infty, \quad x = (x_1, \dots, x_n) \in V_0^n,$$

and

$$\lim_{l \rightarrow \infty} C (2^{sd})^l \prod_{j=1}^n \|x_j\|^{p_j} \|y_j\|^{q_j} = 0, \quad x_j, y_j \in V_0, j \in \{1, \dots, n\},$$

we obtain that there exists a unique function $F^* : V_0^n \rightarrow W$ satisfying Eq. (3) for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n$ and such that

$$\|f(x) - F^*(x)\| \leq \varepsilon^*(x), \quad x \in V_0^n.$$

We define the function $F : V^n \rightarrow W$ in the following way for $x = (x^1, x^2)$

$$F(x) := \begin{cases} F^*(x) & \text{if } x \in V_0^n \\ \lim_{l \rightarrow \infty} \frac{1}{2^{skl}} f(2^{sl} x^1, x^2) & \text{otherwise.} \end{cases}$$

Finally, using Theorem 4 we get that F is k -Cauchy and $n-k$ -Jensen mapping. $\square$

Using the above corollary we can obtain the following hyperstability result.

Corollary 8. Let $C > 0$ and $p_j, q_j \in \mathbb{R}$, $j \in \{1, \dots, n\}$ be such that (25) holds and $p_t + q_t < 0$ with some $t \in \{1, \dots, n\}$. If $f : V^n \rightarrow W$ is a function satisfying the condition (24) for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n$, then f is a k -Cauchy and $n-k$ -Jensen mapping.

Proof. According to Corollary 7, there exists a unique k -Cauchy and $n-k$ -Jensen mapping $F : V^n \rightarrow W$ such that

$$\|f(x) - F(x)\| \leq \varphi(x), \quad x \in V_0^n,$$

where $\varphi(x) := \frac{C}{2^n |1 - 2^d|} \prod_{j=1}^n \|x_j\|^{p_j + q_j}$ for $x = (x_1, \dots, x_n)$.

Let $t \in \{1, \dots, n\}$ be such that $p_t + q_t < 0$. Then at least one of p_t, q_t must be negative. Without loss of generality we can assume that $p_t < 0$. Fix $x = (x^1, x^2) \in V_0^k \times V_0^{n-k}$.

If $t \leq k$, define sequences $(z^m), (w^m)$ of vectors

$z^m := (z_1^m, \dots, z_k^m) \in V_0^k$ such that $z_t^m = (m+1)x_t$ and $z_j^m = \frac{1}{2}x_j$ for $j \neq t$,
and

$w^m := (w_1^m, \dots, w_k^m) \in V_0^k$ such that $w_t^m = -mx_t$ and $w_j^m = \frac{1}{2}x_j$ for $j \neq t$.

Then for every $m \in \mathbb{N}$ and $I \subset [k]$

$$z^m + w^m = x^1 \quad \text{and} \quad [z^m, w^m]_I = \begin{cases} z^m & \text{if } t \in I \\ w^m & \text{if } t \notin I. \end{cases}$$

Therefore for $m \in \mathbb{N}$

$$\begin{aligned} (\Phi f)(z^m, x^2, w^m, x^2) &= 2^{n-k} f(z^m + w^m, x^2) - 2^{n-k} \sum_{I \subset [k]} f([z^m, w^m]_I, x^2) \\ &= 2^{n-k} f(x^1, x^2) - 2^{n-k} \cdot 2^{k-1} (f(z^m, x^2) + f(w^m, x^2)) \end{aligned}$$

and consequently

$$f(x) = \frac{1}{2^{n-k}} (\Phi f)(z^m, x^2, w^m, x^2) + 2^{k-1} (f(z^m, x^2) + f(w^m, x^2)). \quad (26)$$

On the other hand, since F is k -additive and $n-k$ -Jensen we have

$$\begin{aligned} 0 &= (\Phi F)(z^m, x^2, w^m, x^2) = 2^{n-k} F(z^m + w^m, x^2) - 2^{n-k} \sum_{I \subset [k]} F([z^m, w^m]_I, x^2) \\ &= 2^{n-k} F(x^1, x^2) - 2^{n-k} \cdot 2^{k-1} (F(z^m, x^2) + F(w^m, x^2)) \end{aligned}$$

thus

$$F(x) = 2^{k-1} (F(z^m, x^2) + F(w^m, x^2)). \quad (27)$$

Then by (26) and (27), for every $m \in \mathbb{N}$

$$\begin{aligned} \|f(x) - F(x)\| &= \left\| \frac{1}{2^{n-k}} \Phi f(z^m, x^2, w^m, x^2) + 2^{k-1} ((f - F)(z^m, x^2) \right. \\ &\quad \left. + (f - F)(w^m, x^2)) \right\| \\ &\leq \frac{C}{2^{n-k}} (m+1)^{p_t} m^{q_t} \left(\frac{1}{2} \right)^{\sum_{j \neq t} (p_j + q_j)} \prod_{j=1}^n \|x_j\|^{p_j + q_j} \\ &\quad + 2^{k-1} (\varphi(z^m, x^2) + \varphi(w^m, x^2)). \end{aligned}$$

Since

$$\lim_{m \rightarrow \infty} (m+1)^{p_t} m^{q_t} \leq \lim_{m \rightarrow \infty} m^{p_t + q_t} = 0,$$

and

$$\lim_{m \rightarrow \infty} \varphi(z^m, x^2) = \lim_{m \rightarrow \infty} \varphi(w^m, x^2) = 0,$$

letting in the above inequality $m \rightarrow \infty$ we obtain that $f(x) = F(x)$.

We now turn to the case $t > k$ and apply similar arguments with sequences defined as follows

$z^m := (z_{k+1}^m, \dots, z_n^m)$ such that $z_t^m = (m+2)x_t$ and $z_j^m = x_j$ for $j \neq t$,

and

$w^m := (w_{k+1}^m, \dots, w_n^m)$ such that $w_t^m = -mx_t$ and $w_j^m = x_j$ for $j \neq t$.

Then for every $m \in \mathbb{N}$ and $I \subset [k]$

$$\frac{z^m + w^m}{2} = x^2 \quad \text{and} \quad [z^m, w^m]_I = \begin{cases} z^m & \text{if } t \in I \\ w^m & \text{if } t \notin I. \end{cases},$$

and therefore for every $m \in \mathbb{N}$

$$\begin{aligned} \|f(x) - F(x)\| &= \left\| \frac{1}{2^{n-k}} \Phi f\left(\frac{1}{2}x^1, z^m, \frac{1}{2}x^1, w^m\right) \right. \\ &\quad \left. + 2^{n-1} \left[(f - F)\left(\frac{1}{2}x^1, z^m\right) + (f - F)\left(\frac{1}{2}x^1, w^m\right) \right] \right\| \\ &\leq \frac{C}{2^{n-k}} (m+2)^{p_t} m^{q_t} \left(\frac{1}{2}\right)^{\sum_{j=1}^k (p_j + q_j)} \prod_{j=1}^n \|x_j\|^{p_j + q_j} \\ &\quad + 2^{k-1} \left[\varphi\left(\frac{1}{2}x^1, z^m\right) + \varphi\left(\frac{1}{2}x^1, w^m\right) \right]. \end{aligned}$$

Since

$$\lim_{m \rightarrow \infty} (m+2)^{p_t} m^{q_t} \leq \lim_{m \rightarrow \infty} m^{p_t + q_t} = 0,$$

and

$$\lim_{m \rightarrow \infty} \varphi\left(\frac{1}{2}x^1, z^m\right) = \lim_{m \rightarrow \infty} \varphi\left(\frac{1}{2}x^1, w^m\right) = 0,$$

letting in the above inequality $m \rightarrow \infty$ we conclude that $f(x) = F(x)$.

It follows that $f = F$ on V_0^n and finally the application of Theorem 4 finishes the proof. $\square$

The following corollary applied to the case of Cauchy–Jensen equation ($n = 2, k = 1$) is a generalization of [27, Th. 2.2].

Corollary 9. Assume that $f : V^n \rightarrow W$ is a mapping satisfying the inequality

$$\|(\Phi f)(x, y)\| \leq \sum_{j=1}^n (A_j \|x_1\|^{p_j} + B_j \|y_j\|^{q_j}) \quad (28)$$

for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n$, $A_j, B_j \in (0, +\infty)$ and $p_j, q_j \in \mathbb{R}$, $j \in \{1, \dots, n\}$ such that $c_j := p_j - k$ and $d_j := q_j - k$ fulfill a condition

$$\forall_{j \in \{1, \dots, k\}} (c_j < 0 \wedge d_j < 0). \quad (29)$$

Then there exists a unique k -Cauchy and $n - k$ -Jensen mapping $F : V^n \rightarrow W$ such that

$$\|f(x) - F(x)\| \leq \varepsilon^*(x), \quad x = (x_1, \dots, x_n) \in V_0^n,$$

where

$$\varepsilon^*(x) = \frac{1}{2^n} \left(\sum_{j=1}^k \left(\frac{A_j}{2^{c_j}-1} \|x_j\|^{p_j} + \frac{B_j}{2^{d_j}-1} \|x_j\|^{q_j} \right) + \frac{1}{1-2^{-k}} \sum_{j=k+1}^n (A_j \|x_j\|^{p_j} + B_j \|x_j\|^{q_j}) \right).$$

Proof. Put

$$\theta(x, y) := \sum_{j=1}^n (A_j \|x_j\|^{p_j} + B_j \|y_j\|^{q_j}) \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n.$$

From (29) we obtain for each $j \in \{1, \dots, k\}$

$$\sum_{l=0}^{\infty} (2^{c_j})^l = \frac{1}{1-2^{c_j}} \quad \text{and} \quad \sum_{l=0}^{\infty} (2^{d_j})^l = \frac{1}{1-2^{d_j}}.$$

Since

$$\varepsilon^*(x) < +\infty, \quad x \in V_0^n,$$

and

$$\lim_{l \rightarrow \infty} \sum_{j=1}^k \left(A_j 2^{c_j l} \|x_j\|^{p_j} + B_j 2^{d_j l} \|y_j\|^{q_j} \right) + \left(\frac{1}{2} \right)^{kl} \sum_{j=k+1}^n (A_j \|x_j\|^{p_j} + B_j \|y_j\|^{q_j}) = 0,$$

for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n$, therefore in the analogous way like in Corollary 7 we obtain the assertion. $\square$

The above statement and analysis similar to that in the proof of [27, Th. 3.1] lead to the following hyperstability result.

Corollary 10. *Let $p_j, q_j < 0$, $A_j, B_j > 0$ for $j \in \{1, \dots, n\}$. If $f : V^n \rightarrow W$ is a function satisfying condition (28) for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V_0^n$, then f is k -Cauchy and $n - k$ -Jensen mapping.*

Proof. According to the above corollary, there exists a unique k -Cauchy and $n - k$ -Jensen mapping $F : V^n \rightarrow W$ such that

$$\|f(x) - F(x)\| \leq \varphi(x), \quad x = (x_1, \dots, x_n) \in V_0^n,$$

where

$$\varphi(x) := \frac{1}{2^n} \left(\sum_{j=1}^k \left(\frac{A_j}{2^{c_j}-1} \|x_j\|^{p_j} + \frac{B_j}{2^{d_j}-1} \|x_j\|^{q_j} \right) + \frac{1}{1-2^{-k}} \sum_{j=k+1}^n (A_j \|x_j\|^{p_j} + B_j \|x_j\|^{q_j}) \right).$$

Observe that for every $m \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in V_0^n$

$$\begin{aligned}
& \|f(x) - F(x)\| \\
&= \frac{1}{2^{n-k}} \|\Phi f((m+1)x^1, (m+2)x^2, -mx^1, -mx^2) \\
&\quad + \sum_{\substack{a_1, \dots, a_k \in \{m+1, -m\} \\ b_{k+1}, \dots, b_n \in \{m+2, -m\}}} (f - F)(a_1x_1, \dots, a_kx_k, b_{k+1}x_{k+1}, \dots, b_nx_n)\| \\
&\leq \frac{1}{2^{n-k}} \left(\sum_{j=1}^k (A_j(m+1)^{p_j} \|x_j\|^{p_j} + B_j m^{q_j} \|x_j\|^{q_j}) \right. \\
&\quad + \sum_{j=k+1}^n (A_j(m+2)^{p_j} \|x_j\|^{p_j} + B_j m^{q_j} \|x_j\|^{q_j}) \\
&\quad \left. + \sum_{\substack{a_1, \dots, a_k \in \{m+1, -m\} \\ b_{k+1}, \dots, b_n \in \{m+2, -m\}}} \varphi(a_1x_1, \dots, a_kx_k, b_{k+1}x_{k+1}, \dots, b_nx_n) \right).
\end{aligned}$$

Since

$$\lim_{m \rightarrow \infty} \varphi(a_1x_1, \dots, a_kx_k, b_{k+1}x_{k+1}, \dots, b_nx_n) = 0,$$

for $a_1, \dots, a_k \in \{m+1, -m\}, b_{k+1}, \dots, b_n \in \{m+2, -m\}$, letting in the above inequality $m \rightarrow \infty$ we conclude that $f = F$ on V_0^n . Now, using Theorem 4 we obtain our claim. $\square$

Acknowledgements

This work was partially supported by the Faculty of Applied Mathematics AGH UST statutory tasks and dean grant within subsidy of Ministry of Science and Higher Education.

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64–66 (1950)
- [2] Bae, J.-H., Park, W.-G.: On the solution of a bi-Jensen functional equation and its stability. Bull. Korean Math. Soc. **43**, 499–507 (2006)

- [3] Bae, J.-H., Park, W.-G.: On the solution of a multi-additive functional equation and its stability. *J. Appl. Math. Comput.* **22**, 517–522 (2006)
- [4] Bae, J.-H., Park, W.-G.: Solution of a vector variable bi-additive functional equation. *Commun. Korean Math. Soc.* **23**, 191–199 (2008)
- [5] Baker, J.A.: The stability of certain functional equations. *Proc. Am. Math. Soc.* **112**, 729–732 (1991)
- [6] Bae, J.-H., Park, W.-G.: Stability of a Cauchy–Jensen functional equation in quasi-Banach spaces. *J. Inequal. Appl. Art. ID 151547*, 9 pp (2010)
- [7] Bae, J.-H., Park, W.-G.: A fixed point approach to the stability of a Cauchy–Jensen functional equation. *Abstr. Appl. Anal. Art. ID 205160*, 10 pp (2012)
- [8] Bahyrycz, A.: On stability and hyperstability of an equation characterizing multi-additive mappings. *Fixed Point Theory* **18**, 445–456 (2017)
- [9] Bahyrycz, A., Ciepliński, K., Olko, J.: On an equation characterizing multi-Cauchy–Jensen mappings and its Hyers–Ulam stability, *Acta Math. Sci. Ser. B Engl. Ed.* **35B**, 1349–1358 (2015)
- [10] Bahyrycz, A., Ciepliński, K., Olko, J.: On an equation characterizing multi-additive-quadratic mappings and its Hyers–Ulam stability. *Appl. Math. Comput.* **265**, 448–455 (2015)
- [11] Bourgin, D.G.: Classes of transformations and bordering transformations. *Bull. Am. Math. Soc.* **57**, 223–237 (1951)
- [12] Brillouët-Belluot, N., Brzdęk, J., Ciepliński, K.: On some recent developments in Ulam’s type stability. *Abstr. Appl. Anal. Art. ID 716936*, 41 pp (2012)
- [13] Brzdęk, J., Cădariu, L., Ciepliński, K.: Fixed point theory and the Ulam stability. *J. Funct. Spaces. Art. ID 829419*, 16 pp (2014)
- [14] Brzdęk, J., Chudziak, J., Páles, Zs: A fixed point approach to stability of functional equations. *Nonlinear Anal.* **74**, 6728–6732 (2011)
- [15] Brzdęk, J., Ciepliński, K.: Remarks on the Hyers–Ulam stability of some systems of functional equations. *Appl. Math. Comput.* **219**, 4096–4105 (2012)
- [16] Brzdęk, J., Ciepliński, K.: Hyperstability and superstability. *Abstr. Appl. Anal. Art. ID 401756*, 13 pp (2013)
- [17] Brzdęk, J., Fechner, W., Moslehian, M.S., Sikorska, J.: Recent developments of the conditional stability of the homomorphism equation. *Banach J. Math. Anal.* **9**(3), 278–326 (2015)
- [18] Ciepliński, K.: On multi-Jensen functions and Jensen difference. *Bull. Korean Math. Soc.* **45**, 729–737 (2008)
- [19] Ciepliński, K.: Stability of the multi-Jensen equation. *J. Math. Anal. Appl.* **363**, 249–254 (2010)
- [20] Ciepliński, K.: Generalized stability of multi-additive mappings. *Appl. Math. Lett.* **23**, 1291–1294 (2010)
- [21] Ciepliński, K.: Applications of fixed point theorems to the Hyers–Ulam stability of functional equations—a survey. *Ann. Funct. Anal.* **3**, 151–164 (2012)
- [22] Găvruta, P.: A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **184**, 431–436 (1994)

- [23] Hyers, D.H.: On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA* **27**, 222–224 (1941)
- [24] Jun, K.-W., Lee, Y.-H.: On the stability of a Cauchy–Jensen functional equation. *II. Dyn. Syst. Appl.* **18**, 407–421 (2009)
- [25] Jun, K.-W., Lee, J.-R., Lee, Y.-H.: On the Hyers–Ulam–Rassias stability of a Cauchy–Jensen functional equation II. *J. Chungcheong Math. Soc.* **21**, 197–208 (2008)
- [26] Jun, K.-W., Lee, Y.-H., Cho, Y.-S.: On the generalized Hyers–Ulam stability of a Cauchy–Jensen functional equation. *Abstr. Appl. Anal. Art. ID 35151*, 15 pp (2007)
- [27] Jun, K.-W., Lee, Y.-H., Cho, Y.-S.: On the stability of a Cauchy–Jensen functional equation. *Commun. Korean Math. Soc.* **23**, 377–386 (2008)
- [28] Jung, S.-M.: *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*. Springer, New York (2011)
- [29] Kannappan, P.: *Functional Equations and Inequalities with Applications*. Springer, New York (2009)
- [30] Kuczma, M.: *An Introduction to the Theory of Functional Equations and Inequalities*. Birkhäuser Verlag, Basel, Cauchy’s equation and Jensen’s inequality (2009)
- [31] Lee, Y.-H.: On the Hyers–Ulam–Rassias stability of a Cauchy–Jensen functional equation. *J. Chungcheong Math. Soc.* **20**, 163–172 (2007)
- [32] Park, W.-G., Bae, J.-H.: On a Cauchy–Jensen functional equation and its stability. *J. Math. Anal. Appl.* **323**, 634–643 (2006)
- [33] Prager, W., Schwaiger, J.: Multi-affine and multi-Jensen functions and their connection with generalized polynomials. *Aequ. Math.* **69**, 41–57 (2005)
- [34] Prager, W., Schwaiger, J.: Stability of the multi-Jensen equation. *Bull. Korean Math. Soc.* **45**, 133–142 (2008)
- [35] Rassias, ThM: On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **72**, 297–300 (1978)
- [36] Schwaiger, J.: Some remarks on the stability of the multi-Jensen equation. *Cent. Eur. J. Math.* **11**, 966–971 (2013)

Anna Bahyrycz and Jolanta Olko
AGH University of Science and Technology
Faculty of Applied Mathematics
Mickiewicza 30
30-059 Kraków
Poland
e-mail: olko@agh.edu.pl

Anna Bahyrycz
e-mail: bahyrycz@agh.edu.pl

Received: October 6, 2017.

Accepted: March 12, 2018.