Results Math 72 (2017), 491–513

1422-6383/17/010491-23
published online March 30, 2017
DOI 10.1007/s00025-017-0676-6

Results in Mathematics

On Para-Complex Affine Hyperspheres

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Abstract. In this paper we introduce a notion of a para-complex affine hypersphere. We give a complete local classification of such hypersurfaces and give several examples. It turns out that every para-complex affine hypersphere can be constructed from (real) affine hyperspheres. As an application, we classify all 2-dimensional para-complex affine hyperspheres.

Mathematics Subject Classification. 53A15, 53D15.

Keywords. Affine hypersphere, para-complex structure, para-complex hypersphere.

1. Introduction

The main motivation for this paper are results obtained by Dillen et al. [1]. In that paper the authors introduce a notion of a complex affine hypersurface and, in particular, a notion of a complex affine hypersphere. Now, it seems to be natural to ask what happens in a para-complex case. Para-complex structures are widely studied by many authors (see e.g. [2–4]). A concept of a para-complex affine immersion as well as a para-complex affine hypersurface was introduced by Schäfer and Lawn [5].

In this paper we introduce a notion of a para-complex affine hypersphere and give a complete local classification of such hypersurfaces. More precisely, we show that every para-complex affine hypersphere can be locally obtained from two real affine hyperspheres. In particular, we can construct several examples of para-complex affine hyperspheres using well know examples of real affine hyperspheres. As an application we provide examples of 1-dimensional

This Research was financed by the Ministry of Science and Higher Education of the Republic of Poland.

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(in a para-complex sense) para-complex affine spheres and show that they are the only 1-dimensional para-complex affine spheres up to a para-complex equiaffine transformation.

In Sect. 2 we briefly recall basic formulas of affine differential geometry and recall the notion of an affine hypersphere. Since para-complex affine hypersufaces are hypersurfaces of a real codimension two, we recall also a concept of an affine hypersurface of codimension two.

In the first part of Sect. 3 we recall some basic concepts related to para-complex geometry (for details we refer to [5–7]). Later, using similar methods like in [1] we introduce a notion of affine normal fields for para-complex affine hypersurfaces and study several basic properties of hypersurfaces equipped with such vector field.

The Sect. 4 contains main results of this paper. In this section we introduce a notion of a para-complex affine hypersphere and prove classification theorems. Especially, we shall show that there is a strict correspondence between real and para-complex affine hyperspheres. We also give several examples.

2. Preliminaries

We briefly recall the basic formulas for affine differential geometry. For more details, we refer to [8]. Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_*Y = f_*(\nabla_X Y) + h(X,Y)C$$

and

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C,$$

where X, Y are tangent vector fields. For any transversal vector field ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called the second fundamental form, S is a tensor of type $(1,1)$, called the shape operator and τ is a 1-form.

In this paper we assume that h is nondegenerate so that h defines a pseudo-Riemannian metric on M. If h is nondegenerate, then we say that the hypersurface or the hypersurface immersion is *nondegenerate*. We have the following

Theorem 2.1. ([8], Fundamental equations) *For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.1}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.2}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.3}$$

$$h(X, SY) - h(SX, Y) = 2d\tau(X, Y). \tag{2.4}$$

The Eqs. (2.1), (2.2), (2.3), and (2.4) are called the equation of Gauss, Codazzi for h, Codazzi for S and Ricci, respectively.

For an affine hypersurface the cubic form Q is defined by the formula

$$Q(X, Y, Z) = (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z). \tag{2.5}$$

It follows from the equation of Codazzi (2.2) that Q is symmetric in all three variables.

For a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$ a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$). For an affine hypersurface $f : M \to \mathbb{R}^{n+1}$ with a transversal vector field C we consider the following volume element on M:

$$\theta(X_1, \ldots, X_n) = \det[f_*X_1, \ldots, f_*X_n, C]$$

for all $X_1, \ldots, X_n \in \mathcal{X}(M)$. We call θ *the induced volume element* on M.

When f is nondegenerate, there exists a canonical transversal vector field C, called the *affine normal* (or the *Blaschke field*). The affine normal is uniquely determined up to sign by the following conditions:

$$\tau = 0 \quad \text{(i.e. C is equiaffine)},$$
$$\omega_h = \theta,$$

where ω_h is defined by $\omega_h(X_1, \ldots, X_n) = |\det[h(X_i, X_j)]|^{1/2}$, where $X_1, \ldots, X_n$ is positively oriented basis relative to the induced volume form θ. The affine immersion f with a Blaschke field C is called a *Blaschke hypersurface*.

A Blaschke hypersurface M is called an *improper affine hypersphere* if $S = 0$. If $S = \lambda\,\mathrm{id}$, where λ is a nonzero constant, then M is called a *proper affine hypersphere*.

Remark 2.1. Sometimes it is convenient to weak the condition $\omega_h = \theta$ and replace it with $\omega_h = c \cdot \theta$, where $c \in \mathbb{R}\backslash\{0\}$. When for some equiaffine vector field ξ we have $\omega_h = c \cdot \theta$ then ξ is proportional to the Blaschke field. Namely we have that $\xi' := \pm|c|^{\frac{2}{n+2}} \cdot \xi$ is the Blaschke field. Note also that if the shape operator is proportional to identity then f (with ξ') is an affine hypersphere. We will often make use of this observation later in this paper.

Let (M, ∇) and $(\widetilde{M}, \widetilde{\nabla})$ be two differential manifolds of dimension n and $n + p$ with torsion-free affine connections ∇ and $\widetilde{\nabla}$ respectively.

An immersion $f : M \to \widetilde{M}$ is called an affine immersion if there exists around each point of M, a field $\mathcal{N}$ of transversal subspaces of dimension p, denoted by $x \mapsto N_x \subset T_{f(x)}(\widetilde{M})$ and such that

$$T_{f(x)}(\widetilde{M}) = f_*(T_xM) + N_x \tag{2.6}$$

holds and, for all vector fields X and Y on M, we have a decomposition

$$\widetilde{\nabla}_X f_*Y = f_*\nabla_X Y + \alpha(X, Y), \tag{2.7}$$

where $\nabla_X Y \in T_xM$ and $\alpha(X,Y) \in N_x$ at each point x. We call N_x the *transversal space* and α the *affine fundamental form*. If ξ is a vector field with values in $\mathcal{N}$, $\xi_x \in N_x$, then we write

$$\widetilde{\nabla}_X \xi = -f_* S_\xi X + \nabla^\perp_X \xi, \tag{2.8}$$

where $S_\xi X \in T_xM$ and $\nabla^\perp_X \xi \in N_x$ at each point x. We call S_ξ the *shape operator* for ξ, and $\nabla^\perp$ the *normal connection*.

Now, let $\widetilde{M} = \mathbb{R}^{n+2}$ and $\widetilde{\nabla} = D$ be the ordinary flat connection on $\mathbb{R}^{n+2}$. Let $f\colon M \to \mathbb{R}^{n+2}$ be an immersion, and $\mathcal{N}\colon M \ni x \mapsto N_x$ be a transversal bundle for the immersion f. Immersion f together with the transversal bundle $\mathcal{N}$ we call an *affine hypersurface of codimension two*. For any local basis $\{\xi_1, \xi_2\}$ of $\mathcal{N}$, we can write

$$D_X f_* Y = f_*(\nabla_X Y) + h_1(X,Y)\xi_1 + h_2(X,Y)\xi_2, \tag{2.9}$$

$$D_X \xi_1 = -f_*(S_1 X) + \tau_{11}(X)\xi_1 + \tau_{12}(X)\xi_2 \tag{2.10}$$

$$D_X \xi_2 = -f_*(S_2 X) + \tau_{21}(X)\xi_1 + \tau_{22}(X)\xi_2. \tag{2.11}$$

Then ∇ is a torsion-free affine connection on M, which depends only on $\mathcal{N}$ and not on the choice of local basis $\{\xi_1, \xi_2\}$. We call it the affine connection induced by $\mathcal{N}$. The other objects h_i, S_i, τ_{ij}, $i,j \in \{1,2\}$, are respectively the *affine fundamental forms*, the *shape operators* and the *normal connection forms*.

3. Para-Complex Affine Hypersurfaces

Fist we recall some basic concepts related to para-complex geometry. For details see [6,7] and [5].

A *para-complex structure* on a real finite dimensional vector space V is an endomorphism $\widetilde{J} \in \mathrm{End}(V)$, such that $\widetilde{J}^2 = \mathrm{id}$ and the two eigenspaces $V^\pm := \ker(\mathrm{id} \mp \widetilde{J})$ of $\widetilde{J}$ have the same dimension. An *almost para-complex structure* on a smooth manifold M is a (1,1)-tensor $\widetilde{J}$ on M such that, for all $p \in M$, $\widetilde{J}_p$ is a para-complex structure on T_pM. An almost para-complex structure $\widetilde{J}$ on M is called *integrable* if the distributions $D^\pm := \ker(\mathrm{id} \mp \widetilde{J})$ are integrable. An integrable almost para-complex structure on M is called a *para-complex structure* and a manifold M endowed with a para-complex structure is called a *para-complex manifold*.

Lemma 3.1. [7] *An almost para-complex structure $\widetilde{J}$ is integrable if and only if $N_{\widetilde{J}} = 0$, where $N_{\widetilde{J}}$ is the Nijenhuis tensor for $\widetilde{J}$.*

Let us denote by $\widetilde{\mathbb{C}}$ the real algebra of para-complex numbers, which is generated by 1 and the para-complex unit e ($e^2 = 1$). For every $z = x + ey \in \widetilde{\mathbb{C}}$ we have the para-complex conjugation $\overline{x + ey} := x - ey$ and the real and imaginary parts of z: $\Re(z) := x$ and $\Im(z) := y$. The free $\widetilde{\mathbb{C}}$-module $\widetilde{\mathbb{C}}^n$ is a para-complex vector space, where the para-complex structure is just the multiplication by e. The para-complex conjugation extends componentwise to $\widetilde{\mathbb{C}}^n$. The para-complex dimension of a para-complex manifold M is the integer $n = \dim_{\widetilde{\mathbb{C}}} M := \frac{\dim M}{2}$.

Let $(M, \widetilde{J}_M)$ and $(N, \widetilde{J}_N)$ be para-complex manifolds. A smooth function $f\colon (M, \widetilde{J}_M) \to (N, \widetilde{J}_N)$ is called *para-holomorphic* if $df \circ \widetilde{J}_M = \widetilde{J}_N \circ df$. A para-holomorphic map $f\colon (M, \widetilde{J}_M) \to \widetilde{\mathbb{C}}$ is called a *para-holomorphic function.*

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be an immersion and let $\widetilde{J}$ be the standard para-complex structure on $\mathbb{R}^{2n+2}$. That is

$$\widetilde{J}(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) := (y_1, \ldots, y_{n+1}, x_1, \ldots, x_{n+1}).$$

We always identify $(\mathbb{R}^{2n+2}, \widetilde{J})$ with $\widetilde{\mathbb{C}}^{n+1}$.

Assume now that $g_*(TM)$ is $\widetilde{J}$-invariant and $\widetilde{J}|_{g_*(T_xM)}$ is a para-complex structure on $g_*(T_xM)$ for every $x \in M$. Then $\widetilde{J}$ induces an almost para-complex structure on M, which we will also denote by $\widetilde{J}$. Moreover, since $(\mathbb{R}^{2n+2}, \widetilde{J})$ is para-complex then $(M, \widetilde{J})$ is para-complex as well. By assumption we have that $dg \circ \widetilde{J} = \widetilde{J} \circ dg$ that is $g\colon M^{2n} \to \mathbb{R}^{2n+2} \cong \widetilde{\mathbb{C}}^{n+1}$ is a para-holomorphic immersion. Since para-complex dimension of M is n, immersion g is called a *para-holomorphic hypersurface.*

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be an affine hypersurface of codimension 2 with a transversal bundle $\mathcal{N}$. If g is para-holomorphic then it is called *affine para-holomorphic hypersurface.* If additionally the transversal bundle $\mathcal{N}$ is $\widetilde{J}$-invariant then g is called a *para-complex affine hypersurface.*

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. We say that g is *para-complex centro-affine hypersurface* if $\{g, \widetilde{J}g\}$ is a transversal bundle for g.

Now, let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. Then for every $x \in M$ there exists a neighborhood U of x and a transversal vector field $\zeta\colon U \to \mathbb{R}^{2n+2}$ such that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for $g|_U$. That is $g|_U$ considered with $\{\zeta, \widetilde{J}\zeta\}$ is a para-complex affine hypersurface. Indeed, let N_x be any vector space transversal to $g_*(T_xM)$. If N_x is $\widetilde{J}$-invariant then it must be a para-complex vector space, so we can find vector $v \in N_x$ such that $\{v, \widetilde{J}v\}$ is a basis for N_x. If N_x is not $\widetilde{J}$-invariant then $N_x \cap \widetilde{J}N_x$ must be 1-dimensional. In this case we can choose $v \in N_x$ such that $v \notin N_x \cap \widetilde{J}N_x$. Now vector $\widetilde{J}v$ is transversal to $g_*(T_xM)$ and linearly independent with v. That is $\{v, \widetilde{J}v\}$ is a para-complex transversal vector space to $g_*(T_xM)$. Summarizing at x we can always find a transversal vector v such that $g_*(T_xM) \oplus \operatorname{span}\{v, \widetilde{J}v\} = \mathbb{R}^{2n+2}$. Hence, in a neighborhood of x we can find a transversal vector field ζ such that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g in this neighborhood.

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface and let $\zeta\colon U \to \mathbb{R}^{2n+2}$ be a local transversal vector field on $U \subset M$ such that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle to g. So for all tangent vector fields $X, Y \in \mathcal{X}(U)$ we can decompose $D_X Y$ and $D_X\zeta$ into tangent and transversal part. So we have

$$\mathrm{D}_X\, g_*Y = g_*(\nabla_X Y) + h_1(X,Y)\zeta + h_2(X,Y)\widetilde{J}\zeta \quad \text{(formula of Gauss)}$$

$$\mathrm{D}_X\, \zeta = -g_*(SX) + \tau_1(X)\zeta + \tau_2(X)\widetilde{J}\zeta \quad \text{(formula of Weingarten)}$$

where ∇ is a torsion free affine connection on U, h_1 and h_2 are symmetric bilinear forms on U, S is a $(1,1)$-tensor field on U and τ_1 and τ_2 are 1-forms on U.

Using the fact that $D\widetilde{J} = 0$ and the formula of Gauss by straightforward computations we can prove the following

Lemma 3.2. [5]

$$\nabla\widetilde{J} = 0, \tag{3.1}$$

$$h_1(X,\widetilde{J}Y) = h_1(\widetilde{J}X,Y) = h_2(X,Y), \tag{3.2}$$

$$h_2(X,\widetilde{J}Y) = h_1(X,Y). \tag{3.3}$$

We say that a hypersurface is *nondegenerate* if h_1 (and in consequence h_2) is nondegenerate.

Lemma 3.3. *Let $g\colon M \to \mathbb{R}^{2n+2}$ be a para-complex affine hypersurface with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$. Then the induced connection ∇, the affine fundamental forms h_1, h_2, the shape operator S and the transversal connection forms τ_1, τ_2 satisfy the following equations:*

$$\begin{aligned} R(X,Y)Z &= h_1(Y,Z)SX + h_2(Y,Z)\widetilde{J}(SX) \\ &\quad - h_1(X,Z)SY - h_2(X,Z)\widetilde{J}(SY), \end{aligned} \tag{3.4}$$

$$\begin{aligned} (\nabla_X h_1)(Y,Z) - (\nabla_Y h_1)(X,Z) &= \tau_1(Y)h_1(X,Z) + \tau_2(Y)h_2(X,Z) \\ &\quad - \tau_1(X)h_1(Y,Z) - \tau_2(X)h_2(Y,Z), \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\nabla_X h_2)(Y,Z) - (\nabla_Y h_2)(X,Z) &= \tau_1(Y)h_2(X,Z) + \tau_2(Y)h_1(X,Z) \\ &\quad - \tau_1(X)h_2(Y,Z) - \tau_2(X)h_1(Y,Z), \end{aligned} \tag{3.6}$$

$$\begin{aligned} (\nabla_X S)(Y) - (\nabla_Y S)(X) &= \tau_1(X)SY + \tau_2(X)\widetilde{J}(SY) \\ &\quad - \tau_1(Y)SX - \tau_2(Y)\widetilde{J}(SX), \end{aligned} \tag{3.7}$$

$$h_1(X,SY) - h_1(SX,Y) = 2d\tau_1(X,Y), \tag{3.8}$$

$$h_2(X,SY) - h_2(SX,Y) = 2d\tau_2(X,Y). \tag{3.9}$$

Assume now that $\{\widetilde{\zeta}, \widetilde{J}\widetilde{\zeta}\}$ is any other transversal bundle on U. Then there exist functions φ, ψ on U and $Z \in \mathcal{X}(U)$ such that

$$\widetilde{\zeta} = \varphi\zeta + \psi\widetilde{J}\zeta + g_*Z.$$

Since $\{\widetilde{\zeta}, \widetilde{J}\widetilde{\zeta}\}$ is transversal the above formula implies that $\varphi^2 - \psi^2 \neq 0$. Indeed, we have

$$\varphi\widetilde{\zeta} - \psi\widetilde{J}\widetilde{\zeta} = (\varphi^2 - \psi^2)\zeta + \varphi g_*Z - \psi\widetilde{J}g_*Z.$$

If $\varphi^2 - \psi^2 = 0$ then $\varphi\widetilde{\zeta} - \psi\widetilde{J\zeta} \in TU$, but since $\{\widetilde{\zeta}, \widetilde{J\zeta}\}$ is transversal we obtain $\varphi = \psi = 0$, what is impossible because $\widetilde{\zeta}$ is transversal.

By the formulas of Gauss and Weingarten with respect to $\widetilde{\zeta}$ we obtain the objects $\widetilde{\nabla}$, $\widetilde{h_1}$, $\widetilde{h_2}$, $\widetilde{S}$, $\widetilde{\tau}_1$, $\widetilde{\tau}_2$ which satisfy the following relations

Lemma 3.4.

$$h_1(X,Y) = \varphi\widetilde{h_1}(X,Y) + \psi\widetilde{h_2}(X,Y), \tag{3.10}$$

$$h_2(X,Y) = \psi\widetilde{h_1}(X,Y) + \varphi\widetilde{h_2}(X,Y), \tag{3.11}$$

$$\nabla_X Y = \widetilde{\nabla}_X Y + \widetilde{h_1}(X,Y)Z + \widetilde{h_2}(X,Y)\widetilde{J}Z, \tag{3.12}$$

$$-\varphi SX - \psi SX + \nabla_X Z = -\widetilde{S}X + \widetilde{\tau}_1(X)Z + \widetilde{\tau}_2(X)\widetilde{J}Z, \tag{3.13}$$

$$X(\varphi) + \varphi\tau_1(X) + \psi\tau_2(X) + h_1(X,Z) = \varphi\widetilde{\tau}_1(X) + \psi\widetilde{\tau}_2(X), \tag{3.14}$$

$$\varphi\tau_2(X) + X(\psi) + \psi\tau_1(X) + h_2(X,Z) = \psi\widetilde{\tau}_1(X) + \varphi\widetilde{\tau}_2(X), \tag{3.15}$$

$$\widetilde{h_1} = \frac{h_1\varphi - h_2\psi}{\varphi^2 - \psi^2}, \tag{3.16}$$

$$\widetilde{\tau}_1(X) = \frac{1}{2}X(\ln|\varphi^2 - \psi^2|) + \tau_1(X) + \frac{1}{\varphi^2 - \psi^2}(\varphi h_1(X,Z) - \psi h_2(X,Z)). \tag{3.17}$$

Proof. Formulas (3.10)–(3.15) are straightforward. Formulas (3.16) and (3.17) follow at once from (3.10), (3.11), (3.14) and (3.15). □

On U we define the volume form θ_ζ by the formula

$$\theta_\zeta(X_1, \ldots, X_{2n}) := \det(g_*X_1, \ldots, g_*X_{2n}, \zeta, \widetilde{J}\zeta)$$

for tangent vectors X_i, $i = 1, \ldots, 2n$. Then, consider the function H_ζ on U defined by

$$H_\zeta := \det[h_1(X_i, X_j)]_{i,j=1\ldots 2n}$$

where $X_1, \ldots, X_{2n}$ is a local basis in TU such that $\theta_\zeta(X_1, \ldots, X_{2n}) = 1$. This definition is independent of the choice of basis. It is easy to see that ∇, θ_ζ and τ_1 are related by the following formula:

$$\nabla_X \theta_\zeta = 2\tau_1(X)\theta_\zeta. \tag{3.18}$$

If $\{\widetilde{\zeta}, \widetilde{J\zeta}\}$ is other transversal bundle on U then we have the following relations between $\theta_{\widetilde{\zeta}}$, $H_{\widetilde{\zeta}}$ and θ_ζ, H_ζ

Lemma 3.5.

$$\theta_{\widetilde{\zeta}} = (\varphi^2 - \psi^2)\theta_\zeta, \tag{3.19}$$

$$H_{\widetilde{\zeta}} = \frac{1}{(\varphi^2 - \psi^2)^{n+2}} \cdot H_\zeta. \tag{3.20}$$

Proof. Since Formula (3.19) is straightforward it is enough to prove (3.20). Let $\{X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n\}$ be a local basis on TM. Then

$$\theta_\zeta(X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n) = \alpha$$

where $\alpha \neq 0$ (either $\alpha < 0$ or $\alpha > 0$). Now let $\widetilde{X_1} := \frac{X_1}{\sqrt{|\alpha|}}$ then

$$\theta_\zeta(\widetilde{X_1}, \widetilde{J}\widetilde{X_1}, X_2, \widetilde{J}X_2, \ldots, X_n, \widetilde{J}X_n) = \frac{\alpha}{|\alpha|}.$$

It follows that we can choose the basis $\{X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n\}$ such that

$$\theta_\zeta(X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n) = \pm 1.$$

Let $Y_i = \frac{X_i}{|\varphi^2-\psi^2|^{\frac{1}{2n}}}$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}
\theta_{\widetilde{\zeta}}(Y_1, \ldots, \widetilde{J}Y_n) &= (\varphi^2 - \psi^2)\theta_\zeta(Y_1, \ldots, \widetilde{J}Y_n) \\
&= (\varphi^2 - \psi^2) \cdot \frac{1}{|\varphi^2 - \psi^2|}\theta_\zeta(X_1, \ldots, \widetilde{J}X_n) \\
&= sgn(\varphi^2 - \psi^2)\theta_\zeta(X_1, \ldots, \widetilde{J}X_n) = \pm 1,
\end{aligned}$$

and in consequence

$$\begin{aligned}
H_{\widetilde{\zeta}} &= \det \begin{bmatrix} \widetilde{h_1}(Y_1, Y_1) & \widetilde{h_1}(Y_1, \widetilde{J}Y_1) & \cdots & \widetilde{h_1}(Y_1, \widetilde{J}Y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{h_1}(\widetilde{J}Y_n, Y_1) & \widetilde{h_1}(\widetilde{J}Y_n, \widetilde{J}Y_1) & \cdots & \widetilde{h_1}(\widetilde{J}Y_n, \widetilde{J}Y_n) \end{bmatrix} \\
&= \frac{1}{(\varphi^2 - \psi^2)^2} \det \begin{bmatrix} \widetilde{h_1}(X_1, X_1) & \widetilde{h_1}(X_1, \widetilde{J}X_1) & \cdots & \widetilde{h_1}(X_1, \widetilde{J}X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{h_1}(\widetilde{J}X_n, X_1) & \widetilde{h_1}(\widetilde{J}X_n, \widetilde{J}X_1) & \cdots & \widetilde{h_1}(\widetilde{J}X_n, \widetilde{J}X_n) \end{bmatrix}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\det \begin{bmatrix} \widetilde{h_1}(X_k, X_l) & \widetilde{h_1}(X_k, \widetilde{J}X_l) \\ \widetilde{h_1}(X_m, X_l) & \widetilde{h_1}(X_m, \widetilde{J}X_l) \end{bmatrix} \\
&= \frac{1}{(\varphi^2 - \psi^2)^2} \det \begin{bmatrix} \varphi h_1(X_k, X_l) - \psi h_2(X_k, X_l) & \varphi h_1(X_k, \widetilde{J}X_l) - \psi h_2(X_k, \widetilde{J}X_l) \\ \varphi h_1(X_m, X_l) - \psi h_2(X_m, X_l) & \varphi h_1(X_m, \widetilde{J}X_l) - \psi h_2(X_m, \widetilde{J}X_l) \end{bmatrix} \\
&= \frac{1}{(\varphi^2 - \psi^2)^2} \det \begin{bmatrix} \varphi h_1(X_k, X_l) - \psi h_1(X_k, \widetilde{J}X_l) & \varphi h_1(X_k, \widetilde{J}X_l) - \psi h_1(X_k, X_l) \\ \varphi h_1(X_m, X_l) - \psi h_1(X_m, \widetilde{J}X_l) & \varphi h_1(X_m, \widetilde{J}X_l) - \psi h_1(X_m, X_l) \end{bmatrix} \\
&= \frac{1}{(\varphi^2 - \psi^2)^2} \det \begin{bmatrix} \varphi h_1(X_k, X_l) & \varphi h_1(X_k, \widetilde{J}X_l) \\ \varphi h_1(X_m, X_l) & \varphi h_1(X_m, \widetilde{J}X_l) \end{bmatrix} \\
&\quad + \frac{1}{(\varphi^2 - \psi^2)^2} \det \begin{bmatrix} -\psi h_1(X_k, \widetilde{J}X_l) & -\psi h_1(X_k, X_l) \\ -\psi h_1(X_m, \widetilde{J}X_l) & -\psi h_1(X_m, X_l) \end{bmatrix} \\
&= \frac{\varphi^2 - \psi^2}{(\varphi^2 - \psi^2)^2} \det \begin{bmatrix} h_1(X_k, X_l) & h_1(X_k, \widetilde{J}X_l) \\ h_1(X_m, X_l) & h_1(X_m, \widetilde{J}X_l) \end{bmatrix}.
\end{aligned}$$

Now we obtain

$$\det\begin{bmatrix}\widetilde{h_1}(X_k,X_l) & \widetilde{h_1}(X_k,\widetilde{J}X_l)\\ \widetilde{h_1}(X_m,X_l) & \widetilde{h_1}(X_m,\widetilde{J}X_l)\end{bmatrix} = \frac{1}{\varphi^2-\psi^2}\det\begin{bmatrix}h_1(X_k,X_l) & h_1(X_k,\widetilde{J}X_l)\\ h_1(X_m,X_l) & h_1(X_m,\widetilde{J}X_l)\end{bmatrix}.$$

The above implies that

$$H_{\widetilde{\zeta}} = \frac{1}{(\varphi^2-\psi^2)^2}\cdot\frac{1}{(\varphi^2-\psi^2)^n}\cdot H_\zeta$$

and eventually

$$H_{\widetilde{\zeta}} = \frac{1}{(\varphi^2-\psi^2)^{n+2}}\cdot H_\zeta.$$

□

When g is nondegenerate there exist transversal vector fields ζ satisfying the following two conditions:

$$|H_\zeta| = 1,$$
$$\tau_1 = 0.$$

Such vector fields are called *affine normal vector fields*. The first condition is a kind of normalization and the second condition implies that $\nabla\theta_\zeta = 0$ [see (3.18) formula].

Indeed, let $\{\zeta, \widetilde{J}\zeta\}$ be an arbitrary transversal bundle for g. Since g is nondegenerate we have $H_\zeta \neq 0$, so we can find functions φ and ψ such that $\varphi^2-\psi^2 \neq 0$ and

$$|(\varphi^2-\psi^2)^{n+2}| = |H_\zeta|. \tag{3.21}$$

Let $\widetilde{\zeta} := \varphi\zeta + \psi\zeta + Z$ where Z is an arbitrary vector field on M. Lemma 3.5 (Formula (3.20)] and (3.21) imply that $|H_{\widetilde{\zeta}}| = 1$. We shall show that we can choose Z in such a way that $\widetilde{\zeta}$ is an affine normal vector field.

By Lemma 3.4 [Formula (3.17)] we have

$$\widetilde{\tau_1}(X) = \frac{1}{2}X(\ln|\varphi^2-\psi^2|) + \tau_1(X) + \frac{1}{\varphi^2-\psi^2}(\varphi h_1(X,Z) - \psi h_2(X,Z))$$

Now using Lemma 3.2 we obtain

$$\widetilde{\tau_1}(X) = \frac{1}{2}X(\ln|\varphi^2-\psi^2|) + \tau_1(X) + \frac{1}{\varphi^2-\psi^2}\cdot h_1(X,\varphi Z - \psi\widetilde{J}Z).$$

Since h_1 is nondegenerate we can find Z such that $\widetilde{\tau_1}(X) = 0$ for all vector fields X defined on U. In this way we have shown that on every para-holomorphic hypersurface one may find (at least locally) an affine normal vector field.

Lemma 3.6. *Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a nondegenerate para-holomorphic hypersurface and let $\zeta, \widetilde{\zeta}\colon U \to \mathbb{R}^{2n+2}$ be two affine normal vector fields on $U \subset M$. Then $\widetilde{\zeta} = \varphi\zeta + \psi\widetilde{J}\zeta$, where $|\varphi^2-\psi^2| = 1$.*

Proof. Since $\zeta, \widetilde{\zeta}$ are transversal there exist functions $\varphi, \psi \in C^\infty(U)$ and a tangent vector field $Z \in \mathcal{X}(U)$ such that $\widetilde{\zeta} = \varphi\zeta + \psi\widetilde{J}\zeta + Z$. Since $|H_\zeta| = |H_{\widetilde{\zeta}}| = 1$ the Formula (3.20) implies that $|\varphi^2 - \psi^2| = 1$. Now, due to the fact that $\tau_1 = \widetilde{\tau}_1 = 0$ and by Formula (3.17) and Lemma 3.2 we obtain

$$0 = \varphi h_1(X, Z) - \psi h_2(X, Z) = \varphi h_1(X, Z) - \psi h_1(X, \widetilde{J}Z) = h_1(X, \varphi Z - \psi\widetilde{J}Z)$$

for all $X \in \mathcal{X}(U)$. Since h_1 is nondegenerate and $\varphi^2 - \psi^2 \neq 0$ the last formula implies that $Z = 0$. The proof is completed. □

Lemma 3.7. *Let $g \colon M \to \mathbb{R}^{2n+2}$ be a para-complex affine hypersurface with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$. Then for each point $x \in M$ there exists a local coordinate system $x_1, \ldots, x_n, y_1, \ldots, y_n$ with origin at x such that $\partial_{x_1}, \ldots, \partial_{x_n}$ and $\partial_{y_1}, \ldots, \partial_{y_n}$ are local bases for D^+ and D^- respectively and*

$$h_1(\partial_{x_i}, \partial_{y_j}) = 0, \tag{3.22}$$

$$h_2(\partial_{x_i}, \partial_{y_j}) = 0, \tag{3.23}$$

$$\nabla_{\partial_{x_i}}\partial_{y_j} = \nabla_{\partial_{y_j}}\partial_{x_i} = 0, \tag{3.24}$$

$$\nabla_{\partial_{x_i}}\partial_{x_j} \in \mathrm{D}^+, \tag{3.25}$$

$$\nabla_{\partial_{y_i}}\partial_{y_j} \in \mathrm{D}^- \tag{3.26}$$

for $i, j = 1, \ldots, n$.

Proof. Since D^+ and D^- are involutive and $\mathrm{D}^+ \oplus \mathrm{D}^- = TM$ using lemma about direct product of involutive distributions (see Prop. 5.2, p. 182 in [9]) we have that for each $x \in M$ there exists a neighbourhood U of x and a local coordinate system $x_1, \ldots, x_n, y_1, \ldots, y_n$ on U such that $\partial_{x_i} \in \mathrm{D}^+$, $\partial_{y_i} \in \mathrm{D}^-$ for $i = 1, \ldots, n$. Lemma 3.2 implies that

$$h_1(\partial_{x_i}, \widetilde{J}\partial_{y_j}) = h_1(\widetilde{J}\partial_{x_i}, \partial_{y_j}).$$

Since $\widetilde{J}\partial_{x_i} = \partial_{x_i}$ and $\widetilde{J}\partial_{y_j} = -\partial_{y_j}$ we have $h_1(\widetilde{J}\partial_{x_i}, \partial_{y_j}) = h_1(\partial_{x_i}, \partial_{y_j})$ that is $h_1(\partial_{x_i}, \partial_{y_j}) = 0$ for $i, j = 1, \ldots, n$. As an immediate consequence we get that $h_2(\partial_{x_i}, \partial_{y_j}) = 0$ for $i, j = 1, \ldots, n$ as well.

From (3.1) we obtain

$$-\nabla_{\partial_{x_i}}\partial_{y_j} = \nabla_{\partial_{x_i}}\widetilde{J}\partial_{y_j} = \widetilde{J}(\nabla_{\partial_{x_i}}\partial_{y_j})$$

and

$$\nabla_{\partial_{y_j}}\partial_{x_i} = \nabla_{\partial_{y_j}}\widetilde{J}\partial_{x_i} = \widetilde{J}(\nabla_{\partial_{y_j}}\partial_{x_i}),$$

so $\nabla_{\partial_{x_i}}\partial_{y_j} \in \mathrm{D}^-$ and $\nabla_{\partial_{y_j}}\partial_{x_i} \in \mathrm{D}^+$. Since ∇ is torsion free we also have $\nabla_{\partial_{x_i}}\partial_{y_j} = \nabla_{\partial_{y_j}}\partial_{x_i}$ that is

$$\nabla_{\partial_{x_i}}\partial_{y_j} = \nabla_{\partial_{y_j}}\partial_{x_i} = 0.$$

Using again Formula (3.1) we get

$$\nabla_{\partial_{x_i}}\partial_{x_j} = \nabla_{\partial_{x_i}}\widetilde{J}\partial_{x_j} = \widetilde{J}(\nabla_{\partial_{x_i}}\partial_{x_j})$$

and

$$-\nabla_{\partial_{y_i}}\partial_{y_j} = \nabla_{\partial_{y_i}}\widetilde{J}\partial_{y_j} = \widetilde{J}(\nabla_{\partial_{y_i}}\partial_{y_j})$$

that is $\nabla_{\partial_{x_i}}\partial_{x_j} \in \mathrm{D}^+$ and $\nabla_{\partial_{y_i}}\partial_{y_j} \in \mathrm{D}^-$ for $i, j = 1, \ldots, n$. The proof is completed. $\square$

As an immediate consequence of the above lemma we obtain

Corollary 3.1. *Let $g\colon M \to \mathbb{R}^{2n+2}$ be a para-complex affine hypersurface with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$. Then for each $X \in \mathrm{D}^+$, $Y \in \mathrm{D}^-$ we have*

1. *$h_i(X, Y) = 0$ for $i = 1, 2$;*
2. *Distributions D^+ and D^- are ∇ parallel. That is for every $Z \in \mathcal{X}(M)$ we have $\nabla_Z X \in \mathrm{D}^+$ and $\nabla_Z Y \in \mathrm{D}^-$.*

Lemma 3.8. *Let $g\colon M \to \mathbb{R}^{2n+2}$ be a para-complex affine hypersurface with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$. Then for each point $x \in M$ there exists a local coordinate system $x_1, \ldots, x_n, y_1, \ldots, y_n$ with origin at x such that g can be locally expressed in the form*

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n) = A(x_1, \ldots, x_n) + B(y_1, \ldots, y_n),$$

where

$$A\colon U_1 \ni (x_1, \ldots, x_n) \mapsto A(x_1, \ldots, x_n) \in \mathbb{R}^{2n+2}$$

and

$$B\colon U_2 \ni (y_1, \ldots, y_n) \mapsto B(y_1, \ldots, y_n) \in \mathbb{R}^{2n+2}$$

are smooth immersions from open subsets $U_1, U_2 \subset \mathbb{R}^n$. Moreover $\widetilde{J}A = A$ and $\widetilde{J}B = -B$.

Proof. Let $x \in M$ and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be a local coordinate system from Lemma 3.7. By formula of Gauss we have

$$g_{x_i y_j} = D_{\partial_{x_i}} g_* \partial_{y_j} = g_* \nabla_{\partial_{x_i}}\partial_{y_j} + h_1(\partial_{x_i}, \partial_{y_j})\zeta + h_2(\partial_{x_i}, \partial_{y_j})\widetilde{J}\zeta.$$

Now (3.22)–(3.24) imply that $g_{x_i y_j} = 0$ for $i, j = 1, \ldots, n$. Solving this system of partial differential equations we immediately get that there exist open subsets $U_1, U_2 \subset \mathbb{R}^n$ and smooth functions $\overline{A}\colon U_1 \to \mathbb{R}^{2n+2}$, $\overline{B}\colon U_2 \to \mathbb{R}^{2n+2}$ such that

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n) = \overline{A}(x_1, \ldots, x_n) + \overline{B}(y_1, \ldots, y_n)$$

for $(x_1, \ldots, x_n) \in U_1$ and $(y_1, \ldots, y_n) \in U_2$. Since g is an immersion it is obvious that both $\overline{A}$ and $\overline{B}$ are immersions too. To prove the last part of the lemma it is enough to note that since g is para-holomorphic we have $\overline{A}_{x_i} = g_*(\partial_{x_i}) = \widetilde{J}g_*(\partial_{x_i}) = \widetilde{J}\overline{A}_{x_i}$ and $-\overline{B}_{y_i} = -g_*(\partial_{y_i}) = \widetilde{J}g_*(\partial_{y_i}) = \widetilde{J}\overline{B}_{y_i}$ for $i = 1, \ldots, n$. That is there exist constants $C_1, C_2 \in \mathbb{R}^{2n+2}$ such that $\widetilde{J}\overline{A} = \overline{A} + C_1$ and $\widetilde{J}\overline{B} = -\overline{B} + C_2$. Note that $\widetilde{J}C_1 = -C_1$ and $\widetilde{J}C_2 = C_2$. Let us define

$A := \overline{A} + \frac{1}{2}C_1 + \frac{1}{2}C_2$ and $B := \overline{B} - \frac{1}{2}C_1 - \frac{1}{2}C_2$. Then we have $A + B = \overline{A} + \overline{B} = g$ and

$$\begin{aligned}\widetilde{J}A &= \widetilde{J}\overline{A} - \frac{1}{2}C_1 + \frac{1}{2}C_2 = \overline{A} + C_1 - \frac{1}{2}C_1 + \frac{1}{2}C_2\\ &= \overline{A} + \frac{1}{2}C_1 + \frac{1}{2}C_2 = A\\ \widetilde{J}B &= \widetilde{J}\overline{B} + \frac{1}{2}C_1 - \frac{1}{2}C_2 = -\overline{B} + C_2 + \frac{1}{2}C_1 - \frac{1}{2}C_2\\ &= -\overline{B} + \frac{1}{2}C_1 + \frac{1}{2}C_2 = -B.\end{aligned}$$

□

4. Para-Complex Affine Hyperspheres

In this section we focus on a special type of para-complex hypersurfaces. Namely, we study so called para-complex affine hyperspheres. The definition of para-complex affine hypersphere is very similar to definition of a hypersphere in a complex case. The aim of this section is to give a complete local classification of such hypersurfaces. Especially, we shall show that there is a strict correspondence between real and para-complex affine hyperspheres.

A nondegenerate para-complex hypersurface is said to be a *proper para-complex affine hypersphere* if there exists an affine normal vector field ζ such that $S = \alpha I$, where $\alpha \in \mathbb{R}\backslash\{0\}$ and $\tau_2 = 0$. If there exists an affine normal vector field ζ such that $S = 0$ and $\tau_2 = 0$ we say about an *improper para-complex affine hypersphere*.

Remark 4.1. Let $g\colon M \to \mathbb{R}^{2n+2}$ be a proper para-complex affine hypersphere with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$ such that $S = \alpha I$ for ζ. Then g is a para-complex affine hypersphere with a transversal bundle $\{\widetilde{\zeta}, \widetilde{J}\widetilde{\zeta}\}$, where $\widetilde{\zeta} = \frac{1}{2}(\alpha + \frac{1}{\alpha})\zeta + \frac{1}{2}(\frac{1}{\alpha} - \alpha)\widetilde{J}\zeta$ and $\widetilde{S} = \mathrm{id}$.

Now we shall prove a classification theorem for para-complex affine hyperspheres.

Theorem 4.1. *Let $g\colon M \to \mathbb{R}^{2n+2}$ be a para-complex affine hypersphere with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$. Then there exist open subsets $U_1 \subset \mathbb{R}^n$, $U_2 \subset \mathbb{R}^n$ and (real) affine hyperspheres*

$$f_1\colon U_1 \to \mathbb{R}^{n+1}, \quad f_2\colon U_2 \to \mathbb{R}^{n+1}$$

such that g can be locally expressed in the form

$$g = f_1 \times f_2 + \widetilde{J} \circ (f_1 \times (-f_2)). \tag{4.1}$$

Moreover, if g is proper (respectively improper) then both f_1 and f_2 are proper (respectively improper) as well. The converse is also true, in the sense, that for every two proper (respectively improper) affine hyperspheres f_1 and f_2 the

Formula (4.1) *defines a proper (respectively improper) para-complex affine hypersphere.*

Proof. Let $g\colon M \to \mathbb{R}^{2n+2}$ be a para-complex affine hypersphere and let $x \in M$. Since g is a para-complex affine hypersurface the Lemma 3.8 implies that there exist open subsets $U_1, U_2 \subset \mathbb{R}^n$ and smooth immersions $A\colon U_1 \to \mathbb{R}^{2n+2}$, $B\colon U_2 \to \mathbb{R}^{2n+2}$ such that $\widetilde{J}A = A$, $\widetilde{J}B = -B$ and g can be expressed in some neighborhood of x in the form:

$$g\colon U_1 \times U_2 \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto A(x_1, \ldots, x_n) + B(y_1, \ldots, y_n) \in \mathbb{R}^{2n+2}.$$

Let ∇, h_1, S, τ_1 and τ_2 be induced affine objects for g. Since g is a hypersphere we have $\tau_1 = \tau_2 = 0$ and $S = \alpha\,\mathrm{id}$ for some $\alpha \in \mathbb{R}$.

Let $\pi_1\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{n+1}$ be a projection of first $(n+1)$ variables on $\mathbb{R}^{n+1}$ and let $\pi_2\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{n+1}$ be a projection of last $(n+1)$ variables on $\mathbb{R}^{n+1}$. Let us define $f_1\colon U_1 \ni (x_1, \ldots, x_n) \mapsto \pi_1 \circ A(x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ and $f_2\colon U_2 \ni (y_1, \ldots, y_n) \mapsto \pi_2 \circ B(y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$. Since A and B are immersions and $\widetilde{J}A = A$ and $\widetilde{J}B = -B$ we easily verify that f_1 and f_2 are immersions too. We also have

$$g = f_1 \times f_2 + \widetilde{J} \circ (f_1 \times (-f_2)).$$

Now, it is enough to show that f_1 and f_2 are affine hyperspheres. For this purpose we shall consider the following two cases.

Case I $\alpha \neq 0$. In this case we have $\zeta = -\alpha g$. Since ζ and $\widetilde{J}\zeta$ are linearly independent and transversal to g then also $\frac{1}{2}(\zeta + \widetilde{J}\zeta) = -\alpha A$ and $\frac{1}{2}(\zeta - \widetilde{J}\zeta) = -\alpha B$ are transversal to g. In particular $\{A_{x_1}, \ldots, A_{x_n}, A\}$ and $\{B_{y_1}, \ldots, B_{y_n}, B\}$ are linearly independent. Let $\alpha_1, \ldots, \alpha_n, \beta$ be functions on U_1 such that

$$\sum_i \alpha_i f_{1x_i} + \beta f_1 = 0.$$

Then

$$\begin{aligned}\sum_i \alpha_i A_{x_i} + \beta A &= \sum_i \alpha_i (\pi_1 A_{x_i}, \pi_1 A_{x_i}) + \beta(\pi_1 A, \pi_1 A)\\ &= \left(\sum_i \alpha_i f_{1x_i} + \beta f_1, \sum_i \alpha_i f_{1x_i} + \beta f_1\right) = (0,0).\end{aligned}$$

Since $\{A_{x_1}, \ldots, A_{x_n}, A\}$ are linearly independent the above implies that $\alpha_1 = \cdots = \alpha_n = \beta = 0$ that is f_1 is linearly independent with $\{f_{1x_i}\}_{i=1}^n$. Now $\xi_1 := -2\alpha f_1$ is a transversal vector field to f_1. In a similar way we show that $\xi_2 := -2\alpha f_2$ is a transversal vector field to f_2.

The Gauss formula for g implies that

$$\begin{aligned}D_{\partial_{x_i}} g_* \partial_{x_j} &= g_*(\nabla_{\partial_{x_i}} \partial_{x_j}) + h_1(\partial_{x_i}, \partial_{x_j})\zeta + h_2(\partial_{x_i}, \partial_{x_j})\widetilde{J}\zeta\\ &= \Gamma_{ij}^k g_{x_k} + h_1(\partial_{x_i}, \partial_{x_j})(\zeta + \widetilde{J}\zeta)\\ &= \Gamma_{ij}^k A_{x_k} + h_1(\partial_{x_i}, \partial_{x_j}) \cdot (-2\alpha A),\end{aligned} \tag{4.2}$$

where Γ_{ij}^k are Christoffel's symbols for ∇ and we used the fact that $h_1 = h_2$ on D^+. On the other hand we have

$$D_{\partial_{x_i}} g_* \partial_{x_j} = g_{x_i x_j} = A_{x_i x_j} = (f_{1 x_i x_j}, f_{1 x_i x_j}). \tag{4.3}$$

Using (4.3) in (4.2) and applying π_1 projection we get

$$\begin{aligned} f_{1 x_i x_j} &= \Gamma_{ij}^k f_{1 x_k} + h_1(\partial_{x_i}, \partial_{x_j}) \cdot (-2\alpha f_1) \\ &= \Gamma_{ij}^k f_{1 x_k} + h_1(\partial_{x_i}, \partial_{x_j}) \xi_1. \end{aligned} \tag{4.4}$$

For f_1 we have the Gauss formula, that is

$$f_{1 x_i x_j} = D_{\partial_{x_i}} f_{1*} \partial_{x_j} = f_{1*} \left(\nabla^+_{\partial_{x_i}} \partial_{x_j}\right) + h^+(\partial_{x_i}, \partial_{x_j}) \xi_1,$$

where ∇^+ is the induced connection and h^+ is the second fundamental form for f_1. Now (4.4) implies that $\nabla^+ = \nabla|_{TU_1 \times TU_1}$ and $h^+ = h_1|_{TU_1 \times TU_1}$. In particular h^+ is nondegenerate since h_1 is nondegenerate on $TU_1 \times TU_1$. Note also that for f_1 we have the induced volume element θ^+ given by the formula

$$\begin{aligned} \theta^+(\partial_{x_1}, \ldots, \partial_{x_n}) :&= \det[f_{1 x_1}, \ldots, f_{1 x_n}, \xi_1] \\ &= -2\alpha \det[f_{1 x_1}, \ldots, f_{1 x_n}, f_1]. \end{aligned}$$

In a similar way like above (but now using the fact that $h_2 = -h_1$ on D^-) we obtain that $\nabla^- = \nabla|_{TU_2 \times TU_2}$, $h^- = h_1|_{TU_2 \times TU_2}$ and

$$\theta^-(\partial_{y_1}, \ldots, \partial_{y_n}) = -2\alpha \det[f_{2 y_1}, \ldots, f_{2 y_n}, f_2],$$

where ∇^- is the induced connection, h^- is the second fundamental form and θ^- is the induced volume element for f_2.

Let θ be the induced volume element for g, that is

$$\begin{aligned} \theta(\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}) &= \det[A_{x_1}, \ldots, A_{x_n}, B_{y_1}, \ldots, B_{y_n}, \\ &\quad - \alpha(A+B), -\alpha(A-B)] \\ &= \alpha^2 \det[A_{x_1}, \ldots, A_{x_n}, B_{y_1}, \ldots, B_{y_n}, A, -B] \\ &\quad + \alpha^2 \det[A_{x_1}, \ldots, A_{x_n}, B_{y_1}, \ldots, B_{y_n}, B, A] \\ &= -2\alpha^2 \det[A_{x_1}, \ldots, A_{x_n}, B_{y_1}, \ldots, B_{y_n}, A, B] \\ &= -2\alpha^2 \cdot (-1)^n \det[A_{x_1}, \ldots, A_{x_n}, A, B_{y_1}, \ldots, B_{y_n}, B]. \end{aligned}$$

Let us denote

$$\begin{aligned} M &:= [A_{x_1}, \ldots, A_{x_n}, A, B_{y_1}, \ldots, B_{y_n}, B], \\ M^+ &:= [\pi_1 A_{x_1}, \ldots, \pi_1 A_{x_n}, \pi_1 A], \\ M^- &:= [\pi_2 B_{y_1}, \ldots, \pi_2 B_{y_n}, \pi_2 B]. \end{aligned}$$

Then M can be expressed in the following block form:

$$M = \begin{bmatrix} M^+ & -M^- \\ M^+ & M^- \end{bmatrix}.$$

Now replacing the row i with the sum of rows i and $i+n+1$ for $i = 1, \dots, n+1$, we obtain a new matrix

$$M' = \begin{bmatrix} 2M^+ & 0 \\ M^+ & M^- \end{bmatrix}.$$

It is easy to see that

$$\begin{aligned} \det M &= \det M' = \det(2M^+) \cdot \det(M^-) \\ &= 2^{n+1} \det M^+ \cdot \det M^- \\ &= \frac{2^{n-1}}{\alpha^2} \theta^+(\partial_{x_1}, \dots, \partial_{x_n}) \cdot \theta^-(\partial_{y_1}, \dots, \partial_{y_n}). \end{aligned}$$

Finally we get the following relation between θ, θ^+ and θ^-:

$$\theta(\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}) = 2^n \cdot (-1)^{n+1} \theta^+(\partial_{x_1}, \dots, \partial_{x_n}) \theta^-(\partial_{y_1}, \dots, \partial_{y_n}). \tag{4.5}$$

Let $\det h_1$ be the determinant of h_1 in the basis $\{\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}\}$. Since $h_1(\partial_{x_i}, \partial_{y_j}) = 0$ for $i, j = 1, \dots, n$ we have that

$$\det h_1 = \det h^+ \cdot \det h^-, \tag{4.6}$$

where $\det h^+$ is the determinant of h^+ with respect to the basis $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ and $\det h^-$ is the determinant of h^- with respect to the basis $\{\partial_{y_1}, \dots, \partial_{y_n}\}$. Now using (4.5), (4.6) and the fact that $|H_\zeta| = 1$ we obtain

$$\begin{aligned} 1 = |H_\zeta| &= \left| \frac{\det h_1}{\theta^2} \right| = \left| \frac{\det h^+ \cdot \det h^-}{2^{2n} (\theta^+)^2 \cdot (\theta^-)^2} \right| \\ &= \frac{1}{2^{2n}} \left(\frac{\omega_{h^+}(\partial_{x_1}, \dots, \partial_{x_n})}{\theta^+(\partial_{x_1}, \dots, \partial_{x_n})} \right)^2 \cdot \left(\frac{\omega_{h^-}(\partial_{y_1}, \dots, \partial_{y_n})}{\theta^-(\partial_{y_1}, \dots, \partial_{y_n})} \right)^2. \end{aligned}$$

That is

$$\left| \frac{\omega_{h^+}(\partial_{x_1}, \dots, \partial_{x_n})}{\theta^+(\partial_{x_1}, \dots, \partial_{x_n})} \right| \cdot \left| \frac{\omega_{h^-}(\partial_{y_1}, \dots, \partial_{y_n})}{\theta^-(\partial_{y_1}, \dots, \partial_{y_n})} \right| = 2^n.$$

Since ω_{h^+}, θ^+ depends only on $x_1, \dots, x_n$ and ω_{h^-}, θ^- depends only on $y_1, \dots, y_n$ the last equality implies that both ω_{h^+}/θ^+ and ω_{h^-}/θ^- are constant. So there exist constants c^+ and c^- such that $\omega_{h^+} = c^+\theta^+$ and $\omega_{h^-} = c^-\theta^-$.

Case II $\alpha = 0$. Without loss of generality we may assume that $\zeta = (0, \dots, 0, 1) \in \mathbb{R}^{2n+2}$. Let us denote $\xi_1 = \xi_2 = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Since $\{\zeta, \widetilde{J}\zeta\}$ is transversal to g we have that $\zeta + \widetilde{J}\zeta = (\xi_1, \xi_1)$ is transversal to g as well. Let $\alpha_1, \dots, \alpha_n, \beta$ be functions on U_1 such that

$$\sum_i \alpha_i f_{1x_i} + \beta\xi_1 = 0.$$

Then

$$\sum_i \alpha_i A_{x_i} + \beta(\zeta + \widetilde{J}\zeta)$$
$$= \sum_i \alpha_i(\pi_1 A_{x_i}, \pi_1 A_{x_i}) + \beta(\xi_1, \xi_1)$$
$$= \left(\sum_i \alpha_i f_{1x_i} + \beta\xi_1, \sum_i \alpha_i f_{1x_i} + \beta\xi_1\right) = (0, 0).$$

Now, since $\{g_{x_1}, \dots, g_{x_n}, \zeta + \widetilde{J}\zeta\}$ are linearly independent it immediately follows that $\alpha_1 = \cdots = \alpha_n = \beta = 0$ and in consequence ξ_1 is transversal to f_1. In a similar way we show that ξ_2 is transversal to f_2. Like for $\alpha \neq 0$, using the Gauss formulas for g, f_1 and f_2, we obtain that $h^+ = h_1$ on D^+, $h^- = h_1$ on D^- and $\det h_1 = \det h^+ \cdot \det h^-$. In particular we get that both f_1 and f_2 are nondegenerate.

For the induced volume θ we have

$$\theta(\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n})$$
$$= \det[A_{x_1}, \dots, A_{x_n}, B_{y_1}, \dots, B_{y_n}, \zeta, \widetilde{J}\zeta]$$
$$= \frac{1}{2} \cdot (-1)^{n+1} \det[A_{x_1}, \dots, A_{x_n}, \zeta + \widetilde{J}\zeta, B_{y_1}, \dots, B_{y_n}, \zeta - \widetilde{J}\zeta].$$

The above implies that

$$\theta(\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}) = 2^n \cdot (-1)^{n+1} \theta^+(\partial_{x_1}, \dots, \partial_{x_n}) \cdot \theta^-(\partial_{y_1}, \dots, \partial_{y_n}),$$

where θ^+ and θ^- are the induced volume forms for f_1 and f_2 respectively. Now, since ζ is affine normal, we have

$$1 = |H_\zeta| = \left|\frac{\det h_1}{(\theta)^2}\right| = \left|\frac{\det h^+ \cdot \det h^-}{4^n(\theta^+ \cdot \theta^-)^2}\right|$$
$$= \frac{1}{4^n}\left|\frac{\omega_{h^+}}{\theta^+}\right|^2 \cdot \left|\frac{\omega_{h^-}}{\theta^-}\right|^2.$$

Since ω_{h^+}, θ^+ depends only on $x_1, \dots, x_n$ and ω_{h^-}, θ^- depends only on $y_1, \dots, y_n$ the last equality implies that both ω_{h^+}/θ^+ and ω_{h^-}/θ^- are constant and in consequence f_1 and f_2 are improper affine hyperspheres.

In order to prove the converse assume that $f_1\colon U_1 \to \mathbb{R}^{n+1}$ and $f_2\colon U_2 \to \mathbb{R}^{n+1}$ are two affine hyperspheres with the Blaschke field ξ_1 and ξ_2 respectively. Let us denote $U = U_1 \times U_2$ and let $g\colon U \to \mathbb{R}^{2n+2}$ be defined by the Formula (4.1) that is

$$g(x_1, \dots, x_n, y_1, \dots, y_n) = (f_1(x_1, \dots, x_n), f_1(x_1, \dots, x_n)) + (-f_2(y_1, \dots, y_n), f_2(y_1, \dots, y_n)).$$

For the above and similar expressions we will often ommit arguments using the following short notation:

$$g = (f_1, f_1) + (-f_2, f_2).$$

Like in the proof of the first part of the theorem we shall consider two cases.

Case I f_1 and f_2 are proper affine hyperspheres. In this case we have $\xi_1 = -\lambda_1 f_1$ and $\xi_2 = -\lambda_2 f_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}_+$. Let us define $\zeta := -\alpha g$, where

$$\alpha := \left(\frac{1}{2}\right)^{\frac{2n+2}{n+2}} \cdot \sqrt{\lambda_1 \lambda_2}. \tag{4.7}$$

We shall show that g with ζ is a para-complex affine hypersphere. For this purpose let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma, \delta \in C^\infty(U)$ and

$$\sum \alpha_i g_{x_i} + \sum \beta_i g_{y_i} + \gamma\zeta + \delta \widetilde{J}\zeta = 0.$$

Since $g_{x_i} = (f_{1x_i}, f_{1x_i})$ and $g_{y_i} = (-f_{2y_i}, f_{2y_i})$ we obtain

$$\sum(\alpha_i f_{1x_i} - \beta_i f_{2y_i}) - \alpha(\gamma+\delta) f_1 - \alpha(\delta - \gamma) f_2 = 0$$

and

$$\sum(\alpha_i f_{1x_i} + \beta_i f_{2y_i}) - \alpha(\gamma+\delta) f_1 - \alpha(\gamma - \delta) f_2 = 0.$$

The above implies that

$$\sum \alpha_i f_{1x_i} - \alpha(\gamma+\delta) f_1 = 0$$

and

$$\sum \beta_i f_{2y_i} - \alpha(\gamma-\delta) f_2 = 0.$$

Since f_1 and f_2 are proper affine hyperspheres then $\{f_{1x_1}, \ldots, f_{1x_n}, f_1\}$ as well as $\{f_{2y_1}, \ldots, f_{2y_n}, f_2\}$ are linearly independent, that is

$$\alpha_1 = \cdots = \alpha_n = 0, \gamma + \delta = 0$$

and

$$\beta_1 = \cdots = \beta_n = 0, \gamma - \delta = 0.$$

In particular $\gamma = \delta = 0$. In this way we have shown that

$$\{g_{x_1}, \ldots, g_{x_n}, g_{y_1}, \ldots, g_{y_n}, \zeta, \widetilde{J}\zeta\}$$

are linearly independent. Since $\widetilde{J} g_{x_i} = g_{x_i}$ and $\widetilde{J} g_{y_i} = -g_{y_i}$ we see that g is a para-complex hypersurface with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$. The Weingarten formula for g immediately implies that $S = \alpha \,\mathrm{id}$ and $\tau_1 = \tau_2 = 0$, so it is enough to show that g is nondegenerate and $|H_\zeta| = 1$. For this purpose note that since $\partial_{x_i} \in \mathrm{D}^+$ and $\partial_{y_i} \in \mathrm{D}^-$ we have $h_1(\partial_{x_i}\partial_{x_j}) = h_2(\partial_{x_i}, \partial_{x_j})$, $h_1(\partial_{y_i}\partial_{y_j}) = -h_2(\partial_{y_i}, \partial_{y_j})$ and $h_1(\partial_{x_i}, \partial_{y_j}) = h_2(\partial_{x_i}, \partial_{y_j}) = 0$ for $i, j = 1, \ldots, n$. Now using the Gauss formula we get

$$g_{x_i x_j} = g_*(\nabla_{\partial_{x_i}} \partial_{x_j}) - 2\alpha h_1(\partial_{x_i}, \partial_{x_j})(f_1, f_1).$$

On the other hand

$$g_{x_i x_j} = (f_{1x_i x_j}, f_{1x_i x_j}) = \left(f_{1*}\left(\nabla^+_{\partial_{x_i}} \partial_{x_j}\right), f_{1*}\left(\nabla^+_{\partial_{x_i}} \partial_{x_j}\right)\right)$$

$$- \ h^{+}(\partial_{x_i}, \partial_{x_j})\lambda_1(f_1, f_1)$$

where ∇^{+} and h^{+} are the induced affine connection and the second fundamental form for f_1. Now it easily follows that

$$h_1(\partial_{x_i}, \partial_{x_j}) = \frac{\lambda_1}{2\alpha} h^{+}(\partial_{x_i}, \partial_{x_j}). \tag{4.8}$$

Similarly we obtain

$$h_2(\partial_{y_i}, \partial_{y_j}) = \frac{\lambda_2}{2\alpha} h^{-}(\partial_{y_i}, \partial_{y_j}), \tag{4.9}$$

where h^{-} is the second fundamental form for f_2. Formulas (4.8) and (4.9) imply that

$$\det h_1 = \left(\frac{\lambda_1}{2\alpha}\right)^n \cdot \left(\frac{\lambda_2}{2\alpha}\right)^n \det h^{+} \cdot \det h^{-}. \tag{4.10}$$

In particular g is nondegenerate. Now we shall calculate θ_ζ. Namely, we have

$$\begin{aligned}
\theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}) &= \det[g_{x_1}, \ldots, g_{x_n}, g_{y_1}, \ldots, g_{y_n}, \zeta, \widetilde{J}\zeta] \\
&= \alpha^2 \det[g_{x_1}, \ldots, g_{x_n}, g_{y_1}, \ldots, g_{y_n}, g, \widetilde{J}g] \\
&= 2\alpha^2 \det\left[g_{x_1}, \ldots, g_{x_n}, g_{y_1}, \ldots, g_{y_n}, \frac{g + \widetilde{J}g}{2}, \frac{\widetilde{J}g - g}{2}\right] \\
&= -2(-1)^n \alpha^2 \det\left[g_{x_1}, \ldots, g_{x_n}, \frac{g + \widetilde{J}g}{2}, g_{y_1}, \ldots, g_{y_n}, \frac{g - \widetilde{J}g}{2}\right].
\end{aligned}$$

Let us denote

$$M := \left[g_{x_1}, \ldots, g_{x_n}, \frac{g + \widetilde{J}g}{2}, g_{y_1}, \ldots, g_{y_n}, \frac{g - \widetilde{J}g}{2}\right].$$

It is easy to see that M has the following block form:

$$M = \begin{bmatrix} M^{+} & -M^{-} \\ M^{+} & M^{-} \end{bmatrix},$$

where

$$M^{+} = [f_{1x_1}, \ldots, f_{1x_n}, f_1] = \left[f_{1x_1}, \ldots, f_{1x_n}, -\frac{1}{\lambda_1} \cdot \xi_1\right]$$

and

$$M^{-} = [f_{2y_1}, \ldots, f_{2y_n}, f_2] = \left[f_{2y_1}, \ldots, f_{2y_n}, -\frac{1}{\lambda_2} \cdot \xi_2\right].$$

Like in the first part of the proof we see that

$$\begin{aligned}
\det M &= 2^{n+1} \det M^{+} \cdot \det M^{-} \\
&= 2^{n+1} \cdot \frac{-1}{\lambda_1} \cdot \theta^{+}(\partial_{x_1}, \ldots, \partial_{x_n}) \cdot \frac{-1}{\lambda_2} \cdot \theta^{-}(\partial_{y_1}, \ldots, \partial_{y_n}) \\
&= \frac{2^{n+1}}{\lambda_1 \lambda_2} \theta^{+}(\partial_{x_1}, \ldots, \partial_{x_n}) \cdot \theta^{-}(\partial_{y_1}, \ldots, \partial_{y_n}),
\end{aligned}$$

where θ^+ and θ^- are the induced volume elements for f_1 and f_2 respectively. To simplify notation in the forthcoming formulas we will be omitting arguments of θ_ζ, θ^+ and θ^-. Now we obtain

$$\theta_\zeta = (-1)^{n+1}\alpha^2 \cdot \frac{2^{n+2}}{\lambda_1\lambda_2}\theta^+ \cdot \theta^-.$$

Since ξ_1 and ξ_2 are the Blaschke fields we have $\omega_{h^+} = \theta^+$ and $\omega_{h^-} = \theta^-$. In particular $(\theta^+)^2 = |\det h^+|$ and $(\theta^-)^2 = |\det h^-|$. Now using (4.10) we obtain

$$\begin{aligned}(\theta_\zeta)^2 &= \alpha^4 \cdot \frac{2^{2n+4}}{(\lambda_1\lambda_2)^2} \cdot (\theta^+)^2 \cdot (\theta^-)^2 \\ &= \alpha^4 \cdot \frac{2^{2n+4}}{(\lambda_1\lambda_2)^2} \cdot |\det h^+| \cdot |\det h^-| \\ &= \alpha^4 \cdot \frac{2^{2n+4}}{(\lambda_1\lambda_2)^2} \cdot \left(\frac{2\alpha}{\lambda_1}\right)^n \cdot \left(\frac{2\alpha}{\lambda_2}\right)^n \cdot |\det h_1| \\ &= \alpha^{2n+4} \cdot \frac{2^{4n+4}}{(\lambda_1\lambda_2)^{n+2}} \cdot |\det h_1| \\ &= |\det h_1|,\end{aligned}$$

where the last equality is an immediate consequence of (4.7). Summarizing we have shown that

$$|H_\zeta| = \left|\frac{\det h_1}{(\theta_\zeta)^2}\right| = 1,$$

that is g is a proper para-complex affine hypersphere.

Case II f_1 and f_2 are improper affine hyperspheres. In this case we have $\xi_1 = \xi_2 = (0, \ldots, 1) \in \mathbb{R}^{n+1}$. Let us define $\zeta := 2^{\frac{-n}{n+2}}(0, \ldots, 0, 1) \in \mathbb{R}^{2n+2}$ and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma, \delta \in C^\infty(U)$ and

$$\sum \alpha_i g_{x_i} + \sum \beta_i g_{y_i} + \gamma\zeta + \delta\widetilde{J}\zeta = 0.$$

Like for proper hyperspheres we easily compute that

$$\sum(\alpha_i f_{1_{x_i}} - \beta_i f_{2_{y_i}}) + 2^{\frac{-n}{n+2}}\delta\xi_1 = 0$$

and

$$\sum(\alpha_i f_{1_{x_i}} + \beta_i f_{2_{y_i}}) + 2^{\frac{-n}{n+2}}\gamma\xi_2 = 0.$$

The above implies that $\alpha_i = 0$, $\beta_i = 0$, $\gamma = \delta = 0$ and in consequence

$$\{g_{x_1}, \ldots, g_{x_n}, g_{y_1}, \ldots, g_{y_n}, \zeta, \widetilde{J}\zeta\}$$

are linearly independent. It means that g is a para-complex affine hypersurface with a transversal bundle $\{\zeta, \widetilde{J}\zeta\}$. Using similar methods like in the proof for the first case we obtain

$$\det h_1 = 2^{\frac{2n^2}{n+2}} \det h^+ \cdot \det h^- \tag{4.11}$$

and

$$\theta_\zeta = (-1)^{n+1} 2^n \cdot 2^{\frac{-2n}{n+2}} \theta^+ \cdot \theta^- \tag{4.12}$$

where h^+, h^- and θ^+, θ^- are the second fundamental forms and the induced volume elements for f_1 and f_2 respectively. It easily follows from (4.11) that g is nondegenerate. From the Weingarten formula we have $S = 0$, $\tau_1 = 0$ and $\tau_2 = 0$. Now (4.11) and (4.12) implies that

$$|H_\zeta| = \left| \frac{2^{\frac{2n^2}{n+2}} \det h^+ \cdot \det h^-}{\left[(-1)^{n+1} 2^n \cdot 2^{\frac{-2n}{n+2}} \cdot \theta^+ \cdot \theta^-\right]^2} \right| = 1,$$

that is g is an improper para-complex affine hypersphere. The proof is concluded. □

The above theorem gives us a one-to-one correspondence between para-complex affine hyperspheres and pairs of (real) affine hyperspheres. Now, we shall show some examples

Example 4.1. Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x, y) := {\lambda_1}^{-\frac{3}{4}} \begin{pmatrix} \cos x \\ \sin x \\ \cos x \\ \sin x \end{pmatrix} + {\lambda_2}^{-\frac{3}{4}} \begin{pmatrix} -\cos y \\ -\sin y \\ \cos y \\ \sin y \end{pmatrix}, \tag{4.13}$$

where $\lambda_1, \lambda_2 > 0$. It easily follows that g is an immersion. Moreover $\widetilde{J} g_x = g_x$ and $\widetilde{J} g_y = -g_y$, so g is a para-holomorphic hypersurface. If we take $\zeta := -\left(\frac{1}{2}\right)^{\frac{4}{3}} \sqrt{\lambda_1 \lambda_2} \cdot g$ then $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g. By straightforward computations we obtain

$$h_1 = \begin{bmatrix} \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} & 0 \\ 0 & \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} \end{bmatrix}, \quad h_2 = \begin{bmatrix} \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} & 0 \\ 0 & -\frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} \end{bmatrix},$$

$$S = \left(\frac{1}{2}\right)^{\frac{4}{3}} \sqrt{\lambda_1\lambda_2}\, \mathrm{id}, \quad \tau_1 = \tau_2 = 0$$

relative to the canonical basis $\{\partial_x, \partial_y\}$. Moreover, since

$$\theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \widetilde{J}\zeta] = \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}}$$

one may easily compute that $H_\zeta = 1$, that is g is a proper para-complex affine sphere.

Example 4.2. Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x,y) := {\lambda_1}^{-\frac{3}{4}} \begin{pmatrix} \cosh x \\ \sinh x \\ \cosh x \\ \sinh x \end{pmatrix} + {\lambda_2}^{-\frac{3}{4}} \begin{pmatrix} -\cosh y \\ -\sinh y \\ \cosh y \\ \sinh y \end{pmatrix}, \tag{4.14}$$

where $\lambda_1, \lambda_2 > 0$. Exactly like in the previous example we have that g is an immersion and $\widetilde{J} g_x = g_x$ and $\widetilde{J} g_y = -g_y$, so g is a para-holomorphic hypersurface. Again taking $\zeta := -\left(\frac{1}{2}\right)^{\frac{4}{3}} \sqrt{\lambda_1\lambda_2} \cdot g$ we obtain that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g. We also have

$$h_1 = \begin{bmatrix} -\frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} & 0 \\ 0 & -\frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} \end{bmatrix}, \quad h_2 = \begin{bmatrix} -\frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} & 0 \\ 0 & \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} \end{bmatrix},$$

$$S = \left(\frac{1}{2}\right)^{\frac{4}{3}} \sqrt{\lambda_1\lambda_2}\, \mathrm{id}, \quad \tau_1 = \tau_2 = 0$$

relative to the canonical basis $\{\partial_x, \partial_y\}$. Moreover, since

$$\theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \widetilde{J}\zeta] = \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}}$$

we easily compute that $H_\zeta = 1$, that is g is a proper para-complex affine sphere.

Example 4.3. In this example we consider two very similar surfaces. Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ and $g'\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formulas:

$$g(x,y) := {\lambda_1}^{-\frac{3}{4}} \begin{pmatrix} \cosh x \\ \sinh x \\ \cosh x \\ \sinh x \end{pmatrix} + {\lambda_2}^{-\frac{3}{4}} \begin{pmatrix} -\cos y \\ -\sin y \\ \cos y \\ \sin y \end{pmatrix} \tag{4.15}$$

and

$$g'(x,y) := {\lambda_1}^{-\frac{3}{4}} \begin{pmatrix} \cos x \\ \sin x \\ \cos x \\ \sin x \end{pmatrix} + {\lambda_2}^{-\frac{3}{4}} \begin{pmatrix} -\cosh y \\ -\sinh y \\ \cosh y \\ \sinh y \end{pmatrix}, \tag{4.16}$$

where $\lambda_1, \lambda_2 > 0$. Exactly like in the previous examples we prove that g and g' are para-holomorphic hypersurfaces. Let $\zeta := -\left(\frac{1}{2}\right)^{\frac{4}{3}} \sqrt{\lambda_1\lambda_2} \cdot g$ and $\zeta' := -\left(\frac{1}{2}\right)^{\frac{4}{3}} \sqrt{\lambda_1\lambda_2} \cdot g'$ then $\{\zeta, \widetilde{J}\zeta\}$ and $\{\zeta', \widetilde{J}\zeta'\}$ are transversal bundles for g and g' respectively. For g we have

$$h_1 = \begin{bmatrix} -\frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} & 0 \\ 0 & \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} \end{bmatrix}, \quad h_2 = \begin{bmatrix} -\frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} & 0 \\ 0 & -\frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1\lambda_2}} \end{bmatrix},$$

$$S = \left(\frac{1}{2}\right)^{\frac{4}{3}} \sqrt{\lambda_1 \lambda_2}\, \mathrm{id}, \quad \tau_1 = \tau_2 = 0$$

and

$$\theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \widetilde{J}\zeta] = \frac{2^{\frac{1}{3}}}{\sqrt{\lambda_1 \lambda_2}}$$

relative to the canonical basis $\{\partial_x, \partial_y\}$. Now it easily follows that $H_\zeta = -1$ that is g is a proper para-complex affine sphere. In a similar way we show that also g' is a para-complex affine sphere.

Example 4.4. Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x, y) := \begin{pmatrix} x \\ \frac{1}{2}x^2 \\ x \\ \frac{1}{2}x^2 \end{pmatrix} + \begin{pmatrix} -y \\ -\frac{1}{2}y^2 \\ y \\ \frac{1}{2}y^2 \end{pmatrix}. \tag{4.17}$$

It easily follows that g is an immersion and $\widetilde{J}g_x = g_x$ and $\widetilde{J}g_y = -g_y$, so g is a para-holomorphic hypersurface. Let $\zeta := 2^{-\frac{1}{3}}(0, 0, 0, 1)^T$ then $\widetilde{J}\zeta = 2^{-\frac{1}{3}}(0, 1, 0, 0)^T$ and $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g. We compute

$$h_1 = \begin{bmatrix} 2^{\frac{1}{3}} & 0 \\ 0 & 2^{\frac{1}{3}} \end{bmatrix}, \quad h_2 = \begin{bmatrix} 2^{\frac{1}{3}} & 0 \\ 0 & -2^{\frac{1}{3}} \end{bmatrix}, \quad S = 0, \quad \tau_1 = \tau_2 = 0$$

relative to the canonical basis $\{\partial_x, \partial_y\}$. Since

$$\theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \widetilde{J}\zeta] = 2^{\frac{1}{3}}$$

then $H_\zeta = 1$, that is g is an improper para-complex affine sphere.

Using Theorem 4.1 we give a complete local classification of 1-dimensional (in para-complex sense) para-complex affine spheres. Namely we have the following theorem:

Theorem 4.2. *Let $g\colon M^2 \to \mathbb{R}^4$ be a para-complex affine hypersphere. If g is proper then it can be locally expressed in one of the forms* (4.13)–(4.16)*. If g is improper then it can be locally expressed in the form* (4.17)*.*

Proof. It is well known [8] that the only (up to equiaffine transformation) 1-dimensional (real) affine spheres are a circle $\gamma_1(t) = k^{-\frac{3}{4}}(\cos t, \sin t)$, hyperbola $\gamma_2(t) = k^{-\frac{3}{4}}(\cosh t, \sinh t)$ and a parabola $\gamma_3(t) = (t, \frac{1}{2}t^2)$. γ_1 and γ_2 are proper spheres and γ_3 is an improper sphere. Now, applying Theorem 4.1 we easily obtain that there are only four (up to a para-complex equiaffine transformation) proper 1-dimensional para-complex affine spheres, that is spheres from Examples 4.1, 4.2 and 4.3. Similarly the only improper 1-dimensional para-complex affine sphere is the sphere form Example 4.4. □

Remark 4.2. Surfaces (4.13)–(4.17) are examples of so called *translation surfaces* (see [10,11] for details).

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Received: September 17, 2016.
Accepted: March 22, 2017.