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Results in Mathematics

On Sandwich Theorem for Delta-Subadditive and Delta-Superadditive Mappings

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Abstract. In the present paper, inspired by methods contained in Gajda and Kominek (Stud Math 100:25–38, 1991) we generalize the well known sandwich theorem for subadditive and superadditive functionals to the case of delta-subadditive and delta-superadditive mappings. As a consequence we obtain the classical Hyers–Ulam stability result for the Cauchy functional equation. We also consider the problem of supporting delta-subadditive maps by additive ones.

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1. Introduction

We denote by $\mathbb{R}, \mathbb{N}$ the sets of all reals and positive integers, respectively, moreover, unless explicitly stated otherwise, $(Y, \|\cdot\|)$ denotes a real normed space and $(S, \cdot)$ stands for not necessary commutative semigroup.

We recall that a functional $f : S \to \mathbb{R}$ is said to be *subadditive* if

$$f(x \cdot y) \leq f(x) + f(y), \quad x, y \in S.$$

A functional $g : S \to \mathbb{R}$ is called *superadditive* if $f := -g$ is subadditive or, equivalently, if g satisfies

$$g(x) + g(y) \leq g(x \cdot y), \quad x, y \in S.$$

If $a : S \to \mathbb{R}$ is at the same time subadditive and superadditive then we say that it is *additive*, in this case a satisfies the Cauchy functional equation

$$a(x \cdot y) = a(x) + a(y), \quad x, y \in S.$$

The generalizations of the celebrated separation theorem of Rodé [19] (cf. also Köning [16]) which represents a far-reaching generalization of the classical

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Hahn–Banach theorem has been studied by many mathematicians. A survey of results of this type can be found for instance in the book of Buskes [5]. The problem reads as follow:

Suppose $f, g : S \to \mathbb{R}$ are maps with f subadditive, g superadditive, and

$$g(x) \leq f(x), \quad x, y \in S.$$

Does there exist an additive map $a : S \to \mathbb{R}$ separating g from f, that is, satisfying

$$g(x) \leq a(x) \leq f(x),$$

for every $x \in S$?

Results of this type, for commutative semigroups were first obtained by Kaufman [14] and Kranz [17]. The paper of Gajda and Kominek [9] on separations theorems and the paper of Chaljub-Simon and Volkmann [7] on the non-commutative version of Rode's theorems are examples of papers where the assumption of the commutativity is replaced by some essentially weaker conditions.

Recall that a semigroup $(S, \cdot)$ is said to be *weakly commutative* if

$$(x \cdot y)^{2^n} = x^{2^n} \cdot y^{2^n}, \quad x, y \in S.$$

This implies that for any $x, y \in S$ there exists a sequence of positive integers n_k (depending on x and y) such that $n_k \to \infty$ as $k \to \infty$ and

$$(x \cdot y)^{2^{n_k}} = x^{2^{n_k}} \cdot y^{2^{n_k}}, \quad x, y \in S.$$

The definition of weakly commutative semigroups was introduced by Józef Tabor in [21]. It is clear that every Abelian semigroup is weakly commutative, but there exist non-Abelian semigroups and even groups which are weakly commutative. The multiplicative group consisting of the quaternions $1, -1, i, -i, j, -j, k, -k$, with i, j and k being the quaternion imaginary units is an example of non-Abelian weakly commutative group.

The aim of the present paper is to prove some version of separation and support theorem for delta-subadditive mappings. In the second section following Ger [9] we introduce and study a basic properties of delta-subadditive and delta-superadditive maps. Section 3 deals with the definition of Lorenz cone and a partial order generated by this cone. It turns out that there is a close relationship between the above-mentioned order and the concept of delta-subadditivity. The fourth section contains the main result of the paper. We prove the separation theorem for delta-subadditive mappings. Using this theorem we give an easy proof of the Hyers–Ulam stability result for the Cauchy equation. In the last section we give a necessary and sufficient conditions under which a delta-subadditive map can be support at a given point by an additive map.

We recall in this place the following well-known definition.

Definition 1. Let $(S,\cdot)$ and $(Y,+)$ be semigroups and let x_0 be a fixed element of S. A mapping $F : S \to Y$ satisfying

$$F(x_0^n) = nF(x_0) \tag{1}$$

for some $n \in \mathbb{N}$ is said to be n-homogeneous at x_0. If (1) holds for every $n \in \mathbb{N}$, then F is $\mathbb{N}$-homogeneous at x_0. Moreover, if F is n-(resp. $\mathbb{N}$)-homogeneous at every point of S, then we simply say that it is n-(resp. $\mathbb{N}$)-homogeneous.

In the sequel we will need the following result from [9].

Lemma 1. *Let $(S,\cdot)$ be a semigroup and let $f : S \to \mathbb{R}$ be a subadditive or superadditive function. If f is 2-homogeneous then it is $\mathbb{N}$-homogeneous, moreover,*

$$f(x \cdot y) = f(y \cdot x), \quad x, y \in X.$$

2. Delta-Subadditive Maps

In 1989 L. Veselý and L. Zajiček introduced an interesting generalization of functions which are representable as a difference of two convex functions. In the paper [20] the authors have introduced the following definition.

Definition 2. Given two real normed spaces $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ and a non-empty open and convex subset $D \subseteq X$ we say that a map $F : D \to Y$ is delta-convex if there exists a continuous and convex functional $f : D \to \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member y^* of the space Y^* dual to Y with $\|y^*\| = 1$. If this is the case then we say that F is a delta-convex mapping with a control function f.

It turns out that a continuous function $F : D \to Y$ is a delta-convex mapping controlled by a continuous function $f : D \to \mathbb{R}$ if and only if the functional inequality

$$\left\| F\Big(\frac{x+y}{2}\Big) - \frac{F(x) + F(y)}{2} \right\| \le \frac{f(x) + f(y)}{2} - f\Big(\frac{x+y}{2}\Big), \tag{2}$$

is satisfied for all $x, y \in D$. (Corollary 1.18 in [20])

The inequality (2) may obviously be investigated without any regularity assumption upon F and f which, additionally considerably enlarges the class of solutions. Note that the notion of delta-convex mappings has many nice properties (see [20]) and seems to be the most natural generalization of functions which are representable as a difference of two convex functions.

Motivated by the concept of delta-convexity Ger in [10] introduced the following definition.

Definition 3. A map $F : S \to Y$ is called delta-subadditive with a control function $f : S \to \mathbb{R}$ if the following inequality

$$\|F(x) + F(y) - F(x \cdot y)\| \le f(x) + f(y) - f(x \cdot y), \tag{3}$$

holds for all $x, y \in S$.

In a natural way, this allows us to the following definition.

Definition 4. A map $F : S \to Y$ is said to be a delta superadditive, if $-F$ is a delta-subadditive i.e. there exists a control function $f : S \to \mathbb{R}$ such that the inequality

$$\|F(x) + F(y) - F(x \cdot y)\| \leq f(x \cdot y) - f(x) - f(y),$$

is satisfied for all $x, y \in S$.

Observe that, if F is at the same time delta-subadditive and delta-superadditive with a control function f then both maps F and f are additive.

The following result establishes the necessary and sufficient conditions for a given map to be delta-subadditive.

Proposition 1. *For the mappings $F : S \to Y$ and $f : S \to \mathbb{R}$ the following statements are equivalent:*

(i) $y^* \circ F + f$ *is subadditive for any* $y^* \in Y^*$, $\|y^*\| = 1$,
(ii) $\|F(x) + F(y) - F(x \cdot y)\| \leq f(x) + f(y) - f(x \cdot y)$, *for all* $x, y \in S$,
(iii)

$$\Big\| \sum_{i=1}^{n} F(x_i) - F\Big(\prod_{i=1}^{n} x_i\Big) \Big\| \leq \sum_{i=1}^{n} f(x_i) - f\Big(\prod_{i=1}^{n} x_i\Big), \quad \text{for all } x_1, \ldots, x_n \in S, \quad n \in \mathbb{N}. \tag{4}$$

Proof. (i) implies (ii). For every $y^* \in Y^*$, $\|y^*\| = 1$ we have

$$y^*(F(x \cdot y)) + f(x \cdot y) \leq y^*(F(x)) + f(x) + y^*(F(y)) + f(y),$$

and, consequently,

$$\begin{aligned} &\|F(x) + F(y) - F(x \cdot y)\| \\ &\quad = \sup\{y^*(F(x) + F(y) - F(x \cdot y)) : \ y^* \in Y^*, \ \|y^*\| = 1\} \\ &\quad \leq f(x) + f(y) - f(x \cdot y), \ x, y \in S. \end{aligned}$$

(ii) implies (iii). The proof runs by induction on n. The case $n = 1$ is trivial, while for $n = 2$ the inequality (4) is identical with (3). Now suppose (4) to be true for an $n \in \mathbb{N}$, $n > 2$. Take arbitrary $x_1, \ldots, x_{n+1} \in S$. By (ii) and the induction hypothesis we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{n+1} F(x_j) - F\left(\prod_{j=1}^{n+1} x_j\right)\right\| &= \left\|\sum_{j=1}^{n} F(x_j) + F(x_{n+1}) - F\left(\prod_{j=1}^{n} x_j \cdot x_{n+1}\right)\right\| \\ &= \left\|\sum_{j=1}^{n} F(x_j) - F\left(\prod_{j=1}^{n} x_j\right) + F\left(\prod_{j=1}^{n} x_j\right) + F(x_{n+1})\right. \\ &\quad \left. -F\left(\prod_{j=1}^{n} x_j \cdot x_{n+1}\right)\right\| \le \left\|\sum_{j=1}^{n} F(x_j) - F\left(\prod_{j=1}^{n} x_j\right)\right\| \\ &\quad + \left\|F(x_{n+1}) + F\left(\prod_{j=1}^{n} x_j\right) - F\left(\prod_{j=1}^{n+1} x_j\right)\right\| \\ &\le \sum_{j=1}^{n} f(x_j) - f\left(\prod_{j=1}^{n} x_j\right) \\ &\quad + f(x_{n+1}) + f\left(\prod_{j=1}^{n} x_j\right) - f\left(\prod_{j=1}^{n+1} x_j\right) \\ &= \sum_{j=1}^{n+1} f(x_j) - f\left(\prod_{j=1}^{n+1} x_j\right).\end{aligned}$$

(iii) implies (ii). Trivial.

(ii) implies (i). Let $y^* \in Y^*, \|y^*\| = 1$ be arbitrary. For $x, y \in S$, we have

$$\begin{aligned}y^*(F(x) + F(y) - F(x \cdot y)) &\le \|F(x) + F(y) - F(x \cdot y)\| \\ &\le f(x) + f(y) - f(x \cdot y),\end{aligned}$$

or, equivalently,

$$y^*(F(x \cdot y)) + f(x \cdot y) \le y^*(F(x)) + f(x) + y^*(F(y)) + f(y),$$

which completes the proof. □

Immediately from the above proposition we obtain the following result.

Corollary 1. *Under the assumptions of the previous proposition if $F : S \to Y$ is a delta-subadditive map with a control function $f : S \to \mathbb{R}$ then*

(i) $\|nF(x) - F(x^n)\| \le nf(x) - f(x^n), \quad x \in S,\ n \in \mathbb{N},$
(ii) *If $e \in S$ is a neutral element of semigroup S, then* $\|F(e)\| \le f(e)$.

In the sequel we will use the following notation:

$$\begin{aligned}D_s(S) := \{\overline{F} := (F, f) : F : S \to Y \\ \text{is delta-subadditive with control function } f : S \to \mathbb{R}\}\end{aligned}$$

3. A Partially Order connected with the Notion of Delta-Subadditivity

In this place we give the definition and basic properties of so-called the Lorenz cone and partially order generated by this cone. As we will see this partial order appears in a natural way in the connection of the concept of delta-subadditivity.

Let us recall that a nonempty subset $\mathcal{C}$ of a vector space is said to be a pointed, convex cone if it satisfies the following properties:

(i) $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$,
(ii) $\alpha\mathcal{C} \subseteq \mathcal{C}$, for all $\alpha \geq 0$,
(iii) $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

An arbitrary pointed, convex cone $\mathcal{C}$ of a vector space Y induces a vector ordering $\preceq_{\mathcal{C}}$ letting $x \preceq_{\mathcal{C}} y$ whenever $y - x \in \mathcal{C}$. This partial order is compatible with the linear structure of Y in the sense that if $x \preceq_{\mathcal{C}} y$, then

(1) $x + z \preceq_{\mathcal{C}} y + z$, for each $z \in Y$,
(2) $\alpha x \preceq_{\mathcal{C}} \alpha y$, for all $\alpha \geq 0$.

Now, for a normed space $(Y, \|\cdot\|)$ consider the linear space $\overline{Y} := Y \times \mathbb{R}$, where as usual, the addition and scalar multiplication are defined coordinatewise. Given a positive number ε, the convex cone defined by the formula

$$\mathcal{C}_\varepsilon := \{(x,t) \in \overline{Y} :\ \varepsilon\|x\| \leq t\}$$

is called the *Lorenz cone* or *ice cream cone*. More informations about the Lorenz cone can be found for instance in the book [1]. It is easy to check that this cone is closed, pointed and defines a vector ordering on $\overline{Y}$ in the following manner

$$(x_1, t_1) \preceq_{\mathcal{C}_\varepsilon} (x_2, t_2) \ \Leftrightarrow\ \varepsilon\|x_2 - x_1\| \leq t_2 - t_1.$$

In 1962 Bishop and Phelps [4] introduced a slightly different order in functional analysis. The proof of the Bishop–Phelps lemma on the existence of certain minimal elements uses this order concept. As a consequence of this lemma one obtains the celebrated Bishop–Phelps theorem which says that the set of support functionals of a non-empty closed, bounded and convex subset of a real Banach space is dense in its dual space.

Now, for an arbitrary map $F : S \to Y$ and a function $f : S \to \mathbb{R}$ let define a map $\overline{F} : S \to \overline{Y}$ via the formula:

$$\overline{F}(x) := (F(x), f(x)), \quad x \in S.$$

Observe that we can rewrite the inequality defining the notion of delta-subadditivity of F in the form

$$\overline{F}(x \cdot y) \preceq_{\mathcal{C}_1} \overline{F}(x) + \overline{F}(y), \quad x, y \in S,$$

where $\mathcal{C}_1 = \{(x,t) \in \overline{Y} : \|x\| \leq t\}$ is the Lorenz cone. In the sequel for $Y_1, Y_2 \in \overline{Y}$ we will write $Y_1 \preceq Y_2$ instead of $Y_1 \preceq_{\mathcal{C}_1} Y_2$.

4. The Separation Theorem

The following lemma corresponds to the Lemma 2 from [9].

Lemma 2. *Let $(S,\cdot)$ be a weakly commutative semigroup, let $(Y,\|\cdot\|)$ be a real Banach space and assume that $\overline{F}=(F,f)\in D_s(S)$ and $-\overline{G}=(-G,-g)\in D_s(S)$. If*

$$\overline{G}(x)\preceq \overline{F}(x),\quad x\in S, \tag{5}$$

then there exist $\overline{F}_1=(F_1,f_1)$, $\overline{G}_1=(G_1,g_1):D\to\overline{Y}$ such that

(a) $\overline{G}(x)\preceq\overline{G_1}(x)\preceq\overline{F_1}(x)\preceq\overline{F}(x),\quad x\in S,$
(b) $\overline{F}_1,-\overline{G}_1\in D_s(S)$,
(c) $\overline{F}_1,\ \overline{G}_1$ *are* $\mathbb{N}-$ *homogeneous,*
(d) $\overline{F}_1(x\cdot y)=\overline{F}_1(y\cdot x)$ *and* $\overline{G}_1(x\cdot y)=\overline{G}_1(y\cdot x),\quad x,y\in S,$
(e) *moreover, if f (resp. g) is $\mathbb{N}$-homogeneous at a point $x_0\in X$, then $\overline{F}_1(x_0)=\overline{F}(x_0)$ (resp. $\overline{G}_1(x_0)=\overline{G}(x_0)$).*

Proof. On account of condition (i) from Corollary 1 we have

$$\|2^nF(x)-F(x^{2^n})\|\le 2^nf(x)-f(x^{2^n}),\quad x\in S,\quad n\in\mathbb{N}, \tag{6}$$

and

$$\|2^nG(x)-G(x^{2^n})\|\le g(x^{2^n})-2^ng(x),\quad x\in S,\quad n\in\mathbb{N}. \tag{7}$$

Define the sequences $\overline{F}_n,\overline{G}_n:S\to\overline{Y}$ by the formulas

$$\overline{F}_n(x):=(F_n(x),f_n(x)),\quad \overline{G}_n(x):=(G_n(x),g_n(x)),$$

where

$$F_n(x):=\frac{1}{2^n}F(x^{2^n}),\quad f_n(x):=\frac{1}{2^n}f(x^{2^n}),$$

and,

$$G_n(x):=\frac{1}{2^n}G(x^{2^n}),\quad g_n(x):=\frac{1}{2^n}g(x^{2^n}).$$

By virtue of (5), (6), (7) we have

$$\overline{G}(x)\preceq\overline{G}_n(x)\preceq\overline{F}_n(x)\preceq\overline{F}(x),\quad x\in S.$$

Observe that the sequence $\{\overline{F}_n\}_{n\in\mathbb{N}}$ is decreasing, $\{\overline{G}_n\}_{n\in\mathbb{N}}$ is increasing in the sense of an order generated by a Lorenz cone. Indeed, for the sequence $\{\overline{F}_n\}_{n\in\mathbb{N}}$ we get

$$\begin{aligned}\|F_{n+1}(x)-F_n(x)\|&=\left\|\frac{1}{2^{n+1}}F(x^{2^{n+1}})-\frac{1}{2^n}F(x^{2^n})\right\|\\&=\frac{1}{2^{n+1}}\left\|F((x^{2^n})^2)-2F(x^{2^n})\right\|\\&\le\frac{1}{2^{n+1}}\left(2f(x^{2^n})-f(x^{2^{n+1}})\right)\\&=\frac{1}{2^n}f(x^{2^n})-\frac{1}{2^{n+1}}f(x^{2^{n+1}})\\&=f_n(x)-f_{n+1}(x),\end{aligned}$$

and similarly for the sequence $\{\overline{G}_n\}_{n\in\mathbb{N}}$. Therefore

$$\overline{G}(x) \preceq \overline{G}_n(x) \preceq \overline{G}_{n+1}(x) \preceq \overline{F}_{n+1}(x) \preceq \overline{F}_n(x) \preceq \overline{F}(x), \quad x \in S, \quad n \in \mathbb{N}.$$

In particular, we have

$$g(x) \le g_n(x) \le g_{n+1}(x) \le f_{n+1}(x) \le f_n(x) \le f(x), \quad x \in S, \quad n \in \mathbb{N}.$$

For each, fixed $x \in S$, the sequences $\{f_n(x)\}_{n\in\mathbb{N}}$ and $\{g_n(x)\}_{n\in\mathbb{N}}$ being monotone and bounded, are convergent in $\mathbb{R}$. Therefore we may define $f_0, g_0 : S \to \mathbb{R}$ by

$$f_0(x) := \lim_{n\to\infty} f_n(x), \quad g_0(x) := \lim_{n\to\infty} g_n(x).$$

Since the sequences $\{f_n(x)\}_{n\in\mathbb{N}}$ and $\{g_n(x)\}_{n\in\mathbb{N}}$ are convergent, in particular, they are the Cauchy sequences. Observe that $\{F_n(x)\}_{n\in\mathbb{N}}$ and $\{G_n(x)\}_{n\in\mathbb{N}}$ also are the Cauchy sequences. Indeed, for arbitrary $n, k \in \mathbb{N}$ we have

$$\begin{aligned}\|F_{n+k}(x) - F_n(x)\| &= \left\| \frac{1}{2^{n+k}} F(x^{2^{n+k}}) - \frac{1}{2^n} F(x^{2^n}) \right\| \\ &= \frac{1}{2^{n+k}} \|F((x^{2^n})^{2^k}) - 2^k F(x^{2^n})\| \\ &\le \frac{1}{2^{n+k}} \left(2^k f(x^{2^n}) - f(x^{2^{n+k}})\right) \\ &= f_n(x) - f_{n+k}(x).\end{aligned}$$

The proof for the sequence $\{G_n\}_{n\in\mathbb{N}}$ is similar. By the completeness of the space Y we can define functions F_0, G_0 as

$$F_0(x) := \lim_{n\to\infty} F_n(x), \quad G_0(x) := \lim_{n\to\infty} G_n(x).$$

Observe that $(F_0, f_0), (-G_0, -g_0) \in D_s(S)$. To see it, fix $x, y \in S$. Using the week commutativity of S we can find a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and

$$(x \cdot y)^{2^{n_k}} = x^{2^{n_k}} \cdot y^{2^{n_k}}, \quad k \in \mathbb{N}.$$

Then we get

$$\begin{aligned}&\|F_{n_k}(x) + F_{n_k}(y) - F_{n_k}(x \cdot y)\| \\ &= \left\| \frac{1}{2^{n_k}} F(x^{2^{n_k}}) + \frac{1}{2^{n_k}} F(y^{2^{n_k}}) - \frac{1}{2^{n_k}} F(x^{2^{n_k}} \cdot y^{2^{n_k}}) \right\| \\ &\le \frac{1}{2^{n_k}} \left(f(x^{2^{n_k}}) + f(y^{2^{n_k}}) - f(x^{2^{n_k}} \cdot y^{2^{n_k}}) \right) \\ &= f_{n_k}(x) + f_{n_k}(y) - f_{n_k}(x \cdot y).\end{aligned}$$

Tending to the limit with $n \to \infty$ we obtain

$$\|F_0(x) + F_0(y) - F_0(x \cdot y)\| \le f_0(x) + f_0(y) - f_0(x \cdot y), \quad x, y \in S,$$

which means that F_0 is delta-subadditive with a control function f_0. A similar argument ensures the delta-superadditivity of G_0. Moreover, since

$$\|F_n(x) - G_n(x)\| \le f_n(x) - g_n(x), \quad x \in S, \quad n \in \mathbb{N},$$

then

$$\|F_0(x) - G_0(x)\| \le f_0(x) - g_0(x), \quad x \in S,$$

which means that $\overline{G}(x) \preceq \overline{F}(x),\ x \in S$. Further, observe that for each $x \in S$, one has

$$f_0(x^2) = \lim_{n\to\infty} \frac{1}{2^n} f((x^2)^{2^n}) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^{n+1}}) = 2 \lim_{n\to\infty} \frac{1}{2^{n+1}} f(x^{2^{n+1}}) = 2f_0(x),$$

and analogously,

$$g_0(x^2) = 2g_0(x), \quad x \in S.$$

On account of Lemma 1 and condition (i) from Corollary 1 the last two identities guarantee the $\mathbb{N}$-homogeneity of $\overline{F}_0$ and $\overline{G}_0$, whereas Lemma 1 also implies (d). Finally, if f is $\mathbb{N}$-homogeneous at $x_0 \in S$ then $f_n(x_0) = f(x_0),\ n \in \mathbb{N}$. The same argument works for g_0. □

We apply the above lemma to the proof of the following theorem.

Theorem 1. *Let $(S, \cdot)$ be a weakly commutative semigroup,and let $(Y, \|\cdot\|)$ be a real Banach space. Assume that $F : S \to Y$ is a delta-subadditive map with a control function $f : S \to \mathbb{R}$ and $G : S \to Y$ is a delta-superadditive map with a control function $g : S \to \mathbb{R}$. Suppose that $(G, g) \preceq (F, f)$, i.e.*

$$\|F(x) - G(x)\| \le f(x) - g(x), \quad x \in S.$$

If, moreover,

$$\sup\{f(x) - g(x) : \ x \in S\} < \infty,$$

then there exist unique additive mappings $A : S \to Y$ and $a : S \to \mathbb{R}$ such that

$$(G(x), g(x)) \preceq (A(x), a(x)) \preceq (F(x), f(x)), \quad x \in S.$$

Proof. Let $F_1, G_1 : S \to Y$ and $f_1, g_1 : S \to \mathbb{R}$ be the mappings associated with $F, G : S \to Y$ and $f, g : S \to \mathbb{R}$ according to Lemma 2. Using assertions (a) and (c) of that lemma, we obtain

$$n\|F_1(x) - G_1(x)\| = \|F_1(x^n) - G_1(x^n)\| \le f_1(x^n) - g_1(x^n) \le M,$$

for every $x \in S$ and $n \in \mathbb{N}$, where $M = \sup\{f(x) - g(x) : x \in S\}$. Whence

$$\|F_1(x) - G_1(x)\| \le f_1(x) - g_1(x) \le \frac{M}{n},$$

for $x \in S,\ n \in \mathbb{N}$, consequently passing to the limit as $n \to \infty$, we infer that $F_1 = G_1,\ f_1 = g_1$. Obviously $A := F_1 = G_1$ and $a := f_1 = g_1$ are additive, moreover,

$$(G, g) \preceq (A, a) \preceq (F, f).$$

Now, suppose that $A_1 : S \to Y$ and $a_1 : S \to \mathbb{R}$ are another mappings that $\overline{A}_1 := (A_1, a_1)$ separate (F, f) and (G, g). Then

$$n\|A(x) - A_1(x)\| = \|A(x^n) - A_1(x^n)\| \le f(x^n) - g(x^n) \le M,$$

for each $x \in S$ and $n \in \mathbb{N}$. Dividing by n and passing to the limit as $n \to \infty$ we obtain

$$A(x) = A_1(x), \quad \text{and} \quad a(x) = a_1(x).$$

This proves the uniqueness of A and a. □

As an application of this theorem we obtain an easy proof of the classical Hyers–Ulam stability result for the Cauchy equation. For the theory of the stability of functional equations see Hyers et al. [13].

Corollary 2. *Let $(S, \cdot)$ be a weakly commutative semigroup, and let $(Y, \|\cdot\|)$ be a real Banach space. If $F : S \to Y$ is an ε-additive map i.e.*

$$\|F(x) + F(y) - F(x \cdot y)\| \leq \varepsilon, \quad x, y \in S,$$

where $\varepsilon > 0$, then there exists a unique additive map $A : S \to Y$ such that

$$\|F(x) - A(x)\| \leq \varepsilon, \quad x \in S.$$

Proof. Observe that by our assumption F is a delta-subadditive mapping with a control function $f(x) := \varepsilon,\ x \in S$, and it is a delta-superadditive with a control function $g(x) := -\varepsilon,\ x \in S$. Since

$$\sup_{x \in S}[f(x) - g(x)] = 2\varepsilon < \infty,$$

and

$$(F(x), -\varepsilon) \preceq (F(x), \varepsilon), \quad x \in S,$$

then by Theorem 1 there exist unique additive mappings $A : S \to Y$ and $a : S \to \mathbb{R}$ such that

$$\|F(x) - A(x)\| \leq \varepsilon - a(x), \quad \text{and} \quad \|F(x) - A(x)\| \leq a(x) + \varepsilon,$$

which means that

$$\|F(x) - A(x)\| \leq \varepsilon,$$

and finishes the proof of our theorem. □

5. Support Theorem

In this section $(X, +)$ is assumed to be a uniquely 2-divisible abelian group. It means that the mapping $\omega : X \to X,\ \omega(x) = 2x,\ x \in X$ is bijective. Then both ω and ω^{-1} are authomorphism of $(X, +)$, and we write $\frac{1}{2}x$ for $\omega^{-1}(x)$. In the sequel we will use an additive notation $+$ for a group operation and 0 stands for a neutral element of X.

We consider the following problem: Whether for a given delta-subadditive map $F : X \to Y$ with a control function $f : X \to \mathbb{R}$ and a given point $y \in X$ there exist additive functions $A_y : X \to Y,\ a_y : X \to \mathbb{R}$ such that a map

$\overline{A_y} = (A_y, a_y) : X \to \overline{Y}$ support $\overline{F} = (F, f)$ at y in the sense of an order generated by a Lorenz cone. It means that

$$\|F(x) - A_y(x)\| \leq f(x) - a_y(x), \quad x \in X \tag{8}$$

and

$$A_y(y) = f(y), \quad a_y(y) = f(y). \tag{9}$$

In the proof of our main result of this section we apply the following theorem, which is a particular case of Theorem 4 proved in [18] (Actually this theorem has been proved in the case when X is a real Banach space but its proof in our case runs without any essential changes).

Theorem 2. *Assume that $(X, +)$ is a uniquely divisible by 2 abelian group, and $(Y, \|\cdot\|)$ is a real Banach space. If $F : X \to Y$ is a delta Jensen-convex map with a control function $f : X \to \mathbb{R}$ that is*

$$\left\|F(x) + F(z) - 2F\Big(\frac{x+z}{2}\Big)\right\| \leq f(x) + f(z) - 2f\Big(\frac{x+z}{2}\Big), \quad x, z \in X, \tag{10}$$

then for an arbitrary point $y \in X$ there exist affine maps $B_y : X \to Y$ and $b_y : X \to \mathbb{R}$ such that

$$\|F(x) - B_y(x)\| \leq f(x) - b_y(x), \quad x \in X, \tag{11}$$

moreover,

$$B_y(y) = f(y), \quad b_y(y) = f(y). \tag{12}$$

Main result of this section reads as follows.

Theorem 3. *Let $(X, +)$ be a uniquely divisible by 2 abelian group and let $(Y, \|\cdot\|)$ be a real Banach space. Let $F : X \to Y$ be a delta-subadditive map with a control function $f : X \to \mathbb{R}$, and let $y \in Y$ be arbitrary. Then there exist additive maps $A_y : X \to Y$ and $a_y : X \to \mathbb{R}$ such that (8) and (9) hold if and only if f is $\mathbb{N}$-homogeneous at y.*

Proof. Assume that $(F, f) \in D_s(X)$, moreover,

$$f(y^n) = nf(y), \quad n \in \mathbb{N}.$$

Observe, that $\mathbb{N}$-homogeneity of F at y follows from the condition (i) from Corollary 1. Let define the map $\overline{G} : X \to \overline{Y}$ by formula

$$\overline{G}(x) = (G(x), g(x)) := (-F(-x), -f(-x)), \quad x \in X.$$

Obviously, $-\overline{G} \in D_s(X)$, moreover, using a delta-subadditivity of F and condition (ii) from Corollary 1 we obtain

$$\begin{aligned}
\|F(x) - G(x)\| &= \|F(x) + F(-x) - F(0) + F(0)\| \\
&\leq \|F(x) + F(-x) - F(0)\| + \|F(0)\| \\
&\leq f(x) + f(-x) - f(0) + f(0) \\
&= f(x) + f(-x) = f(x) - g(x),
\end{aligned}$$

which means that

$$\overline{G}(x) \preceq \overline{F}(x), \quad x \in X.$$

On account of Lemma 2 there exist $\mathbb{N}$-homogeneous maps $\overline{F}_0, \overline{G}_0 : X \to \overline{Y}$ such that $\overline{F}_0, -\overline{G}_0 \in D_s(X)$ and

$$\overline{F}(x) \preceq \overline{F}_0(x) \preceq \overline{G}_0(x) \preceq \overline{G}(x), \quad x \in X.$$

Since f and g are $\mathbb{N}$-homogeneous at y then by condition (e) of Lemma 2 we know that

$$\overline{F}(y) = \overline{F_0}(y), \quad \text{and} \quad \overline{G}(y) = \overline{G_0}(y).$$

Observe that F_0 is a delta Jensen-convex map with a control function f_0. Indeed, for any $x, z \in X$ by 2-homogeneity of F_0 and f_0 we get

$$\begin{aligned} \left\| F_0(x) + F_0(z) - 2F_0\Big(\frac{x+z}{2}\Big) \right\| &= \|F_0(x) + F_0(z) - F_0(x+z)\| \\ &\le f_0(x) + f_0(z) - f_0(x+z) \\ &= f_0(x) + f_0(z) - 2f_0\Big(\frac{x+z}{2}\Big). \end{aligned}$$

Now, we are able to apply the Theorem 2. By virtue of this theorem there are affine maps $B_y : X \to Y$ and $b_y : X \to \mathbb{R}$ such that

$$\|F_0(x) - B_y(x)\| \le f_0(x) - b_y(x), \quad x \in X, \tag{13}$$

moreover,

$$B_y(y) = F_0(y), \quad b_y(y) = f_0(y). \tag{14}$$

It is known (see for instance [15]) that the maps B_y and b_y have the form

$$B_y(x) = A_y(x) + C, \quad b_y(x) = a_y(x) + c, \quad x \in X, \tag{15}$$

where $A_y : X \to Y, a_y : X \to \mathbb{R}$ are additive maps and $C \in X$ and $c \in \mathbb{R}$ are constants. We need to show that $C = 0$ and $c = 0$. By (13), (14) and $\mathbb{N}$-homogeneity of A_y, a_y, F_0 and f_0 for all $x \in X$ we obtain

$$\|F_0(2^n x) - A_y(2^n x) - C\| = \|2^n F_0(x) - 2^n A_y(x) - C\| \le 2^n f_0(x) - 2^n a_y(x) - c.$$

The above inequality is equivalent to the following one

$$\left\| F_0(x) - A_y(x) - \frac{C}{2^n} \right\| \le f_0(x) - a_y(x) - \frac{c}{2^n}, \quad x \in X,$$

therefore,

$$\|F_0(x) - A_y(x)\| \le f_0(x) - a_y(x), \quad x \in X.$$

By (14) and (15) we have

$$C = F_0(y) - A_y(y), \quad \text{and} \quad c = f_0(y) - a_y(y),$$

whence,

$$\|C\| = \|F_0(y) - A_y(y)\| \le f_0(y) - a_y(y) = c. \tag{16}$$

On the other hand, using again the $\mathbb{N}$-homogeneity of A_y, a_y, F_0 and f_0 for all $x \in X$ we obtain

$$\left\| \frac{1}{2^n} F_0(x) - \frac{1}{2^n} A_y(x) - C \right\| \leq \frac{1}{2^n} f_0(x) - \frac{1}{2^n} a_y(x) - c,$$

or, equivalently,

$$\|F_0(x) - A_y(x) - 2^n C\| \leq f_0(x) - a_y(x) - 2^n c.$$

In particular,

$$2^n c \leq f_0(x) - a_y(x), \quad x \in X, \quad n \in \mathbb{N},$$

hence $c \leq 0$ and this together with (16) implies that $c = 0$ and $C = 0$.

To end the proof of sufficiency it is enough to observe that

$$\begin{aligned} \|F(x) - A_y(x)\| &\leq \|F(x) - F_0(x)\| + \|F_0(x) - A_y(x)\| \\ &\leq f(x) - f_0(x) + f_0(x) - a_y(x) = f(x) - a_y(x), \quad x \in X. \end{aligned}$$

Conversely, observe that the $\mathbb{N}$-homogeneity at $y \in X$ is necessary for existence of additive map (A_y, a_y) supporting (F, f) at y. Indeed, if $(F, f) \in D_s(X)$ and additive map (A_y, a_y) supports (F, f) at y, then for any $n \in \mathbb{N}$ we have

$$\begin{aligned} \|F(ny) - nF(y)\| &\leq \|F(ny) - A_y(ny)\| + \|nA_y(y) - nF(y)\| \\ &= \|F(ny) - A_y(ny)\| \leq f(ny) - a_y(ny) \\ &= f(ny) - na_y(y) \leq nf(y) - na_y(y) = 0. \end{aligned}$$

□

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