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Results in Mathematics

Erratum

Erratum to: Differentiated Generalized Voronovskaja's Theorem in Compact Disks

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The original article has been published with an error in the proof of Theorem 2.2, case (ii) (when $k-1 \geq 2p$). Additional assumptions are needed for the inverse estimate in Theorem 2.2, case $k-1 \geq 2p$ to hold. The correct version of Theorem 2.2 for the case $k-1 \geq 2p$ is given below.

1. Introduction

For $f : \mathbb{D}_R \to \mathbb{C}$ analytic in $\mathbb{D}_R$, say $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{D}_R$, let us consider the sequence of complex Bernstein polynomials attached to f by $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n), n \in \mathbb{N}, z \in \mathbb{D}_R$.

Also, denote $T_{n,j}(z) := \sum_{k=0}^{n}(k-nz)^j \binom{n}{k} z^k (1-z)^{n-k}$ and $E_{n,p,k}(f)(z) := [B_n(f)(z) - f(z) - \sum_{j=1}^{2p} \frac{f^{(j)}(z)}{j!} n^{-j} T_{n,j}(z)]^{(k)}$.

Theorem 2.2 in [1] states that for all $p, k \in \mathbb{N}$ and under the hypothesis that f is not a polynomial of degree $\leq \max\{k-1, 2p\}$, for any $1 \leq r < R$ we have the exact estimate $\|E_{n,p,k}(f)\|_r \sim \frac{1}{n^{p+1}}$, $n \in \mathbb{N}$, where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$.

While the above exact estimate is correct for $k < 2p+1$, unfortunately for $k \geq 2p+1$ is valid the upper estimate only (by Theorem 2.1 in [1]), as the following counterexample shows. For $p = 1$ and $k = 3$ (that is for $k = 2p+1$), choose

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$f(z) = z^3$. Denoting $e_m(z) = z^m$, by $B_n(e_3)(z) = z^3 + \frac{3z^2(1-z)}{n} + \frac{z(1-z)(1-2z)}{n^2}$, a simple calculation implies $\|E_{n,1,3}(e_3)\|_r \sim \frac{1}{n}$, while Theorem 2.2 in [1] would imply $\|E_{n,1,3}(e_3)\|_r \sim \frac{1}{n^2}$.

2. Main Result

Analysing the case $k \geq 2p+1$ (Case (ii)) in the proof of Theorem 2.2 in [1], it can be recovered under some simple additional hypothesis, as follows.

Theorem 2.1. *Let $R > 1, p, k \in \mathbb{N}$ and $f : \mathbb{D}_R \to \mathbb{C}$ be an analytic function in $\mathbb{D}_R$, say $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) *Let $k = 2p+1$. If f is not a polynomial of degree $\leq k-1$ and $\{f^{(2p+1)}(0) = 0$ or $f^{(2p+1)}(1) = 1\}$, then for any $1 \leq r < R$ we have*

$$\left\| \left[B_n(f) - f - \sum_{j=1}^{2p} \frac{f^{(j)}}{j!} n^{-j} T_{n,j} \right]^{(k)} \right\|_r \sim \frac{1}{n^{p+1}}, \quad n \in \mathbb{N}, \tag{2.1}$$

where the constants in the equivalence depend only on f, r, p, k but are independent of n.

(ii) *Let $k = 2p+2$ or $k = 2p+3$. If f is not a polynomial of degree $\leq k-1$ and $\{f^{(2p+1)}(0) = f^{(2p+1)}(1) = f^{(2p+2)}(1/2) = 0\}$, then again (2.1) holds.*

(iii) *More general, let $k = 2p+s$ with $s \geq 4$. If f is not a polynomial of degree $\leq k-1$ and*

$$f^{(2p+1)}(0) = f^{(2p+2)}(0) = \cdots = f^{(2p+s-1)}(0) = f^{(2p+2)}(1/2) = f^{(2p+1)}(1) = 0,$$

or

$$f^{(2p+1)}(1) = f^{(2p+2)}(1) = \cdots = f^{(2p+s-1)}(1) = f^{(2p+2)}(1/2) = f^{(2p+1)}(0) = 0,$$

then again (2.1) holds.

Proof. By Theorem 2.1 in [1], in the above points (i), (ii) and (iii) what remain to be proved is the lower estimate in (2.1). Then, keeping the notations for U in the proof of Theorem 2.2 in [1], namely

$$U(z) := \left[\frac{a_p}{(2p+1)!}(1-2z)[z(1-z)]^p f^{(2p+1)}(z) + \frac{[z(1-z)]^{p+1}}{2^{p+1}(p+1)!} f^{(2p+2)}(z) \right]^{(k)},$$

where $a_p > 0$, what remains to prove is that $\|U\|_r > 0$ (see the reasoning in the proof of Theorem 2.2 in [1]).

For that purpose, supposing the contrary we get that f necessarily satisfies the differential equation $U(z) = 0$ for all $|z| \leq r$, which by the substitution $f^{(2p+1)}(z) := y(z)$, implies that $y(z)$ necessarily is analytic in $\mathbb{D}_R$ (since f is supposed analytic there) and is a solution of the differential equation

$$\frac{a_p}{(2p+1)!}(1-2z)[z(1-z)]^p y(z) + \frac{[z(1-z)]^{p+1}}{2^{p+1}(p+1)!} y'(z) = P_{k-1}(z), \quad |z| \le r,$$

where $P_{k-1}(z)$ is a polynomial of degree $\le k-1$.

But $k \ge 2p+1$ necessarily implies that $P_{k-1}(z) = [z(1-z)]^p Q_l(z)$, where $l = k-1-2p \ge 0$ and $Q_l(z)$ is a polynomial of degree $\le l$. Dividing by $[z(1-z)]^p$, we obtain that $y(z)$ is an analytic function in $\mathbb{D}_R$, satisfying the differential equation (here recall that $a_p > 0$)

$$\frac{a_p}{(2p+1)!}(1-2z)y(z) + \frac{z(1-z)}{2^{p+1}(p+1)!} y'(z) = Q_l(z), \quad |z| \le r,\ z \ne 0,\ z \ne 1. \quad (2.2)$$

(i) Let $k = 2p+1$, that is above we have $l = 0$ and $Q_l(z)$ is a constant. By hypothesis, in (2.2) we have $y(0) = 0$ or $y(1) = 0$, which implies $Q_l(0) = 0$ or $Q_l(1) = 0$, that is (2.2) necessarily becomes

$$\frac{a_p}{(2p+1)!}(1-2z)y(z) + \frac{z(1-z)}{2^{p+1}(p+1)!} y'(z) = 0, \quad |z| \le r.$$

Writing $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} b_k z^k$ and replacing in the above differential equation, by comparison of coefficients we easily obtain that $b_k = 0$, for all $k = 0, 1, \ldots$, that is $y(z) = 0$ for all $\overline{\mathbb{D}}_r$. But from the identity theorem of analytic functions, it necessarily follows that $y(z) = 0$ for all $|z| < R$.

Therefore, $f(z)$ necessarily is a polynomial of degree $\le 2p = k-1$, a contradiction with the hypothesis.

(ii) Let $k = 2p+2$ or $k = 2p+3$, that is $l = 1$ or $l = 2$ in the differential equation (2.2). By the hypothesis on f it follows $Q_l(0) = Q_l(1) = Q_l(1/2) = 0$, where $Q_l(z)$ is a polynomial of degree ≤ 2, which necessarily implies that Q_l is identically equal to zero. In continuation, reasoning exactly as in the proof of the above point (i), it necessarily follows that $y(z) = 0$ for all $|z| < R$, that is $f(z)$ necessarily is a polynomial of degree $\le 2p < k-1$, contradicting the hypothesis.

(iii) Let $k = 2p+s$ with $s \ge 4$. It follows that in the differential equation (2.2), we have $l = s-1$. Differentiating successively (2.2) until the order $s-3$ (including $s-3$), it is easy to check that we get equalities of the form

$$c_1^{(1)} y(z) + c_2^{(1)}(1-2z)y'(z) + c_3^{(1)} z(1-z)y''(z) = Q_l'(z),$$
$$c_1^{(2)} y'(z) + c_2^{(2)}(1-2z)y''(z) + c_3^{(2)} z(1-z)y'''(z) = Q_l''(z),$$

and so on, finally we get

$$c_1^{(s-3)} y^{(s-4)}(z) + c_2^{(s-3)}(1-2z)y^{(s-3)}(z) + c_3^{(s-3)} z(1-z)y^{(s-2)}(z) = Q_l^{(s-3)}(z),$$

where $c_k^{(j)}$ are real constants.

Taking now into account the hypothesis on f and that $y(z) = f^{(2p+1)}(z)$, we immediately obtain

$$Q_l(0) = Q_l'(0) = \cdots = Q_l^{(s-3)}(0) = 0, \quad Q_l(1/2) = 0,\ Q_l(1) = 0,$$

or

$$Q_l(1) = Q_l'(1) = \cdots = Q_l^{(s-3)}(1) = 0, \quad Q_l(1/2) = 0,\ Q_l(0) = 0,$$

respectively, which immediately implies that Q_l is identically equal to zero. In continuation, reasoning exactly as in the proof of the above point (i), it necessarily follows that $y(z) = 0$ for all $|z| < R$, that is $f(z)$ necessarily is a polynomial of degree $\leq 2p < k - 1$, contradicting the hypothesis.

So in all the three cases (i), (ii) and (iii) we necessarily have $\|U\|_r > 0$.

For $n \in \{1, \ldots, n_0 - 1\}$, in all the cases (i), (ii) and (iii) we get $\|E_{n,p,k}(f)\|_r \geq \frac{M_{r,n}(f)}{n^{p+1}}$ with $M_{r,n}(f) = n^{p+1} \cdot \|E_{n,p,k}(f)\|_r > 0$, which implies $\|E_{n,p,k}(f)\|_r \geq \frac{C_{p,r}(f)}{n^{p+1}}$, for all $n \in \mathbb{N}$, where $C_{p,r}(f) = \min\{M_{r,1}(f), \ldots, M_{r,n_0-1}(f), \frac{1}{2}\|U\|_r\}$. □

Remark 2.2. Simple functions f satisfying the hypothesis of Theorem 2.1 are of the form $f(z) = z^{2p+s}(1-z)(z-1/2)^2 g(z)$, or $f(z) = (1-z)^{2p+s} z(z-1/2)^2 g(z)$, where $s \geq 4$ and g is an arbitrary analytic function in $\mathbb{D}_R$. We see that $f(z) = z^3$ does not satisfy any hypothesis in Theorem 2.1.

Reference

[1] Gal, S.G.: Differentiated generalized Voronovskaja's theorem in compact disks. Results Math. **61**, 347–353 (2012)

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