Results. Math. 64 (2013), 1–12

1422-6383/13/010001-12
*published online* August 9, 2012
DOI 10.1007/s00025-012-0275-5

**Results in Mathematics**

# On Selections of Set-Valued Inclusions in a Single Variable with Applications to Several Variables

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**Abstract.** We present some applications of the result corresponding to the existence of a unique selection of a set-valued function satisfying inclusions in a single variable to the inclusions in several variables, especially the general linear inclusions or quadratic inclusions.

**Mathematics Subject Classification (2000).** 39B05, 39B82, 54C60, 54C65.

**Keywords.** Set-valued map, selection, inclusion.

## 1. Introduction

The stability theory of functional equations has developed in connection with a problem set by S.M. Ulam during his talk at a conference at the Wisconsin University in 1940. The first answer was given in 1941 by Hyers [5] who proved the following theorem:

*Let $X$ be a linear normed space, $Y$ a Banach space and $\epsilon > 0$. Then for every function $f\colon X \to Y$ satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon, \quad x, y \in X \tag{1}$$

*there exists a unique additive function $g\colon X \to Y$ such that*

$$\|f(x) - g(x)\| \le \epsilon, \quad x \in X. \tag{2}$$

From now on the subject has been intensively investigated by many authors (see for example: [1,3,6,7,10,11,16]).

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Smajdor [18] and Gajda and Ger [4] observed that if $f$ satisfies (1), then the set-valued function $F\colon X \to n(Y)$ ($n(Y)$ denotes the family of all non-empty subsets of $Y$) given by

$$F(x) = f(x) + \overline{B}(0, \epsilon), \quad x \in X,$$

where $\overline{B}(0, \epsilon)$ is the closed ball of radius $\epsilon$ centered at 0, is subadditive (i.e., $F(x + y) \subset F(x) + F(y), x, y \in X$) and the function $g$ from the relation (2) is an additive selection of $F$ (i.e., $g(x + y) = g(x) + g(y)$ and $g(x) \in F(x)$ for $x, y \in X$).

Now one may ask under what conditions a subadditive set-valued function admits an additive selection. We recall the result of Gajda and Ger [4] ($\delta(F(x))$ denotes the diameter of the set $F(x)$).

**Theorem 1.** *Let $(S, +)$ be a commutative semigroup with zero, $X$ a real Banach space and $F\colon S \to 2^X$ a set-valued map with convex and closed values such that*

$$F(x + y) \subset F(x) + F(y), \quad x, y \in S \tag{3}$$

*and* $\sup\{\delta(F(x)) : x \in S\} < \infty$*. Then $F$ admits a unique additive selection.*

Later the above result was extended by Nikodem and Popa to the set-valued functions satisfying the following general linear inclusions:

$$F(ax + by + c) \subset pF(x) + qF(y) + C,$$
$$pF(x) + qF(y) \subset F(ax + by + c) + C,$$

where $a, b, p, q \in \mathbb{R}$, $X$ is a real vector space, $Y$ is a real Banach space, $F\colon X \to n(Y)$, $c \in X, C \in 2^Y$ (see [9,13–15]).

The aim of this paper is to give some modification of Theorem 1 in [12] and its applications. We also show that our theorem generalizes the above results.

## 2. Main Results

Let $(Y, d)$ be a metric space. We will denote by $n(Y)$ the family of all non-empty subsets of $Y$. We understand the convergence of sets with respect to the Hausdorff metric derived from the metric $d$. The number $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ is said to be the diameter of $A \subset Y$. For $F\colon K \to n(Y)$ we denote by $\operatorname{cl} F$ the multifunction defined as $(\operatorname{cl} F)(x) = \operatorname{cl} F(x), x \in K$. A function $f\colon K \to Y$ such that $f(x) \in F(x)$ for all $x \in K$ is called a selection of the multifunction $F$. We write $a^0(x) = x$ for $x \in K$ and $a^{n+1} = a^n \circ a$ for all $n \in \mathbb{N}_0$.

The subsequent theorem is a simple modification of Theorem 1 in [12]. However, we prove it for the convenience of the readers.

**Theorem 2.** *Assume that $K$ is a nonempty set, $(Y, d)$ is a metric space. Let $F\colon K \to n(Y), \Psi\colon Y \to Y, a\colon K \to K, \lambda \in (0, +\infty)$,*

$$d(\Psi(x), \Psi(y)) \le \lambda d(x, y) \quad \text{for } x, y \in Y \tag{4}$$

*and*

$$\lim_{n\to\infty} \lambda^n \delta(F(a^n(x))) = 0 \quad \text{for } x \in K.$$

(1) *If $Y$ is complete and*

$$\Psi(F(a(x))) \subset F(x), \qquad x \in K, \tag{5}$$

*then, for each $x \in K$, the limit $\lim_{n\to\infty} \operatorname{cl} \Psi^n \circ F \circ a^n(x) = f(x)$ exists and $f$ is a unique selection of the multifunction $\operatorname{cl} F$ such that $\Psi \circ f \circ a = f$.*

(2) *If*

$$F(x) \subset \Psi(F(a(x))), \quad x \in K, \tag{6}$$

*then $F$ is a single-valued function and $\Psi \circ F \circ a = F$.*

*Proof.* (1) Let us fix $x \in K$. Replacing $x$ by $a^n(x)$ in (5) we get

$$\Psi(F(a^{n+1}(x))) \subset F(a^n(x))$$

for all $n \in \mathbb{N}_0$. Hence

$$\Psi^{n+1}(F(a^{n+1}(x))) \subset \Psi^n(F(a^n(x))) \qquad \text{for } n \in \mathbb{N}_0.$$

Thus $(\operatorname{cl} \Psi^n(F(a^n(x))))_{n\in\mathbb{N}_0}$ is a decreasing sequence of closed sets in a complete metric space. Moreover, in virtue of (4),

$$\delta(\operatorname{cl} \Psi^n(F(a^n(x)))) \le \lambda^n \delta(F(a^n(x))),$$

so $\lim_{n\to\infty} \delta(\operatorname{cl} \Psi^n(F(a^n(x)))) = 0$. Therefore

$$\lim_{n\to\infty} \operatorname{cl} \Psi^n(F(a^n(x))) = \bigcap_{n\in\mathbb{N}_0} \operatorname{cl} \Psi^n(F(a^n(x))) =: f(x)$$

is a singleton. Of course, $f(x) \in \operatorname{cl} F(x)$ and as $\Psi$ is continuous we have

$$\begin{aligned}\Psi(f(a(x))) &= \Psi(\lim_{n\to\infty} \operatorname{cl} \Psi^n(F(a^n(a(x)))) \subset \lim_{n\to\infty} \operatorname{cl} \Psi^{n+1}(F(a^{n+1}(x)))\\ &= f(x),\end{aligned}$$

thus $\Psi \circ f \circ a = f$.

It remains to show the uniqueness of $f$. Suppose that $f, g$ are selections of $\operatorname{cl} F$ and $\Psi \circ f \circ a = f, \Psi \circ g \circ a = g$. By induction we obtain that $\Psi^n \circ f \circ a^n = f$ and $\Psi^n \circ g \circ a^n = g$ for $n \in \mathbb{N}_0$. Hence, for $x \in K, n \in \mathbb{N}_0$,

$$\begin{aligned}d(f(x), g(x)) &= d\big(\Psi^n \circ f \circ a^n(x), \Psi^n \circ g \circ a^n(x)\big)\\ &\le \lambda^n d(f(a^n(x)), g(a^n(x))) \le \lambda^n \delta(F(a^n(x))).\end{aligned}$$

As $\lim_{n\to\infty} \lambda^n \delta(F(a^n(x))) = 0$, we have $f = g$.

(2) By (6) we obtain

$$F(x) \subset \Psi^n(F(a^n(x))) \subset \Psi^{n+1}(F(a^{n+1}(x))), \quad n \in \mathbb{N},\ x \in K.$$

It follows that $(\Psi^n(F(a^n(x))))_{n\in\mathbb{N}_0}$ is an increasing sequence of sets in a metric space satisfying

$$\delta(\Psi^n(F(a^n(x)))) \le \lambda^n \delta(F(a^n(x))).$$

Hence $\delta(\Psi^n(F(a^n(x))))$ converges to 0 as $n \to \infty$. Consequently, $\Psi^n \circ F \circ a^n(x)$ is single-valued for all $n \in \mathbb{N}_0, x \in K$ and $\Psi \circ F \circ a = F$. □

Obviously, if $\Psi$ is a contraction and $\sup\{\delta(F(x)) : x \in K\} < \infty$, then the limit $\lim_{n\to\infty} \lambda^n \delta(F(a^n(x))) = 0$ and the assertions of Theorem 2 are satisfied.

From now on we assume that $Y$ is a real normed space. By $ccl(Y)$ we denote the family of all nonempty, convex and closed subsets of $Y$. For $A, B \in n(Y)$ and $\lambda \in \mathbb{R}$ we define

$$A + B = \{a + b : a \in A, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a : a \in A\}.$$

It is known (see [8]) that

$$\lambda(A + B) = \lambda A + \lambda B \quad \text{and} \quad (\lambda + \mu)A \subset \lambda A + \mu A$$

for $A, B \in n(Y)$ and $\lambda, \mu \in \mathbb{R}$. If additionally $A$ is convex and $\lambda\mu \ge 0$, then

$$(\lambda + \mu)A = \lambda A + \mu A.$$

Now we give some applications of Theorem 2 to the problem of the stability of set-valued functional equations in several variables.

Notice that Theorem 1 follows from Theorem 2. Indeed, setting $y = x$ in (3) we get

$$F(2x) \subset F(x) + F(x), \quad x \in K.$$

As the set $F(x)$ is convex we have

$$F(2x) \subset 2F(x), \quad x \in K$$

and

$$\frac{1}{2}F(2x) \subset F(x), \quad x \in K.$$

By Theorem 2, with $\Psi(x) = \frac{1}{2}x$ and $a(x) = 2x$, the limit $\lim_{n\to\infty} \Psi^n(F(a^n(x))) = \lim_{n\to\infty} \frac{1}{2^n}F(2^n x) = f(x)$ exists and $f$ is the selection of $F$. Moreover,

$$\frac{1}{2^n}F(2^n(x + y)) \subset \frac{1}{2^n}F(2^n x) + \frac{1}{2^n}F(2^n y)$$

for $n \in \mathbb{N}$, so letting $n \to \infty$ we obtain $f(x + y) = f(x) + f(y)$. Theorem 2 gives the uniqueness of $f$ as well.

If the inverse inclusion is satisfied, i.e,

$$F(x) + F(y) \subset F(x + y) \quad \text{for } x, y \in K,$$

then $F$ must be single-valued. This comes out from Theorem 2, too. We have

$$F(x) \subset \frac{1}{2}F(2x), \quad x \in K,$$

thus, with $\Psi(x) = \frac{1}{2}x$ and $a(x) = 2x$, we obtain that $F$ is single-valued and $F(x+y) = F(x) + F(y)$ for $x, y \in K$.

Next corollaries concern the general linear inclusions and correspond to the results in [9,13].

**Corollary 1.** *Let $X$ be a real vector space, $Y$ be a real Banach space, $K$ be a convex cone in $X, a, b, p, q > 0, F\colon K \to ccl(Y)$,*

$$F(ax+by) \subset pF(x) + qF(y) \quad \textit{for } x, y \in K \tag{7}$$

*and* $\sup\{\delta(F(x)) : \ x \in K\} < \infty$.

(1) *If $p+q > 1$, then there exists a unique selection $f\colon K \to Y$ of the multifunction $F$ such that*

$$f(ax+by) = pf(x) + qf(y) \qquad \textit{for } x, y \in K.$$

(2) *If $p+q < 1$, then $F$ is single-valued.*

*Proof.* (1) Setting $y = x$ in (7) we get

$$F((a+b)x) \subset (p+q)F(x), \qquad x \in K.$$

Dividing both sides of the last inclusion by $p+q$ we have

$$\frac{1}{p+q}F((a+b)x) \subset F(x), \quad x \in K.$$

By Theorem 2, with $\Psi(x) = \frac{1}{p+q}x, a(x) = (a+b)x$, there exists the limit $\lim_{n\to\infty} \Psi^n(F(a^n(x))) = \lim_{n\to\infty} \frac{1}{(p+q)^n}F((a+b)^n x) = f(x), f$ is single-valued and $f(x) \in F(x)$ for $x \in K$. Moreover, the inclusion

$$\frac{F((a+b)^n(ax+by))}{(p+q)^n} \subset p\frac{F((a+b)^n x)}{(p+q)^n} + q\frac{F((a+b)^n y)}{(p+q)^n}, \quad x, y \in K,$$

with $n \to \infty$, yields

$$f(ax+by) = pf(x) + qf(y), \quad x, y \in K.$$

The uniqueness also follows from Theorem 2.

(2) Putting $y = x$ in (7) we have

$$F((a+b)x) \subset (p+q)F(x), \quad x \in K.$$

Now, replacing $x$ by $\frac{1}{a+b}x$ in the last inclusion we obtain

$$F(x) \subset (p+q)F\left(\frac{1}{a+b}x\right), \quad x \in K.$$

Using Theorem 2, with $\Psi(x) = (p+q)x, a(x) = \frac{1}{a+b}x$, we get that $F$ is single-valued and satisfies the equality $F(ax+by) = pF(x) + qF(y)$ for $x, y \in K$. □

By the same method as in the proof of Theorem 2.1 in [13] we can also obtain the same result for the inclusion

$$F(ax+by+k) \subset pF(x)+qF(y), \qquad x,y \in K,$$

where $k \in K, a+b \neq 1$. Taking $x_0 = \frac{k}{1-a-b}$ and defining a multifunction $G\colon K - x_0 \to ccl(Y)$ by $G(x) = F(x+x_0)$ we obtain

$$G(ax+by) \subset pG(x)+qG(y) \quad \text{for } x,y \in K.$$

If $F\colon K \to ccl(Y)$ satisfies, instead of (7), the inclusion

$$F(ax+by+k) \subset pF(x)+qF(y)+C, \quad x,y \in K,$$

where $C$ is a compact and convex subset of $Y, a+b \neq 1, p+q > 1$, then there exists a unique single-valued function $f\colon K \to Y$ satisfying the equation

$$f(ax+by+k) = pf(x)+qf(y), \quad x,y \in K$$

and

$$f(x) \in F(x) + \frac{1}{p+q-1}C, \quad x \in K.$$

It is sufficient, as in [13], to consider the multifunction $G(x) = F(x) + \frac{1}{p+q-1}C$ and use Corollary 1.

**Corollary 2.** *Let $X$ be a real vector space, $Y$ be a real Banach space, $K$ be a convex cone in $X, a, b, p, q > 0, F\colon K \to ccl(Y)$,*

$$pF(x)+qF(y) \subset F(ax+by) \quad \textit{for } x,y \in K \tag{8}$$

*and* $\sup\{\delta(F(x)) :\ x \in K\} < \infty$.

(1) *If $p+q < 1$, then there exists a unique selection $f\colon K \to Y$ of the multifunction $F$ such that*

$$f(ax+by) = pf(x)+qf(y), \quad x,y \in K.$$

(2) *If $p+q > 1$, then $F$ is single-valued.*

*Proof.* (1) Putting $y = x$ in (8) and taking into account that $F$ has convex values we get

$$(p+q)F(x) \subset F((a+b)x), \quad x \in K.$$

Replacing $x$ by $\frac{1}{a+b}x$ in the last inclusion we have

$$(p+q)F\Big(\frac{1}{a+b}x\Big) \subset F(x), \qquad x \in K.$$

Again by Theorem 2, with $\Psi(x) = (p+q)x$ and $a(x) = \frac{1}{a+b}x$, we get that the limit $\lim_{n\to\infty}(p+q)^n F(\frac{1}{(a+b)^n}x) = f(x)$ exists and $f$ is the selection of $F$.

Moreover, by

$$\begin{aligned} & p(p+q)^n F\Big(\frac{1}{(a+b)^n}x\Big) + q(p+q)^n F\Big(\frac{1}{(a+b)^n}y\Big) \\ & \subset (p+q)^n F\Big(\frac{1}{(a+b)^n}(ax+by)\Big) \end{aligned}$$

we obtain

$$pf(x) + qf(y) = f(ax+by) \quad \text{for } x, y \in K.$$

(2) Setting $y = x$ in (8) and dividing both sides of (8) by $p + q$ we get

$$F(x) \subset \frac{1}{p+q} F((a+b)x), \quad x \in K.$$

By Theorem 2, $F$ must be single-valued. □

We can also obtain a similar result if $F$ satisfies

$$pF(x) + qF(y) \subset F(ax + by + k) + C, \quad x, y \in K + x_0,$$

where $x_0 = \frac{k}{1-a-b}, a+b \neq 1, p+q < 1$. Then there exists a unique single-valued map $f\colon K + x_0 \to Y$ such that

$$pf(x) + qf(y) = f(ax + by + k), \quad x, y \in K + x_0$$

and

$$f(x) \in F(x) + \frac{1}{1-a-b}C, \quad x \in K + x_0$$

(see [9]). To obtain this, we define a multifunction $G\colon K \to ccl(Y)$ by

$$G(x) = F(x + x_0) + \frac{1}{1-a-b}C, \quad x \in K.$$

Since the multifunction $G$ satisfies (8) we can use Corollary 2.

Notice that if $p + q = 1$ the above method breaks down. Moreover, if $a = b = \frac{1}{2}$ and $p = q = \frac{1}{2}$, then we get the Jensen inclusions

$$F\Big(\frac{x+y}{2}\Big) \subset \frac{F(x)+F(y)}{2} \quad \text{or} \quad \frac{F(x)+F(y)}{2} \subset F\Big(\frac{x+y}{2}\Big).$$

It easy to see that a multifunction $F\colon \mathbb{R} \to ccl(\mathbb{R})$ given by $F(x) = [x-1, x+1]$ satisfies

$$F\Big(\frac{x+y}{2}\Big) = \frac{F(x)+F(y)}{2}, \quad x, y \in \mathbb{R}$$

and each function $f(x) = x + b$, where $b \in [-1, 1]$ is a Jensen selection of $F$.

Observe also that a constant set-valued function $F(x) = M$, where $M \in ccl(X)$ satisfies inclusions (7), (8) (in fact, $F$ satisfies even the equality) if $p + q = 1$ and each constant function $f(x) = m$, where $m \in M$ satisfies $f(ax + by) = pf(x) + qf(y)$.

Let $(T,\star)$ be a groupoid, where $\star$ is square symmetric, i.e, $(x\star y)\star(x\star y) = (x\star x)\star(y\star y)$ for $x,y\in T$. Then the map $\rho\colon T\to T$ given by $\rho(x):=x\star x$ for $x\in T$ is an endomorphism of the grupoid $(T,\star)$. It is easy to check that

$$x\star y := ax+by+k, \qquad a,b>0, \quad x,y,k\in K,$$

where $K$ is a convex cone, is square symmetric. The operation

$$x\star y := \alpha(x)+\beta(y)+\gamma_0, \quad x,y,\gamma_0\in T$$

is square symmetric as well, where $\alpha,\beta\colon T\to K$ are homomorphisms with $\alpha\circ\beta=\beta\circ\alpha$. Next corollaries complement the above results and correspond to the Corollary 2.8 in [2].

**Corollary 3.** *Let $(T,\star)$ be a grupoid, $S\subset T, \rho(S)\subset S, a,b>0, Y$ be a real Banach space, $F\colon S\to ccl(Y)$,*

$$F(x\star y)\subset pF(x)+qF(y) \quad \text{for } x,y\in S,\ x\star y\in S \tag{9}$$

*and* $\sup\{\delta(F(x)):\ x\in S\}<\infty$.

(1) *If $p+q>1$, then there exists a unique selection $f\colon S\to Y$ of the multifunction $F$ such that*

$$f(x\star y)=pf(x)+qf(y) \qquad \text{for } x,y\in S,\ x\star y\in S.$$

(2) *If $p+q<1$ and $\rho$ is an invertible function, then $F$ is single-valued.*

*Proof.* (1) Setting $y=x$ in (9) and dividing both sides of (9) by $p+q$ we get

$$\frac{1}{p+q}F(\rho(x))\subset F(x), \quad x\in S.$$

Then, by Theorem 2 with $\Psi(x)=\frac{1}{p+q}x, a(x)=\rho(x)$, there exists a limit $\lim_{n\to\infty}\frac{F(\rho^n(x))}{(p+q)^n}=f(x)$ and $f$ is a unique selection of the multifunction $F$ such that

$$f(x\star y)=pf(x)+qf(y), \qquad x,y\in S, \quad x\star y\in S.$$

(2) Putting $y=x$ in (9) we get

$$F(\rho(x))\subset(p+q)F(x), \quad x\in S.$$

As $\rho$ is invertible we have

$$F(x)\subset(p+q)F(\rho^{-1}(x)), \quad x\in S.$$

By Theorem 2, $F$ must be single-valued, which establishes the proof. $\square$

We observe that if

$$F(x\star y)\subset pF(x)+qF(y)+C \quad \text{for } x,y\in S,\ x\star y\in S,$$

where $p+q>1, C$ is a compact and convex subset of $Y$, then $G(x)=F(x)+\frac{1}{p+q-1}C, x\in S$, satisfies the inclusion (9) (see [2]). Thus, by Corollary 3, there exists a unique selection $f$ of $G$ (that is $f(x)\in F(x)+\frac{1}{p+q-1}C, x\in S$) such that

$$f(x \star y) = pf(x) + qf(y), \qquad x, y \in S, \quad x \star y \in S.$$

**Corollary 4.** *Let* $(T, \star)$ *be a grupoid,* $S \subset T, \rho(S) \subset S, a, b > 0, Y$ *be a real Banach space,* $F \colon S \to ccl(Y)$

$$pF(x) + qF(y) \subset F(x \star y), \qquad x, y \in S,\ x \star y \in S \tag{10}$$

*and* $\sup\{\delta(F(x)) :\ x \in S\} < \infty$*.*

(1) *If* $p + q < 1$ *and* $\rho$ *is an invertible function, then there exists a unique selection* $f \colon S \to Y$ *of the multifunction* $F$ *such that*

$$f(x \star y) = pf(x) + qf(y), \qquad x, y \in S, \quad x \star y \in S.$$

(2) *If* $p + q > 1$*, then* $F$ *is single-valued.*

*Proof.* (1) Putting $y = x$ in (10) we get

$$(p + q)F(x) \subset F(\rho(x)), \quad x \in S.$$

As $\rho$ is an invertible function we have

$$(p + q)F(\rho^{-1}(x)) \subset F(x), \quad x \in S.$$

In the same manner by Theorem 2, with $\Psi(x) = (p + q)x, a(x) = \rho^{-1}(x)$, we get the assertion.

(2) Setting $y = x$ in (10) and dividing both sides of the (10) by $p + q$ we get

$$F(x) \subset \frac{1}{p + q} F(\rho(x)), \quad x \in S.$$

Therefore, by Theorem 2, the proof is complete. □

We can also obtain a result similar to the above for $F$ satisfying

$$pF(x) + qF(y) \subset F(x \star y) + C, \qquad x, y \in S,\ x \star y \in S,$$

where $p + q < 1, \rho$ is invertible and $C$ is a compact and convex subset of $Y$. Then, for $G(x) = F(x) + \frac{1}{p+q-1}C, x \in S$, we have

$$pG(x) + qG(y) \subset G(x \star y), \qquad x, y \in S, \quad x \star y \in S$$

and by Corollary 4 there exists a unique selection $f$ of the multifunction $G$ (that is $f(x) \in F(x) + \frac{1}{p+q-1}C, x \in S$) such that

$$f(x \star y) = pf(x) + qf(y), \qquad x, y \in S, \quad x \star y \in S,$$

We end presenting an application of Theorem 2 to the quadratic inclusions.

**Corollary 5.** *Let* $X$ *be a real vector space,* $Y$ *be a real Banach space,* $K$ *be a set in* $X$ *such that for* $x, y \in K, x + y \in K$ *and* $x - y \in K, F \colon K \to ccl(Y)$ *and* $\sup\{\delta(F(x)) :\ x \in K\} < \infty$*.*

(1) *If*

$$F(x + y) + F(x - y) \subset 2F(x) + 2F(y), \quad x, y \in K, \tag{11}$$

*then there exists a unique selection $f\colon K \to Y$ of the multifunction $F$ such that*

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x,y\in K.$$

(2) *If*

$$2F(x)+2F(y)\subset F(x+y)+F(x-y), \quad x,y\in K, \tag{12}$$

*then $F$ is single-valued.*

*Proof.* (1) Setting $y=0$ in (11) we have

$$F(x)+F(x)\subset 2F(x)+2F(0) \quad \text{for } x\in K.$$

By the Rådström cancelation lemma [17] we get

$$\{0\}\subset F(0).$$

Next setting $y=x$ in (11) and using the last inclusion we obtain

$$F(2x)\subset F(2x)+F(0)\subset 4F(x), \quad x\in K$$

and

$$\frac{F(2x)}{4}\subset F(x) \quad \text{for } x\in K.$$

By Theorem 2, with $\Psi(x)=\frac{1}{4}x, a(x)=2x$, there exists the limit $\lim_{n\to\infty}\Psi^n(F(a^n(x)))=\frac{F(2^nx)}{4^n}=f(x), f(x)\in F(x)$ for $x\in K$ and as

$$\frac{F(2^n(x+y))}{4^n}+\frac{F(2^n(x-y))}{4^n}\subset 2\frac{F(2^nx)}{4^n}+2\frac{F(2^ny)}{4^n}$$

we get $f(x+y)+f(x-y)=2f(x)+2f(y)$ for $x,y\in K$. Moreover, $f$ is unique.

(2) Setting $y=0$ in (12) and using the Rådström cancelation lemma we get

$$F(x)+F(0)\subset F(x), \quad x\in K.$$

Thus and by (12) with $y=x$ we have

$$4F(x)\subset F(2x)+F(0)\subset F(2x) \quad x\in K$$

and

$$F(x)\subset\frac{F(2x)}{4} \quad \text{for } x\in K.$$

By Theorem 2, with $\Psi(x)=\frac{1}{4}x, a(x)=2x, F$ must be single-valued. □

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Received: June 1, 2012.
Accepted: July 27, 2012.