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Results in Mathematics

Erratum to: Integration by Parts for Perron Type Integrals of Order 1 and 2 in Riesz Spaces

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Abstract. We note that the proof of differentiability of the integral function I_2 given in [1, Theorem 7.9] needs some corrections which we present here.

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The proof of differentiability of the integral function I_2 in [1, Theorem 7.9] is based on the sufficient condition for global differentiability given in [2, Theorem 4.3] and stated in the paper as Proposition 4.5. But this last statement is not justified in [2, Theorem 4.3] properly. In fact the following question seems to be open: if the "componentwise differentiability" in the complement of a meager set does imply global differentiability, where the involved "components" are taken according to the Maeda–Ogasawara–Vulikh representation theorem for Riesz spaces. To overcome this gap we present here a new proof of differentiability of the integral function I_2, which is based on a comparison theorem for (g)-differentiable functions (Lemma 1 below), and on a Cauchy-type criterion ([2, Theorem 3.10]), which states that the global limit $(g)\lim_{h\to 0}\phi(x,h)$ (in $[a,b]$) exists in R if and only if there exists an

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(o)-net $(p_\gamma)_{\gamma\in\Gamma}$ (that is a net with $p_\gamma \downarrow 0$) with the property that, for all $\gamma \in \Gamma = (\mathbb{R}^+)^{[a,b]}$, we have:

$$\sup_{x\in[a,b]} \left(\sup\{\phi(x,h): \quad 0<|h|\le\gamma(x)\} - \inf\{\phi(x,l): \quad 0<|l|\le\gamma(x)\}\right)$$
$$= \sup_{x\in[a,b]} \left(\sup\{|\phi(x,h)-\phi(x,l)|: \quad 0<|h|,|l|\le\gamma(x)\}\right) \le p_\gamma. \tag{1}$$

The technical comparison lemma is as follows:

Lemma 1. *Let $f_j : [a,b] \to R$, $j = 1,2,3$, be such that f_1, f_2 are (g)-differentiable in $[a,b]$ and $f_1 - f_3$, $f_3 - f_2$ are convex in $[a,b]$, then f_3 is (g)-differentiable in $[a,b]$.*

Proof. Fix arbitrarily x,h,l with $h,l \in \mathbb{R}\backslash\{0\}$, x, $x+h$, $x+l \in [a,b]$. By convexity of $f_1 - f_3$, when $h \ge l$ we get

$$\frac{(f_1-f_3)(x+h)-(f_1-f_3)(x)}{h} \ge \frac{(f_1-f_3)(x+l)-(f_1-f_3)(x)}{l},$$

and hence

$$\frac{f_1(x+h)-f_1(x)}{h} - \frac{f_1(x+l)-f_1(x)}{l} \ge \frac{f_3(x+h)-f_3(x)}{h} - \frac{f_3(x+l)-f_3(x)}{l}.$$

Analogously, using convexity of $f_3 - f_2$, we obtain, for $h \ge l$:

$$\frac{f_3(x+h)-f_3(x)}{h} - \frac{f_3(x+l)-f_3(x)}{l} \ge \frac{f_2(x+h)-f_2(x)}{h} - \frac{f_2(x+l)-f_2(x)}{l}.$$

If $h < l$, by reversing the above inequalities we have:

$$\begin{aligned} &\frac{f_1(x+h)-f_1(x)}{h} - \frac{f_1(x+l)-f_1(x)}{l} \\ &\le \frac{f_3(x+h)-f_3(x)}{h} - \frac{f_3(x+l)-f_3(x)}{l} \\ &\le \frac{f_2(x+h)-f_2(x)}{h} - \frac{f_2(x+l)-f_2(x)}{l}. \end{aligned}$$

In any case we get:

$$\begin{aligned} 0 &\le \left|\frac{f_3(x+h)-f_3(x)}{h} - \frac{f_3(x+l)-f_3(x)}{l}\right| \\ &\le \sum_{j=1}^{2} \left|\frac{f_j(x+h)-f_j(x)}{h} - \frac{f_j(x+l)-f_j(x)}{l}\right|. \end{aligned} \tag{2}$$

Since f_1, f_2 are (g)-differentiable in $[a,b]$, by (1) applied to $\phi(x,h) = \dfrac{f_j(x+h)-f_j(x)}{h}$, $j = 1,2$, two (o)-nets $(p^{(j)})_{\gamma\in\Gamma}$ can be found, with

$$\sup_{x\in[a,b]} \left(\sup_{0<|h|,|l|\le\gamma(x)} \left|\frac{f_j(x+h)-f_j(x)}{h} - \frac{f_j(x+l)-f_j(x)}{l}\right| \right) \le p_\gamma^{(j)}$$

for all $\gamma \in \Gamma$. From this and from (2) we obtain

$$0 \le \left| \frac{f_3(x+h) - f_3(x)}{h} - \frac{f_3(x+l) - f_3(x)}{l} \right| \le p_\gamma^{(1)} + p_\gamma^{(2)}$$

for all $\gamma \in \Gamma$, whenever x, $x+h$, $x+l \in [a,b]$ and $0 < |h|, |l| \le \gamma(x)$. Since the net $(p_\gamma^{(1)} + p_\gamma^{(2)})_{\gamma \in \Gamma}$ is an (o)-net, then, again by (1) used for $\phi(x,h) = \dfrac{f_3(x+h) - f_3(x)}{h}$, we get (g)-differentiability of f_3 in $[a,b]$. □

Now we are ready to give the new proof of differentiability of the integral function I_2. In fact we replace the text on page 18 of [1, Theorem 7.9] from line 5 up to the end of the page 18 with the following one. The new fragment of the proof of [1, Theorem 7.9] is:

Proof. Now we prove the differentiability of I_2. Let Ψ (resp. Φ) be any fixed major (minor) function of order 2 for f. We know that Ψ, Φ are (g)-differentiable in $[a,b]$ and $\Psi - I_2$, $I_2 - \Phi$ are convex in $[a,b]$. In fact, for example, if we take $J := I_2 - \Phi$, observe that $J(b) = \inf_\Psi[\Psi(b) - \Phi(b)]$, where the infimum is taken along all the major functions Ψ. Fix any pair (x,y) in $[a,b]$, choose any number $c \in]0,1[$, and any major function Ψ. Writing $\Psi_1 := \Psi - \Phi$, thanks to convexity of Ψ_1 and since $J - \Psi_1 = I_2 - \Psi$ is decreasing, we have:

$$\begin{aligned} cJ(x) + (1-c)J(y) &= c\Psi_1(x) + (1-c)\Psi_1(y) + c[J(x) - \Psi_1(x)] \\ &\quad + (1-c)[J(y) - \Psi_1(y)] \\ &\ge \Psi_1(cx + (1-c)y) + c[J(b) - \Psi_1(b)] \\ &\quad + (1-c)[J(b) - \Psi_1(b)] \\ &\ge J(cx + (1-c)y) + [J(b) - \Psi_1(b)]. \end{aligned}$$

Now, taking the supremum with respect to Ψ, we get

$$cJ(x) + (1-c)J(y) \ge J(cx + (1-c)y).$$

So, $I_2 - \Phi$ is convex. Thus, (g)-differentiability of I_2 is an immediate consequence of Lemma 1 with $f_1 = \Psi$, $f_2 = \Phi$, $f_3 = I_2$. □

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