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Results in Mathematics

ERRATUM

Erratum to: Convergence in Variation and Rate of Approximation for Nonlinear Integral Operators of Convolution Type

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In the present erratum the authors want to mark out that the term I_2 of Theorem 1 (convergence theorem) in [1] has to be estimated in a different way. A similar problem occurs in Theorem 2 of [1], where in order to get the required result one has to assume, instead of (2) of the original paper (involving assumption $K_w.3)$ of [1]), condition (I) below (involving assumption $K_w.3)'$ mentioned below). Since it is easy to see that the two assumptions $K_w.3)$ and $K_w.3)'$ cannot be compared, in order to unify our approach, here we prove Theorem 1 by using $K_w.3)'$ even if, as mentioned in the Remark of Sect. 1, $K_w.3)$ is still valid in order to prove the result. Similar reasonings hold also for the multidimensional case.

1. Periodic Case

For the notations we refer to [1]. Assumption $K_w.3)$ of the original paper ([1]) has to be replaced by the following

$\mathbf{K_w.3)'}$ denoted by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$,

$$\frac{V_J[G_w]}{m(J)} \to 0, \quad \text{as } w \to +\infty,$$

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uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$, that is, for every $\varepsilon > 0$ there exists $\overline{w} > 0$ such that $\frac{V_J[G_w]}{m(J)} < \varepsilon$, for every $w \geq \overline{w}$ and for every bounded interval $J \subset \mathbb{R}$.

We remark that it is easy to provide examples of kernels which fulfill assumption $K_w.3)'$, besides all the other assumptions of our theory. For example, let us consider the kernel functions of Example 3 of [1], namely $K_w(t,u) = L_w(t)H_w(u)$, $t \in \mathbb{R}_0^+$, $u \in \mathbb{R}$, $w > 0$, where $\{L_w(t)\}_{w>0}$ is an approximate identity,

$$H_w(u) = \begin{cases} u + \log\left(1 + \frac{u}{w}\right), & 0 \leq u < 1, \\ u + \log\left(1 + \frac{1}{wu}\right), & u \geq 1, \end{cases}$$

and the definition of $H_w(u)$ is extended in odd-way for $u < 0$. Then

$$G_w(u) = \begin{cases} \log\left(1 + \frac{u}{w}\right), & 0 \leq u < 1, \\ \log\left(1 + \frac{1}{wu}\right), & u \geq 1, \end{cases}$$

and it is easy to see that $G_w(u)$ is increasing in $[0,1]$, decreasing in $[1,+\infty)$. Hence, for every interval $[a,b] \subset [0,1]$ there holds

$$\frac{V_{[a,b]}[G_w]}{m([a,b])} = \frac{\log\left(1 + \frac{b}{w}\right) - \log\left(1 + \frac{a}{w}\right)}{b-a} \leq \frac{1}{w} \rightarrow 0$$

as $w \rightarrow +\infty$, and, for every $[a,b] \subset [1,+\infty)$,

$$\frac{V_{[a,b]}[G_w]}{m([a,b])} = \frac{\log\left(1 + \frac{1}{aw}\right) - \log\left(1 + \frac{1}{bw}\right)}{b-a} \leq \frac{1}{wab} \leq \frac{1}{w} \rightarrow 0$$

as $w \rightarrow +\infty$. If $[a,b] \subset \mathbb{R}$ is such that $0 \leq a < 1 < b$, it is sufficient to notice that $V_{[a,b]}[G_w] = V_{[a,1]}[G_w] + V_{[1,b]}[G_w]$. This implies that $K_w.3)'$ holds.

It is possible to prove the following

Lemma 1. *Let $f \in BV_{2\pi} \cap C^0_{2\pi}$, where $C^0_{2\pi}$ denotes the space of continuous functions over $[-\pi,\pi]$. If $K_w.3)'$ is satisfied, then*

$$V_{2\pi}[H_w \circ f - f] \rightarrow 0, \quad as \quad w \rightarrow +\infty.$$

Proof. Since $f \in C^0_{2\pi}$ f has at most countably infinitely many proper points of maximum/minimum ([2]). Let us consider the most general case in which f has countably infinitely many proper points of maximum/minimum, that we will denote as $\overline{x}_i$, $i = 0,1,2,\ldots$. Let $D = \{x_0 = -\pi, x_1, \ldots, x_n = \pi\}$ be a division of the interval $[-\pi,\pi]$ and let $\widetilde{D} \equiv \{y_0, y_1, \ldots\}$ be the (infinite) division obtained adding the points $\overline{x}_i$ to D. Without any loss of generality let us assume that $\lim_{i \rightarrow +\infty} \overline{x}_i = \pi$ (in the other cases it is sufficient to split the interval) and that $\overline{x}_0 \equiv x_0 = -\pi$ (if $\overline{x}_0 > -\pi$, then f is constant in $[-\pi, \overline{x}_0]$ and so $V_{2\pi}[f] = V_{[\overline{x}_0,\pi]}[f]$).Then obviously $V_{2\pi}[f] = \sum_{i=1}^{+\infty} |f(\overline{x}_i) - f(\overline{x}_{i-1})|$.

Hence in each of the intervals $A_i := [\overline{x}_{i-1}, \overline{x}_i]$, $i = 1, 2, \ldots$, f is monotone, and so

$$V_{A_i}[H_w \circ f - f] \leq V_{I_i}[G_w], \quad i = 1, 2, \ldots,$$

where $I_i := [\min\{f(\overline{x}_{i-1}), f(\overline{x}_i)\}, \max\{f(\overline{x}_{i-1}), f(\overline{x}_i)\}]$. By $K_w.3)'$, for a fixed $\varepsilon > 0$, there exists $\overline{w} > 0$ such that $V_{I_i}[G_w] \leq \varepsilon m(I_i)$, for every $w \geq \overline{w}$ and $i = 1, 2, \ldots$, and so

$$\begin{aligned}\sum_{i=1}^{+\infty} |(H_w \circ f - f)(y_i) - (H_w \circ f - f)(y_{i-1})| &\leq \sum_{i=1}^{+\infty} V_{A_i}[H_w \circ f - f] \\ &\leq \sum_{i=1}^{+\infty} V_{I_i}[G_w] \\ &\leq \varepsilon \sum_{i=1}^{+\infty} m(I_i) = \varepsilon V_{2\pi}[f].\end{aligned}$$

Then, passing to the supremum over all the possible divisions of $[-\pi, \pi]$, we obtain that, for every $\varepsilon > 0$, there exists $\overline{w} > 0$ such that for every $w \geq \overline{w}$, $V_{2\pi}[H_w \circ f - f] \leq \varepsilon V_{2\pi}[f]$, and so the thesis follows, since by assumption $f \in BV_{2\pi}$. □

Remark. We remark that, in order to obtain the previous convergence result, it is sufficient to assume condition $K_w.3)$ of the original paper (obviously using a different proof). However, since a condition of the form $K_w.3)'$ is needed in order to obtain the order of approximation and all the further results in the multidimensional frame, for a sake of simplicity we use directly condition $K_w.3)'$.

Using Lemma 1, the proof of Theorem 1 of the original paper follows. Indeed, it is sufficient to replace the estimate of I_2 (page 10, line 9) with the following

$$I_2 \leq \int_{-\pi}^{\pi} |L_w(t)| V_{2\pi}[H_w \circ f - f]\, dt \leq A V_{2\pi}[H_w \circ f - f],$$

by assumption $K_w.1)$, and so, by Lemma 1, $I_2 \leq A\varepsilon$, for sufficiently large $w > 0$.

About the order of approximation, in a similar fashion, in Theorem 2 of the original paper instead of condition (2) we have to assume that

$$\frac{V_J[G_w]}{m(J)} = O(\xi(w^{-1})), \quad w \to +\infty, \tag{I}$$

uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$, i.e., there exists an absolute constant $M > 0$ and $\overline{w} > 0$ such that for every $w \geq \overline{w}$ and for every bounded interval $J \subset \mathbb{R}$, $\frac{V_J[G_w]}{m(J)} \leq M\xi(w^{-1})$. It is not difficult to provide examples of kernels which satisfy (I): among them, for example, the family of kernel functions of Example 3 of the original paper (i.e., the above example of the present section) satisfy the above condition with $\xi(w) = w^{\alpha}$, $0 < \alpha \leq 1$. Moreover, in the estimate of $V_{2\pi}[T_w f - f]$ (page 12, line 8), instead of the term $AV_J[G_w]$ we now have $AV_{2\pi}[H_w \circ f - f]$. As concerns this term, it is not difficult to see that, following the same reasoning of Lemma 1, by assumption (I), $V_{2\pi}[H_w \circ f - f] = O(\xi(w^{-1}))$, as $w \to +\infty$.

2. Multidimensional Case

We refer again to the original paper for the notations that we shall use in the multidimensional case. In an analogous way to the periodic case, assumption $K_w.3)$ has to be replaced by $K_w.3)'$ and it is now possible, with this condition, to prove the following convergence result:

Lemma 2. *Let $f \in AC(\mathbb{R}^N)$. If $K_w.3)'$ is satisfied, then*

$$V[H_w \circ f - f] \to 0, \quad as \quad w \to +\infty.$$

Proof. Let $I = \prod_{i=1}^{N} [a_i, b_i]$ and let $\{J_1, \ldots, J_m\}$ be a partition of I, with $J_k = \prod_{j=1}^{N} [{}^{(k)}a_j, {}^{(k)}b_j]$, $k = 1, 2, \ldots, m$. Since $f \in AC(\mathbb{R}^N)$, for every $k = 1, \ldots, m$ in particular $f(x'_j, \cdot)$ is continuous in $[{}^{(k)}a_j, {}^{(k)}b_j]$ for almost every $x'_j \in [{}^{(k)}a'_j, {}^{(k)}b'_j]$ and for every $j = 1, \ldots, N$, $w > 0$. Hence, in a similar way to Lemma 1, using $K_w.3)'$ it is possible to prove that, for every fixed $\varepsilon > 0$ there exists $\overline{w} > 0$ (depending only on ε) such that

$$V_{[{}^{(k)}a_j, {}^{(k)}b_j]}[(H_w \circ f - f)(x'_j, \cdot)] \leq \varepsilon V_{[{}^{(k)}a_j, {}^{(k)}b_j]}[f(x'_j, \cdot)]$$

for every $w \geq \overline{w}$. Then, for $w \geq \overline{w}$ there holds

$$\begin{aligned} \Phi_j(H_w \circ f - f, J_k) &= \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V_{[{}^{(k)}a_j, {}^{(k)}b_j]}[(H_w \circ f - f)(x'_j, \cdot)]\, dx'_j \\ &\leq \varepsilon \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V_{[{}^{(k)}a_j, {}^{(k)}b_j]}[f(x'_j, \cdot)]\, dx'_j = \varepsilon\, \Phi_j(f, J_k), \end{aligned}$$

for every $j = 1, \ldots, N$. Hence $\Phi(H_w \circ f - f, J_k) \leq \varepsilon\, \Phi(f, J_k)$ and, passing to the supremum over all the possible partitions of I, $V_I[H_w \circ f - f] \leq \varepsilon V_I[f]$, for all $I \subset \mathbb{R}^N$. This implies that

$$V[H_w \circ f - f] \leq \varepsilon V[f],$$

for every $w \geq \overline{w}$, which completes the proof, since by assumption in particular $f \in BV(\mathbb{R}^N)$. □

Now, Theorem 3 of the original paper follows, replacing the estimate of $V[T_w f - f]$ with the following

$$
\begin{aligned}
V[T_w f - f] \leq & \int_{0 \leq |\mathtt{t}| \leq \delta} |L_w(\mathtt{t})| V[(H_w \circ f)(\cdot - \mathtt{t}) - (H_w \circ f)(\cdot)] d\mathtt{t} \\
& + 2KV[f] \int_{|\mathtt{t}| \geq \delta} |L_w(\mathtt{t})| d\mathtt{t} \\
& + V[H_w \circ f - f] \int_{\mathbb{R}^N} |L_w(\mathtt{t})| d\mathtt{t} + |\bar{A}_w - 1| V[f] \\
=: & \; J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

and taking into account that, by Lemma 2, $J_3 \leq A\varepsilon$. We finally remark that, following this proof, it is no more necessary to assume that $f \in L^\infty(\mathbb{R}^N)$ and so Theorem 3 of the original paper holds assuming only that $f \in AC(\mathbb{R}^N)$.

About the order of approximation, as before assumption (2) of Theorem 2 (used in Theorem 4) has to be replaced with condition (I) and estimate (7) has to be modified in a similar fashion to the periodic case, taking into account that, as in the proof of Lemma 2, it can be proved that $V[H_w \circ f - f] = O(\xi(w^{-1}))$, as $w \to +\infty$. We remark that also in Theorem 4 of the original paper the assumption that $f \in L^\infty(\mathbb{R}^N)$ can be dropped.

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