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Results in Mathematics

Erratum to "Divisors of Bernoulli Sums", 51 no. 1–2, 141–179

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Abstract. We give the correct computation here. The computation of the multiplicative function $\rho_k(D)$ in Proposition 2.10 was incorrect.

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The computation of the multiplicative function $\rho_k(D)$ in Proposition 2.10 was incorrect. In place we have

Proposition 0.1. *For any $\varepsilon > 0$, there exists a constant C_ε depending on ε only, such that for any positive integers n, m and D*

$$\left|\mathbb{P}\{D|B_nB_m\} - \frac{1}{D2^{m-n}}\sum_{k=0}^{m-n} C_{m-n}^k \rho_k(D)\right| \le C_\varepsilon \left(\frac{D^{1+\varepsilon}}{n}\right)^{1/2},$$

where

$$\rho_k(D) = \begin{cases} \prod_{p|D} p^{\lfloor \frac{v_p(D)}{2}\rfloor} & \text{if} \quad k = 0\,, \\ \prod_{v_p(k)<v_p(D)/2}(2p^{v_p(k)}) \cdot \prod_{v_p(k)\ge v_p(D)/2} p^{\lfloor \frac{v_p(D)}{2}\rfloor} & \text{if} \quad k \ge 1\,. \end{cases}$$

And $\mathbb{P}\{D|B_nB_m\} \le \frac{1}{2^{m-n}\sqrt{D}} + \frac{2^{\omega(D)}(m-n)}{D} + C_\varepsilon \frac{D^{1/2+\varepsilon}}{\sqrt{n}}$. *Further, for any $\varepsilon > 0$, there exists a constant C_ε depending on ε only, such that*

$$\mathbb{P}\{d|B_n\,,\,\delta|B_m\} \le C(m-n)\frac{2^{\omega(d\delta)}}{d\delta} + C_\varepsilon \frac{(d\delta)^{(1+\varepsilon)/2}}{\sqrt{n}}\,.$$

A modification of the computation of the quantity B_1^1 in the proof of Theorem 3.1 is necessary. Let us first indicate how to compute $\rho_k(D)$. Assume $1 \le k \le m-n$. It suffices to compute $\rho_k(p^r)$. First observe for $r \ge 1$ that $\rho_k(p^r) = 2$ if $p \not| k$. If $p|k$, the solutions of the equation $p^r|y(y+k)$ are of type $y = p^sY$, with $(Y,p) = 1$. Suppose $r \ge 4$ and put $r' = \lfloor \frac{r}{2} \rfloor$. We have $\rho_k(p^r) = \#\{1 \le y \le p^r : \ p^r|y(y+k)\}$. There is no solution if $(y,p) = 1$, and apart from the trivial solution $y = p^r$, the other solutions are of type $y = p^s Y$, $(Y, p) = 1$, $1 \le s < r$. We distinguish three cases: (i) $r' < s < r$ (ii) $s = r'$ (iii) $1 \le s < r'$.

(i) Then $p^r | y(y+k)$ means $p^{r-s} | p^s Y + k$, which is possible if and only if $r - s \le v_p(k)$. Thus $\max(r'+1, r - v_p(k)) \le s < r$. We have $Y \le p^{r-s}$, $(Y, p^{r-s}) = 1$. Their number is $\phi(p^{r-s}) = p^{r-s}(1 - \frac{1}{p})$, where ϕ is Euler's totient function, and the corresponding number of solutions is

$$\sum_{\max(r'+1, r-v_p(k)) \le s \le r-1} p^{r-s}\left(1 - \frac{1}{p}\right) = p \frac{p^{(\lfloor \frac{r-1}{2} \rfloor) \wedge v_p(k)} - 1}{p-1}\left(1 - \frac{1}{p}\right) = p^{\lfloor \frac{r-1}{2} \rfloor \wedge v_p(k)} - 1\,.$$

The other cases are treated similarly, and one find for (ii): $p^{r'}$ solutions if r is odd and $p^{r'}(1 - \frac{1}{p})$ if r is even. For (iii), the only case providing solutions is $s = v_p(k) < r'$, providing then $p^{v_p(k)}$ solutions. The cases $r = 2, 3$ are easily checked. Therefore for $r \ge 2$

$$\rho_k(p^r) = \begin{cases} 2 & \text{if } \ p \not| k\,, \\ 2p^{v_p(k)} & \text{if } \ v_p(k) < \frac{r}{2}\,, \\ p^{\lfloor \frac{r}{2} \rfloor} & \text{if } \ v_p(k) \ge \frac{r}{2}\,. \end{cases}$$

If $k = 0$, the number of solutions is: $\#\{Y \le p^{r-s} : (Y, p) = 1\} = \phi(p^{r-s})$, and so $\rho_0(p^r) = p^{\lfloor \frac{r}{2} \rfloor}$.

We deduce for any positive integer D and any integer k, $\rho_k(D) \le 2^{\omega(D)}(k \wedge \sqrt{D})$ and $\rho_0(D) \le \sqrt{D}$. Since

$$\sum_{k=0}^{m-n} \frac{C_{m-n}^k}{2^{m-n}} \frac{\rho_k(D)}{D} \le \frac{1}{2^{m-n}\sqrt{D}} + \sum_{k=1}^{m-n} \frac{C_{m-n}^k}{2^{m-n}} \frac{2^{\omega(D)} k}{D} \le \frac{1}{2^{m-n}\sqrt{D}} + \frac{2^{\omega(D)}(m-n)}{D}\,,$$

the second assertion follows. For the last assertion, notice that $\mathbb{P}\{d|B_n\,, \delta|B_m\} > 0$ only if $m - n \ge (d, \delta)$, since $\mathbb{P}\{d|B_n\,, \delta|B_m\} \le \mathbb{P}\{d|B_n\}\mathbb{P}\{(d,\delta)|B_{m-n}\}$. Now

$$\begin{aligned} \mathbb{P}\{d|B_n\,, \delta|B_m\} &\le \mathbb{P}\{d|B_n\,, \delta|B_n\} + \mathbb{P}\Big\{d\delta\big|\big(B_n^2 + B_n(B_m - B_n)\big),\ B_m - B_n > 0\Big\} \\ &= \mathbb{P}\{[d,\delta]|B_n\} + \frac{1}{2^n 2^{m-n}} \sum_{k=1}^{m-n} C_{m-n}^k \frac{1}{d\delta} \sum_{j=0}^{d\delta-1} \sum_{h=0}^{n} C_n^h e^{2i\pi \frac{j}{d\delta}(h^2+kh)}\,. \end{aligned}$$

Using (1.15) gives

$$\begin{aligned} \left| \frac{1}{2^m} \sum_{k=1}^{m-n} C_{m-n}^k \frac{1}{d\delta} \sum_{j=0}^{d\delta-1} \sum_{h=0}^{n} C_n^h e^{2i\pi \frac{j}{d\delta}(h^2+kh)} \right| &\le \frac{C_\varepsilon (d\delta)^{(1+\varepsilon)/2}}{\sqrt{n}} + \frac{1}{2^{m-n}} \sum_{k=1}^{m-n} \frac{\rho_k(d\delta)}{d\delta} \\ &\le C_\varepsilon \frac{(d\delta)^{(1+\varepsilon)/2}}{\sqrt{n}} \end{aligned}$$

$$+\frac{1}{2^{m-n}}\sum_{k=1}^{m-n}\frac{2^{\omega(d\delta)}(k\wedge\sqrt{d\delta})}{d\delta}$$
$$\le C_\varepsilon\frac{(d\delta)^{(1+\varepsilon)/2}}{\sqrt{n}}+(m-n)\frac{2^{\omega(d\delta)}}{d\delta}\,.$$

We deduce $\mathbb{P}\{d|B_n\,,\,\delta|B_m\}\le C(m-n)\frac{2^{\omega!(d\delta)}}{d\delta}+C_\varepsilon\frac{(d\delta)^{(1+\varepsilon)/2}}{\sqrt{n}}$, as claimed.

For the sum B_1^1, we have

$$\sum_{\substack{d\le n^\theta,\delta\le m^\theta\\ d,\delta\in\mathcal{D}}}\frac{2^{\omega(d\delta)}}{d\delta}\le 2\sum_{D\le(nm)^\theta}\#\big\{d\le n^\theta,\delta\le m^\theta:D=d\delta\big\}\cdot\frac{2^{\omega(D)}}{D}\,.$$

Given d, if $D=d\delta$, at most one choice of δ is possible. Thus $\#\big\{d\le n^\theta,\delta\le m^\theta:D=d\delta\big\}\le\#\big\{d,\delta:D=d\delta\big\}\le d(D)$. As $d(D)\le C_cD^c\le C_cn^c$, if D large, say $D\ge\Delta_c$, we get

$$\sum_{D<\Delta_c}\#\big\{d\le n^\theta,\delta\le m^\theta:D=d\delta\big\}\cdot\frac{2^{\omega(D)}}{D}\le\sum_{D<\Delta_c}d(D)\cdot\frac{2^{\omega(D)}}{D}\le K(\Delta_c)\,,$$

whereas

$$\sum_{\Delta_c\le D\le(nm)^\theta}\#\big\{d\le n^\theta,\delta\le m^\theta:D=d\delta\big\}\cdot\frac{2^{\omega(D)}}{D}\le C_cn^c\sum_{D\le 2n^{2\theta}}\frac{2^{\omega(D)}}{D}\,.$$

As $F(x):=\sum_{k\le x}\frac{2^{\omega(k)}}{k}=\mathcal{O}(x(\log x)^2)$, by Abel summation, we deduce $\sum_{2\le D\le N}\frac{2^{\omega(D)}}{D}\le C'(\log N)^3$. Hence

$$\sum_{\substack{d\le n^\theta,\delta\le m^\theta\\ d,\delta\in\mathcal{D}}}\frac{2^{\omega(d\delta)}}{d\delta}\le\sum_{D\le(nm)^\theta}\frac{2^{\omega(D)}}{D}\le C_\theta(\log nm)^3\,.$$

For n large

$$B_1^1=\sum_{i\le n\le j}\sum_{n<m\le n+n^H}\sum_{\substack{d\le n^\theta,\delta\le m^\theta\\ d,\delta\in\mathcal{D}}}\frac{2^{\omega(d\delta)}(m-n)}{d\delta}$$
$$\le C_{\theta,c}\sum_{i\le n\le j}n^c\sum_{n<m\le n+n^H}(m-n)(\log nm)^3$$
$$\le C_{\theta,c}\sum_{i\le n\le j}n^c(\log 2n^2)^3\sum_{n<m\le n+n^H}(m-n)\le C_{\theta,c}\sum_{i\le n\le j}n^{2H+c}(\log 2n^2)^3\,.$$

Thereby for n large, $B_1\le C_{\theta,c}\sum_{i\le n\le j}n^{12c}$.

An improvment of these results was recently obtained using a slightly simplified approach. This will be published elsewhere.

References

[1] M. Weber, *Divisors of Bernoulli sums*, Result. Math. **51** (2007), no. 1–2, 141–179.

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