Pure Appl. Geophys. 175 (2018), 1227–1229

https://doi.org/10.1007/s00024-018-1812-2

Pure and Applied Geophysics

Book Review

Ordinary Differential Equations: Basics and Beyond by D. G. Schaeffer, J. W. Cain, Springer Science+Business Media New York, 2016. ISBN 978-1-4939-6387-4

Andrzej Icha[1]

The formulation of a real-world problem in mathematical terms often leads to ordinary differential equations, the foundations of which were laid by Isaac Newton, who said: "It is useful to solve differential equations" (Arnold V. I., *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, 1988). An ordinary differential equation (ODE) is an equation for a function of a single variable that relates the values of the function to the values of its derivatives. This book is conceived to provide a bridge between theoretical and applied aspects of ODEs.

The book consists of ten chapters, and three appendices. Chapter 1, *Introduction*, presents a terse survey of the standard topics encountered in an introductory course of ODE. The following issues are addressed: the intuitive notion of ODE and its solution, the order of ODEs, the notion of linearity, the importance of numerical solutions, and the methods for finding explicit solutions of certain ODEs. Also, a few examples that illustrate the process of translating scientific laws into physically based second-order ODEs are demonstrated, together with some terminology regarding systems of ODEs.

Chapter 2, *Linear Systems with Constant Coefficients*, concerns the basic theory for homogeneous linear systems of ODEs. To study such systems, the apparatus of linear algebra is used. A key concept in matrix analysis, the matrix exponential, is discussed first. Then, it is shown how it produces the solution to ODEs with constant coefficients, $\mathrm{d}x/\mathrm{d}t = Ax$, $x \in R^n$ (*), where A is a real $n \times n$ (constant) matrix. In fact, the problem of constructing a fundamental set of solutions of ODEs is a problem to linear algebra, and it is connected with the eigenvalues and eigenvectors of the matrix A. The use of the Jordan form for computing the exponential of a matrix is explained and demonstrated. The remaining three sections concern the asymptotic behavior of solutions of systems (*), the classification and phase portraits for planar, homogeneous, autonomous systems with A a constant 2×2 matrix, according to the character of its stationary point (the origin), and the formulas for solving an inhomogeneous linear system of ODEs with constant coefficients.

Chapter 3, *Nonlinear Systems: Local Theory*, deals with the issues regarding the initial value problem (IVP) for systems of nonlinear ODEs, namely, the existence of solutions, and the uniqueness of solutions. The autonomous systems of the form $x' = F(x)$, where F: $R^n \to R^n$ is locally Lipschitz, are thoroughly analyzed. The proof of the existence of the solutions based on the contraction-mapping principle is presented, together with the construction of converging Picard iterates. Next, the uniqueness of the solution is proved, using the celebrated lemma of Gronwall. The extensions of the theory to nonautonomous systems are moved to exercises.

Chapter 4, *Nonlinear Systems: Global Theory*, focuses on the problems of global existence of solutions, the continuous dependence on the initial data, and the differentiability properties of the solution. Two sections concentrate on the maximal interval of existence for a given solution to the IVP and two sufficient conditions for global existence of solutions.

[1] Institute of Mathematics, Pomeranian Academy in Słupsk, ul. Kozietulskiego 6-7, 76-200 Słupsk, Poland. E-mail: majorana38@gmail.com

Birkhäuser

The concept of trapping regions is introduced and used to prove the global existence for Duffing's equation, the chemostat, and the torqued pendulum. Next, the idea of nullclines and its usefulness in finding trapping regions is explained and illustrated by applying it to specific ODEs, including an activator–inhibitor system, Sel'kov's model for glycolysis, Van der Pol's equation, and the Michaelis–Menten kinetics. The subsequent sections present the results concerning the dependence of solutions on the initial conditions and other parameters, using the formalism from the dynamical systems theory. An appendix provides the terse overview of the numerical scheme for the IVP for systems of ODEs, namely, the simplest numerical method, known as Euler's method.

Dimensional analysis and scaling are most effective when the researcher already knows something about a physical situation or process. Chapter 5, *Nondimensionalization and Scaling*, discusses the dimensional scaling concepts, and methods which provide an auxiliary techniques for simplifying ODEs, appearing in the various physical and natural situations. The dimensional scaling techniques are applied to the problems grouped into several subject areas: mechanics, electrical sciences, biology, and ecology (the bathtub models). The examples considered illustrate the possibilities and limitations of dimensional analysis.

The qualitative study of ODEs is concerned with how to predict the fundamental characteristics of the solutions of ODEs without knowing explicit solutions. Chapters 6–9 present a comprehensive introduction to the theory of ODEs with a focus on dynamical systems theory. Chapter 6, *Trajectories Near Equilibria*, describes the linearization technique for analyzing the behavior of a nonlinear system in a neighborhood of an equilibrium point. The main theorem concerning the asymptotic stability of solutions of ODEs is formulated and proved, together with the basic terminology for classifying equilibrium points. The several interesting applications include: the Lotka–Volterra system, the Turing instability, and the classic control problem of the inverted pendulum. The notion of Lyapunov function and the basic theorems on Lyapunov stability are introduced and discussed, and then applied to the Duffing's equation through the Lasalle's invariance principle. The method of construction of Lyapunov functions for the Lotka–Volterra equations is also demonstrated. In the next section, the stable manifold theorem for ODEs is formulated and illustrated for sketching phase portraits for several equations. Some important proofs (e.g., the Hartman–Grobman theorem and the stable manifold theorem) are moved to the exercises and the appendices. Chapter 7, *Oscillations in ODEs*, covers topics connected with the techniques that predict and describe the oscillatory behavior of the solutions of ODEs. The notion of the periodic solutions of ODEs is introduced, together with the illustrative examples. The topological structures of two-dimensional systems of ODEs are studied, and the celebrated Poincaré–Bendixson theorem and its generalization are formulated and explained. The applications of this theorem and of the Dulac's criterion, which refer to the problem of the existence and nonexistence of periodic solutions and limit cycles, are briefly discussed. The next section presents a number of results connected with the stability of periodic solutions of ODEs. A very powerful concept in the study of periodic orbits, the return map (or the Poincaré map), is introduced and described. Next, the Poincaré–Bendixson theorem is applied to show that the torqued pendulum equation has a periodic, asymptotically stable solution. Also, the asymptotic perturbation theory is used to approximate limit cycles of the van der Pol system, and the Poincaré map technique is applied to prove the stability for the solutions of this equation. Some ideas from Floquet theory to explain how vibration can stabilize an inverted pendulum are presented and placed in an appendix.

The notion of *bifurcation* of a dynamical system refers to a qualitative change in its dynamics produced by varying parameters. Chapter 8, *Bifurcation from Equilibria*, explores some of the basic philosophies of the local bifurcation theory and discusses how bifurcation problems arise in ODEs. The first section illustrates, through the examples, the pitchfork bifurcations (supercritical and subcritical) referring to the bead on a rotating wire hoop, the Lorenz system, and the laterally supported inverted pendulum. Several types of bifurcations, saddle node, transcritical, hysteresis-point, isola-center, and Hopf are also discussed and explained. To supplement the examples-oriented sections, the chapter is completed by the theorem that unifies many steady-state bifurcation phenomena, and the important technique called Lyapunov–Schmidt reduction. Chapter 9, *Examples of Global Bifurcation*, continues with bifurcations of periodic motions of ODEs. The representative examples of five different types of global bifurcation are demonstrated and discussed. These include: the homoclinic and heteroclinic bifurcations; the saddle node bifurcation of limit cycles; the mutual annihilation bifurcation of two limit cycles; the Neimark–Sacker bifurcation; and the period-doubling bifurcation. A brief theoretical background on the use of the Poincaré maps technique to analyze these three bifurcations is sketched. Also, in the two sections, some applications of the theory to problems in which global bifurcation plays a crucial role are presented. First, the famous Lorenz system is described, and then the bursting oscillations in the denatured Morris–Lecar equations, which are one of the simplest models for the production of action potentials in neuroscience.

Chapter 10, *Epilogue*, covers briefly a number of issues connected with the ODEs which not presented in the main body of the book. The first section shows how to formulate and solve a boundary value problems (BVP) for ODEs. Typical BVPs problems are considered, including a calculus-of-variations examples, and the eigenvalue problems. The next section focuses on the use of stochastic differential equation (SDE) models to describe a variety of population growth dynamics. Subsequently, the standard numerical approach, the backward Euler method, for ODEs is discussed. To illustrate the significance of periodic and aperiodic motions, the generic behavior for ODEs on a torus is presented briefly. A large and important class of dynamical systems generated by the delay differential equations is also discussed. The last section presents a gentle introduction to the main aspects of the chaotic dynamics with the aid of specific examples, namely, the quadratic map and the Lorenz system.

Exercises are an essential part of this book. Some of these are standard, while many others are the materials written in the form of "exercise section". The electronic material on the book website http://www.math.duke.edu/ode-book is also provided to complement and supplement the material in the text. It is an important resource for the book. Thirteen pertinent examples that use the freely available XPPAUT software are on the website, in the form of source files (.ode), along with auxiliary pdf files and additional commentary on these examples. XPPAUT is a powerful tool for simulating, animating, and analyzing dynamical systems. The syntax of XPPAUT program for setting up differential equations is amazingly simple compared to the other programs and I strongly recommend the use of it! For helping the reader to get an orientation in various aspects of XPPAUT, I suggest reading the book by B. Ermentrout, *Simulating, Analyzing, and Animating Dynamical Systems.* A Guide to XPPAUT for Researchers and Students, SIAM, 2002. There is also some additional materials in the appendices, ranging from background material on advanced calculus (Appendix B) to linear algebra (Appendix C).

In sum, this book is very well written and to a certain extent is self-contained. It is aimed at students of applied mathematics, theoretical physics, geophysics, engineering, and information science, who already possess a solid knowledge of calculus and a sufficient knowledge of linear algebra. I would definitely use this book to teach an undergraduate and graduate course! The book is also useful for Ph.D. students, whose dissertation areas are related to ODEs.