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Book Review

The Navier–Stokes Problem in the 21st Century. Lemarié-Rieusset P. G., CRC Press, 2016. ISBN-13: 978-1-4665-6621-7

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The Navier–Stokes equations (NSE) are the system of non-linear partial differential equations governing the motion of a Newtonian fluid. The NSE occupy a central position in the studies of nonlinear phenomena, including the basic problems in nonlinear equations connected with well posedness, oscillations, discontinuities, and nonlinear dynamics. Because of the complexity and variety of fluid dynamical phenomena, the mathematical theory of the NSE is far from being finished, and, in practice, is still essentially open. On the other hand, the fact that many of the most important questions in the theory of NSE remain yet to be answered may be for all a source of pleasure and fascination. So, the celebrated American Clay Mathematics Institute created the Navier–Stokes Millennium Price Problem and offered one million dollar for its solution, stating: "Although [the Navier–Stokes equation] were written down in the nineteenth Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier–Stokes equations." (http://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation).

The book consists of long *Preface*, 21 chapters, including *Notations and glossary*, and the *Bibliography*.

Chapter 1, *Presentation of the Clay Millennium Prizes*, pertains to the Millennium Prize Problems and their importance to a broad audience by developing the mathematical background necessary to understand them, and their historical context, range from solving the P versus NP problem, the classical fluid flow equations that were formulated over 170 years ago, and to proving the Birch and Swinnerton-Dyer conjecture. The sixth Millennium Problem formulated by the Clay Mathematics Institute, Navier–Stokes Existence and Smoothness Problem asks for a proof of global existence of smooth solutions for all smooth data, or a proof of the contrary non-global existence of a smooth solution for some smooth data, referred to as breakdown or blow-up. The chapter concisely explains why the existence and smoothness of the NSE have remained an outstanding open problem in partial differential equations theory for more than 80 years.

The Navier–Stokes equations describe the balance between the rate of change of momentum of an element of fluid and the forces on it, as does Newton's second law of motion for a particle, where the stress tensor is a linear function of the rate of strain. Chapter 2, *The physical meaning of the Navier–Stokes equations*, presents briefly some of the basic ideas of continuum mechanics applied to the Newtonian fluid in a physically, and mathematically attractive manner. Several sections deal with standard material on kinematics, conservation laws, the equations of hydrodynamics and vorticity dynamics. Next, the important notions as pressure, strain, and stress are introduced, and elucidated. Then, the model of the Newtonian, isotropic, homogeneous and incompressible fluid flow is analyzed in some detail, leading to the Navier–Stokes equations, and the role of boundary conditions is clarified. At last, the blow-

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up and turbulence phenomena are sketched, in the context of NSE.

Chapter 3, *History of the equation*, gives a short presentation of the engineering and theoretical developments in fluid mechanics, beginning with significant inventions in the ancient times, the crucial role of the NSE for fluid flows and their complex solutions as well as the birth of mathematical fluid dynamics, until the end of twentieth century. It begins with Archimedes, and then traces how the hydrodynamics became transformed into a rudimentary science through the stimulated works of Leonardo, Newton, Leibniz, Bernoulli, D'Alembert, Euler, Laplace, Navier (in 1823), Cauchy (in 1828), Poisson (in 1829), Saint–Venant (in 1843), and Stokes (in 1845). In twentieth century, the science of hydrodynamics was referred to pure mathematics, and to mathematical physics. This direction has been initiated by many researchers, including Kolmogorov, Oseen, Leray, Lichtenstein, Hopf, Ladyzhenskaya, Monin, Yaglom, Visik, Foias, and Temam.

Chapter 4, *Classical solutions*, concerns the classical approach to the Cauchy problem for the Navier–Stokes equations in R^3. The presented method is in the spirit of the paper by Knightly (Arch. Ration. Mech. Anal., 21: 211–245, 1966), and uses only classical tools of differential calculus. To study the problem, some necessary formulas are introduced and proved first, including the heat equation and the heat kernel, the Poisson equation and the Green function, the Helmholtz decomposition theorem, and the Leray projection operator. This mathematical apparatus is sufficient to solve the Stokes problem, i.e., the NSE when the convective bilinear term is neglected. After these preliminaries, and use of the Oseen tensor, the NSE has been transformed into an integro-differential equation, for which, the following facts are proved: the existence of a classical solution on $[0, T] \times R^3$, $T > 0$, for prescribed regular initial data, and the global existence of the solution when the initial data are sufficiently small. In addition, some results related to the asymptotic behavior of incompressible fluid equations are analyzed as time tends to infinity—the phenomenon of instantaneous spreading for the velocities, and the localization of the vorticity.

Chapter 5, *A capacitary approach of the Navier–Stokes integral equations*, concentrates on solving the NSE treated as the fixed point problem of an integro-differential transform, through Picard's iterative scheme. For this purpose, a basic framework to describe global solutions for a Cauchy problem with small initial values concerning a wide class of semilinear parabolic equations with quadratic nonlinearity is developed. These include such important notions and tools as Kalton and Verbitsky's theorem, parabolic Riesz potentials, Hardy–Littlewood maximal function, and Hedberg's inequality, dominating functions (essential to establish the existence of solutions to the integral NSE) as well as the suitable function spaces, Triebel–Lizorkin–Morrey-Campanato spaces. For helping the reader to get an orientation in various function spaces, I recommend to read the book by W. Yuan, W. Sickel, and D. Yang (*Morrey and Campanato Meet Besov*, *Lizorkin and Triebel*, Springer, 2010).

Chapter 6, *The differential and the integral Navier–Stokes equations*, discusses the various definitions of a solution of the Cauchy initial value problem for the NSE, where the pressure term has been eliminated through the Leray projection operator. The essential role in this study plays the uniform local estimates, connected with the invariance of the NSE under space translation. The uniform local estimates for the heat equation are proved first. Then, the Stokes problem is considered (with some restrictions for the class of forcing terms), and the Oseen's tensor formalism, i.e., the equivalence between the differential and the integral formulation of the NSE, is established. After introducing various functional analytic tools in the analysis of the NSE, the concepts of a very weak solution, and weakly regular very weak solution are presented. Also, the criterion is given to ensure that a very weak solution is indeed an Oseen solution. A special case of Oseen solutions, called the mild solutions, is also discussed. These are the solutions that may be obtained by Picard's iteration method. The remaining section deals with the notions of the suitable solutions, and the Leray weak solution for the NSE, i.e., the class of weak solutions that satisfy a localized version of the energy inequality, and the Leray energy inequality, respectively.

The next three chapters are devoted to mild solutions in various function spaces. Such solutions are obtained by rewriting the NSE as the integral equation, and then proving that, in suitable function spaces, the right-hand side defines a contraction. Chapter 7, *Mild solutions in Lebesgue or Sobolev spaces*, discusses the classical results of Kato and Fujita on mild solutions in Sobolev spaces, and in Lebesgue spaces together with the relatively recent proof of uniqueness by Furioli et al. The local solutions of NSE in the Hilbertian setting are analyzed first. Then, the criterion for the existence of global solutions is formulated, and proved. The subsequent section extends the Fujita and Kato's approach to the case of Sobolev spaces, while the another section studies the mild solution of NSE in the Lebesgue spaces. The chapter is concluded with the proof of the uniqueness of mild solutions of NSE by means of an appropriate use of Besov type estimates. Chapter 8, *Mild solutions in Besov or Morrey spaces*, focuses on the analysis of a class of function spaces, to obtain mild solutions for the NSE in Besov or Morrey spaces. After establishing some key estimates and properties of the bilinear operator in the Morrey spaces, two alternative proofs for existence of mild solutions of the NSE are presented. Then, the local uniqueness of Morrey solutions is shown. The similar analysis is made to the Navier–Stokes problem to guarantee existence of solutions to the NSE with initial values in Besov spaces. The case of Besov spaces with positive regularity indexes is also discussed. The Cauchy problem for the NSE in Triebel–Lizorkin spaces is considered briefly, as well as the existence of global mild solutions of NSE in critical Fourier–Herz spaces. The term *BMO* denotes the space of functions of Bounded Mean Oscillation, while the space BMO^{-1} is defined as the space of derivatives of functions in *BMO*. Chapter 9, *BMO^{-1} and the Koch and Tataru theorem*, concerns the important theorem stating the fact that the NSE in R^n are globally well posed for small initial data in BMO^{-1} (R^n). The space BMO^{-1} (R^n) plays a special role since it is the largest critical space where such existence results are available. In addition, the following important facts are presented and proved (for the Navier–Stokes problem with a null force): ill-posedness of the NSE in a certain Besov spaces; the problem of global mild solutions associated to large initial value; the time and space analyticity of mild solution of the NSE, and the persistency phenomenon, i.e., the property connected with the propagation of initial regularity for mild solutions in BMO^{-1}.

The NSE are invariant under the action of various discrete and continuous groups of transformations, and this invariance leads to certain symmetries for the NSE. Chapter 10, *Special examples of solutions*, analyzes the solutions of NSE that are invariant with respect to mentioned symmetries. The groups of transformations under which the NSE are invariant, (for the Navier–Stokes problem with a null force), include: translations of time and space; space rotation; Galilean transformations; the group of transformations due to scalings of time and space, and change of orientation. The so-called "two-and-a-half dimensional flows" are considered first. The global existence of unique solutions to the Navier–Stokes problem, and the stability of the solutions are proved. Then, the axisymmetrical solutions of Navier–Stokes problem are analyzed using the theory of Muckenhoupt weights. In the case of axisymmetric flows with no swirl, the global existence under some regularity assumptions on the initial velocity and forces, but without any size requirements on the data, is formulated and proved. The next section deals with the helical symmetry of Navier–Stokes problem. Global existence of helical symmetrical solutions is formulated, and proved. Subsequently, the Brandolese's symmetrical solutions are considered, i.e., the solutions of Navier–Stokes problem which are left invariant under the action of discrete group generated by the certain isometries. The successive section concerns the self-similar solutions, namely, the solutions of Navier–Stokes problem that are invariant under the action of time–space rescalings. It is shown that the existence of self-similar solutions for the NSE is an immediate consequence of the theory of mild solutions for small data in Besov spaces. Two sections deal with the steady solutions of the NSE. After the formulation of the stationary NSE problem in the form of the integral equation with small data, the existence of steady solutions is proved, together with the proof of the stability of steady solutions under small perturbations. Then, the Landau's solutions are discussed, a celebrated family of explicit solutions of the steady-state NSE. The next section is devoted to time-periodic solutions of the NSE. The time-periodic Navier–Stokes problems in Sobolev, and Morrey spaces are analyzed, and the existence of solutions for NSE is proved. The last section presents a brief introduction to the Beltrami flows, and the Trkalian flows. This term is associated with the name of outstanding Czech physicist Viktor Trkal (1888–1956), who made the significant contribution in fluid mechanics, time-harmonic electromagnetism and astrophysics. The Trkal solution of the NSE forms an important subclass of Beltrami flows. In particular, the famous Arnold–Beltrami–Childress (ABC) flow, a helical steady solution of Euler equations for ideal incompressible flow, is the classical example of the Trkalian flow.

Chapter 11, *Blow-up?* concentrates on the exciting, open question of whether the NSE can generate a finite-time singularity from smooth initial data. In fact, this issue is of the essence of the Clay Millennium problem. The simplified model for the NSE, called the cheap NSE, is analyzed first. For this equation, the blow-up in finite time for sufficiently large initial data is proved. Next, the Serrin's criterion for blow-up of the NSE solutions together with the several generalizations of it is discussed and proved. The connection between blow-up and vorticity is examined, and it is stated that, whenever the direction of vorticity evolves regularly in the areas where the vorticity is large, the solution cannot blow up. The squirt singularities are also presented, and the inequality is proved that precludes the possibility of squirt singularities at the blow-up time for hydrodynamic type equations.

Chapter 12, *Leray's weak solutions*, provides a concise overview of the classical theory on existence, and weak-strong uniqueness of Leray solutions, and the extensions of the Prodi–Serrin criterion to larger classes of solutions. Two sections are on the existence of weak solutions, beginning with the Rellich–Lions theorem (with proof), and contain the following strategy for the main proof: the use of the Picard algorithm to solve on a small interval of time the mollified NSE; the establishing an energy estimate on this solution, and use of the Rellich–Lions theorem to relax the mollification and get a solution to the NSE. The next section deals with the weak–strong uniqueness problem. The problem relies to find conditions on a strong solution u of NSE such that all weak solutions which share the same initial condition u_0 equal u. The Prodi–Serrin uniqueness criterion is presented and proved, as well as the recent, generalized conditions, formulated and discussed by several authors. The problem of the uniqueness for almost strong solutions, and the novel results on stability of mild solutions of the NSE through weak perturbations, are also analysed.

Chapter 13, *Partial regularity results for weak solutions*, concerns the interior regularity criteria for weak solutions of the NSE, and the famous theorem of Caffarelli, Kohn, and Nirenberg (C–K–N) that gives the best estimate for Hausdorff's dimension of the singular set for a class of weak Leray–Hopf solutions to the Cauchy problem. The terse review of the state of the regularity theory for the NSE including the Serrin's, and the O'Leary's theorems on the interior regularity, is presented. The key points of the strategy of Scheffer and C–K–N to achieve an estimate of the Hausdorff dimension of the singular set, together with the new proofs and results of the partial regularity theorems for solutions of the NSE are described, and discussed.

Two chapters are devoted to solutions of NSE in Lebesgue spaces. Chapter 14, *A theory of uniformly locally L^2 solutions*, summarizes the state of knowledge about local existence (in time) of suitable local square-integrable weak solutions for NSE. The existence of uniformly locally square integrable solutions, in four steps, is proven first. Then, the local inequalities for local Leray solutions, together with the asymptotic behavior of local Leray solutions, are discussed, and proved. The specific variant of the C–K–N regularity criterion is formulated, and the inequalities in the L^∞ norm for local Leray solutions, are obtained and clarified. Also, the generalization of the weak–strong uniqueness theorem for Leray solutions is analyzed. Chapter 15, *The L^3 theory of suitable solutions* explores in an expository manner, the $L^\infty{}_t\, L^3_x$ regularity results for suitable solutions of the NSE, and the possibility of the existence of initial data with minimal L^3-norm for potential Navier–Stokes singularities.

Chapter 16, *Self-similarity and the Leray–Schauder principle*, deals with the Leray–Schauder theorem and its applications to existence problem of large steady-state solutions, and the existence of large self-similar solutions for NSE. At the beginning, the case of steady-state solutions of NSE is considered. The simple proof of existence is presented, based upon particular form of the Leray–Schauder principle, known as Schaefer's fixed-point theorem. The remaining sections concern the theory of self-similar solutions of NSE. The problem of the existence of forward self-similar solution for any large homogeneous initial value is formulated and proved. Also, the non-existence of backward self-similar solutions is shown, i.e., that the only self-similar solution satisfying the global energy estimates is the null solution. It is an answer on 1934 Leray's original problem, but some important questions regarding self-similarity were left open. As an additional reading, I recommend the recent, interesting paper, by Chae and Jörg Wolf (*On the Liouville Type Theorems for Self-Similar Solutions to the Navier–Stokes Equations*. Arch. Rational Mech. Anal. 225 (2017) 549–572).

The next two chapters concentrate on the approximate models of the NSE. In Chapter 17, *α-Models*, the four models are introduced and discussed, namely, the Leray-α model; the Navier–Stokes α-model, also known as viscous Camassa–Holm equations; the Clark-α model, and the simplified Bardina model. The strategy refers to the 1934 Leray paper, and relies on solving a mollified system, and then showing that the limit of the solutions of such mollified systems is, in some sense, a solution of the NSE. The following steps for the solutions are considered: the local existence; the energy estimates and global existence; the weak convergence; the global energy estimates for the weak limit; the local energy estimates for the weak limit, and the strong convergence. The approximate formulas for the Reynolds stress tensor in such models are also presented, and discussed briefly. Chapter 18, *Other approximations of the Navier–Stokes equations*, continues the subject by describing the other approximations and corrections for the NSE. These include: the Faedo–Galerkin method; the frequency cut-off technique; a higher order artificial viscosity approach (hyperviscosity); the Ladyzhenskaya's model, and the model with nonlinear damping. I think, that the way of enforcing uniqueness and global existence of weak solutions, connected with the introduce a damping term in the NSE is very promising (see also H. Liu, H. Gao, *Decay of solutions for the 3D Navier–Stokes equations with damping*. Appl. Math. Lett. 68 (2017) 48–54).

To overcome the computational difficulties connected with the incompressibility constraints, the "so-called" artificial compressibility approximation was developed. Chapter 19, *Artificial compressibility*, concerns the Leray weak solutions of the NSE, constructed by the artificial compressibility method. Two classical models, Temam's model and Vishik and Fursikov's model, are introduced and analyzed. Also, the correction to the NSE, combining the relaxation time term and Vishik and Fursikov's artificial compressibility to get a hyperbolic model with finite speed of propagation, (Hachicha model), is investigated.

The final chapter, *Conclusions*, recapitulates briefly the key results concerning solvability of the Cauchy problem for the NSE, together with a catalogue of what has been to prove in order to solve the sixth Millennium Problem.

To summarize, I would strongly recommend this book to anyone seriously interested in developments on Navier–Stokes equations theory in the second half of the twentieth century, and in the first 17 years of the twenty-first century. The book is on a source of extremely valuable information on NSE for both the mathematicians, and the mathematically oriented theoretical physicists. However, in my opinion, only those who professionally work in the area will have the fortitude to exploration it in all details! The prerequisites for the study are solid backgrounds in advanced analysis, theory of function spaces, distribution theory, and nonlinear partial differential equations. In particular, concerning the solutions of the NSE, I think that, from the physical point of view, a very important question is the obtain of at least one explicit, nontrivial solution of the full Navier–Stokes equations… in future!

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