Book Review

Analysis and Mathematical Physics by Shaun Bullett, Tom Fearn, and Frank Smith, LTCC Advanced Mathematics Series, vol. 6, World Scientific Publishing Europe Ltd., 2017; ISBN: 978-178634-098-6

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Physicists try to understand the nature of physical reality by constructing mathematical models. There is no logical reason why the mathematical modelling is possible at all, but "the steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced" (P.A.M. Dirac. *Quantised Singularities in the Electromagnetic Field*. Proc. Roy. Soc. A 133, 60, 1931).

The books published in the LTCC series are intended for graduate students and all physicists and mathematicians who wish to broaden their research horizons or who want to get a better knowledge of what is going on in a given field. The present book (volume six) concerns the topics in advanced analysis and mathematical physics.

The book consists of seven chapters. The first chapter of the book, written by Andrew Hone and Steffen Krusch, is devoted to a terse study of basic ideas of differential geometry. The notion of a manifold, which is one of the central concepts in modern theoretical physics, together with the examples, is presented. Useful concepts of fibre bundles, as well as the tangent vectors, tangent bundle, sections, metric tensor, and connections, are described and explained. The Yang–Mills theory in terms of fibre bundles philosophy is also sketched. Next, the elementary calculus of differential forms, including the calculus of integration and a discussion of the generalized Stokes theorem, Hodge decomposition, etc., is presented. This is applied to Ginzburg–Landau vortices

and Maxwell's equations. Some elements of complex manifolds theory is also provided. Three sections develop some of the geometrical concepts and tools that are helpful in understanding the formulations of Lagrangian mechanics, Hamiltonian mechanics, generalized Hamiltonian mechanics (inspired by the infinite-dimensional dynamical systems theory), classical field theory, and the dynamics of topological solitons. The emphasis is put on the first-order Lagrangian densities for field theories, the canonical Hamiltonian formalism, as well as the general Hamiltonian formulations of evolutionary partial differential equations. Finally, the relativistic vortex dynamics is discussed, from the point of view of Bogomolny formalism, a powerful approach in the study of solitons [see also: *On Bogomol'nyi Equations of Classical Solutions*, A.N. Atmaja, H. Ramadhan. Physical Review D 90, 105009 (2014)].

The second chapter, written by Yuri Safarov, provides a concise introduction to the microlocal analysis. Generally speaking, the microlocal analysis is an analysis of functions and systems of differential equations on the cotangent bundle, i.e. in the "phase space". First, the general framework of the theory of the Fourier transform and tempered distributions is recalled. Next, the pseudodifferential operators, their symbols, and the pseudolocal property are defined and analysed. Some applications of pseudodifferential operators in the theory of partial differential equations are described in some detail. It leads to the key notion of microlocal analysis: the wave front set of a distribution. As a result, the wave front set is used to analyse the problem of the propagation of singularities. Some familiarity with the modern functional analysis is a prerequisite for reading this

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chapter. Much of the subtle technical work in what follows has been shifted onto the reader, in the form of exercises (with solutions!). This chapter is my first favourite chapter of this book.

The third chapter, written by Cho-Ho Chu, contains a brief presentation of the theory of operator algebras (C^* -algebras and von Neumann algebras). First, the basic definitions of Banach algebras and C^* -algebras, some of the important related terminology and theorems, and several facts about functional calculus and spectrum of elements are collected. Moreover, the basic structure of von Neumann algebras and the techniques for working with them are discussed. Every von Neumann algebra is a C^* -algebra, and all the results and techniques developed earlier are applied to von Neumann algebras, but there is a whole set of techniques which are special to von Neumann algebras, many of which are inspired by ideas from measure theory. Some proofs are only sketched only and relegated to exercises and references.

The fourth chapter, written by Rod Halburd, treats special functions. The Weierstrass and Jacobi elliptic functions are described, and a number of identities are derived, including differential equations and addition laws. Several exercises are also proposed. Subsequently, special functions as solutions of Fuchsian ordinary differential equations (ODE) are examined (an ODE is said to be Fuchsian if all its singular points in the extended complex plane are regular). Discussion is confined to the basic properties of the hypergeometric equation. The Riemann equation, the most general second-order homogeneous linear equation with three regular singular points in the extended complex plane, is derived and examined. It is showed that Riemann's equation can be mapped to the hypergeometric equation. The series solutions and Kummer's solutions are analysed in some detail, together with integral representations. Other issues concern the monodromy's studies of Fuchsian hypergeometric differential equation. The basic problem in monodromy theory (now called the Riemann–Hilbert problem) can be formulated as follows: having given singularities and corresponding monodromy transformations, find a differential equation which realizes these data (see e.g. Żołądek H., *The Monodromy Group*. Birkhauser Verlag,

Basel–Boston–Berlin, 2006). For matrix Fuchsian equations with regular singular points, the Schlesinger equations are derived, with emphasis on how the sixth Painlevé equation arises from symmetries of Schlesinger's equations. The chapter ends with the bibliographical remarks and the solutions and hints for selected exercises.

The fifth chapter, written by Shahn Majid, gives a comprehensive overview of the basic ideas of non-commutative differential geometry, as a mathematical theory, with some remarks on possible physical applications. Roughly speaking, the basic idea of non-commutative geometry is to reformulate the geometry of a space in terms of an algebra of functions defined on it, and then to generalize the corresponding results of differential geometry to the case of a non-commutative algebra. The mathematical structure of quantum groups (or Hopf algebras) and some idea of the meaning of Hopf algebras for physics (to models of quantum spacetime) are presented and discussed. Some remarks on the non-commutative geometric approach to Planck scale physics emerging from quantum groups are also outlined. The idea of non-commutative spacetime encounters, however, has some serious conceptual problems. Therefore, for physically oriented readers, I recommend the additional materials (e.g. Eckstein, *The Geometry of Noncommutative Spacetimes*. Universe, 2017, 3(1), 25, <http://www.mdpi.com/search?authors=Micha%C5%82%20Eckstein&orcid=>). Five exercises with solutions supplemented the text.

The sixth chapter, written by Juan A. Valiente Kroon, discusses the general relativity as an initial value problem. A compact introduction to Einstein's general theory of relativity is offered first. Next, the direct and strong connection between Einstein's general theory of relativity and the theory of hyperbolic partial differential equations is illustrated. Some prototypical examples include the scalar wave equation, the source-free Maxwell equations, and the vacuum Einstein field equation. The following two sections contain key steps in studying the Einstein's equations as an initial value problem. The formalism discussed here is called the ADM (Arnowitt–Deser–Misner) formulation, or 3 + 1 formulation. This is an approach to general relativity and to Einstein equations that relies on the slicing of the four-dimensional

spacetime by three-dimensional surfaces. In such formulations, the initial data cannot be chosen freely; they have to satisfy certain constraint equations; so some aspects of the constraint equations are also explored. A brief discussion of time-independent solutions to the Einstein field equations, that is the solution in the form of Minkowski spacetime, and the Killing initial data for the Einstein's equations finishes the chapter.

The last, seventh chapter, written by Rod Halburd, contains an informative exposition of the Nevanlinna theory of meromorphic functions. Rolf Nevanlinna (1895–1980) was a Finnish mathematician, who made a crucial contribution to complex analysis. After necessary preparations (Jensen's formula, the Nevanlinna characteristic, counting and proximity functions, etc.), the first fundamental theorem of the value distribution theory is formulated and proved. The Lemma on the Logarithmic Derivative is then formulated (without a proof), and it is used to prove some global results about meromorphic solutions of some differential equations. The key consequence of the Lemma is the Nevanlinna's second fundamental theorem, the deepest and the most important result of

the value distribution theory. The proof of theorem is presented, with its corollaries (e.g. Picard's theorem), together with the deficiency relations. This chapter is my second favourite chapter of this book, but for a nice intuitive introduction to this field. I also refer the reader to an article of K.S. Charak (*Value Distribution Theory of Meromorphic Functions*. Mathematics Newsletter, Vol. 18 4, March 2009).

Overall, this is a very nicely and uniformly written book, but it is not for beginners. The book's presentation requires from the reader a fairly high level of mathematical sophistication. I think also that this book would be an excellent choice for a lecturer wishing to teach selected branches of mathematical physics to theoretically oriented geophysicists.

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