Book Review

“*Differential Geometry of Curves and Surfaces*”, by M. Umehara, and K. Yamada, World Scientific Publishing Co., 2017, ISBN 978-981-4740-24-1

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This book is intended to bring the reader through classical differential geometry of curves and surfaces into the modern differential geometry of manifolds. The classical differential geometry studies the local properties of the curves and surfaces by using techniques based on differential calculus. Chapters I and II, the 11 sections in total, offer a concise guide for the study of classical theory of curves and surfaces, and are designed as a textbook for a one-semester course for undergraduates, assuming only experience in calculus and linear algebra at the undergraduate level. Chapter III, eight sections in sum, provides an introduction to the theory of differentiable manifolds—the natural generalization of regular curves and surfaces to higher dimensions. It is addressed to advanced undergraduate readers. Each section ends with a large set of carefully selected exercises that are suitable for self-study. The solutions to them require often inventiveness and geometrical intuition. Two appendices contain the necessary background from linear algebra and calculus, that is helpful for reading the first 11 sections of this book, and more advanced topics within curve and surface theory.

Chapter I, *Curves*, deals with the preliminary concepts in plane curves and space curves, respectively. The mathematical tools needed to study the curves in the plane, are developed first. The subjects covered are the implicit function representation for curves, parameterizations of curves, the concept of length of curves, the notion of curvature, the construction of the evolute, and the Frenet formula with the applications. Most of the

subjects are subsequently explained in some detail from the geometrical viewpoints (with proofs). The chapter is supplemented by the discussion of the properties of closed regular curves, and the geometry of spirals. The notion of rotation index for closed regular curves is introduced, and two main theorems are formulated and proved, together with the Whitney’s formula which shows how to calculate the rotation index by counting self-intersections of the curve. Two spectacular examples of spiral curves, namely, the Archimedean spiral and the logarithmic spiral, are presented and investigated. A Möbius transformation is described and used to prove the four-vertex theorem for simple, not necessarily convex, curves. Other properties of spiral curves are left to readers, as interesting exercises. The last section refers to the local properties of space curves similarly to the theory of plane curves. The fundamental theorem for space curves is formulated and proved, and the concept of a knotted curve is illuminated briefly.

Chapter II, *Surfaces*, concerns the local theory of surfaces. First, the basic ways to describe surfaces are presented including graphs, implicit function representation and parameterizations. The notion of a tangent space, normal vector and the concept of the area of a surface, are introduced also. An interesting example of regular surface with self-intersections, namely, the Klein bottle is illustrated briefly. The next section is devoted to the first fundamental form of a surface, which contains information needed to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions, etc.). Another section deals with the second fundamental form and focuses on the two fundamental geometric invariants—on the Gaussian curvature and on the mean curvature, respectively. The geometric

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meanings of these curvatures are explained and illustrated. There is a special example studying the properties of surfaces of revolution with constant Gaussian curvature. Then, the geometric explanation of the principal curvatures on regular surfaces is presented. Particular attention is paid to geodesics and the Gauss–Bonnet theorem, which relates the sum of the interior angles of a triangle with its Gaussian curvature, an intrinsic feature of the geometry of the space that the triangle is drawn on. The famous fact known as *Theorema Egregium* of Gauss, is also discussed. The applications of the Gauss–Bonnet theorem to closed surfaces, euclidean, and hyperbolic geometry are outlined. At the end of chapter, the detailed proof of the Gauss–Bonnet theorem is presented.

Roughly speaking, manifolds are generalizations of our imaginations about curves and surfaces to arbitrary dimensional objects. Chapter III, *Surfaces from the Viewpoint of Manifolds*, presents the selected aspects of differential geometry by studying manifolds. The mathematical machinery connected with differential forms is presented first. After defining differential forms, an exterior differentiation, wedge product, orientability, and the important notion of a manifold-with-boundary, which is the setting for Stokes' theorem, are discussed. The notion of Levi-Civita connections, which is of central importance for defining Gaussian curvature for Riemannian 2-manifolds, is introduced and elucidated. In the next section, the global Gauss–Bonnet theorem for 2-manifolds is formulated and proved. As illustration of this theorem, the Poincaré–Hopf theorem, which relates vector fields on compact oriented 2-manifolds to the Euler characteristic, is stated and proved. Also, the index of umbilics on a surface is studied. The subsequent section shows the existence of conformal coordinates for surfaces. Another section concerns the fundamental equations of surfaces and presents relations between the first and second fundamental forms—the Gauss equations and the Codazzi equations. The theorem regarding the characterization of the sphere (the Hopf theorem), is presented and proved also. The classical

brachistochrone problem from the viewpoint of Riemannian geometry is illustrated. The chapter ends with the proof of existence of geodesic triangulations on surfaces.

The main topics included in Appendix A (Supplements) are the following: Taylor's formula, L'Hôpital's rule, the hyperbolic functions, the triangle inequalities for integrals and for vector-valued functions, the inverse and implicit function theorems, and change of variables in double integrals. It also presents the fundamental theorems for ordinary differential equations and the elements of theory of Euclidean spaces.

Appendix B, *Advanced Topics on Curves and Surfaces*, consists of ten concise sections. Based on classical geometric concepts, the geometry of envelopes, evolutes and involutes is discussed first. Also, the Huygens's cycloidal pendulum and some secrets of the cycloid curves, are presented. Then, the convex curves and curves of constant width together with the construction of such curves, are investigated. The Green's theorem, which connects the line integral with the double integral is formulated and the isoperimetric inequality is proved. The next section focuses on the world maps of the Earth. It is known that the first fundamental form gives information about lengths, angles and areas, on the map treated as a map of the parametrized surface. A several examples of map projections are considered, including Mercator's world map, the Lambert cylindrical equal-area projection, the stereographic projection, and the central projection. The last section contains the proof of the fundamental theorem of surface theory. After necessary preparations (auxiliary lemmas, and theorems), the proof of existence and uniqueness theorem for surfaces, is presented. The book ends with the solutions and hints for almost all exercises.

The book will serve as a very useful reference for a broad range of applied mathematicians, physicists, as well as theoretical geophysicists seeking a precise, systematic presentation of the differential geometry underlying much of modern theory. After reading the book the reader should have a good feeling for the material presented, provided that he has done all the

exercises! For mathematically oriented readers, I recommend also the additional material. I think that a good companion in reading Umehara and Yamada book could be the classical textbook by Marcel Berger and Bernard Gostiaux *Differential Geometry: Manifolds, Curves, and Surfaces*, GTM 115, Springer-Science + Business Media, LLC, 1988.

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