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Kasner-Like Behaviour for Subcritical Einstein-Matter Systems

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Abstract. Confirming previous heuristic analyses à la Belinskii-Khalatnikov-Lifshitz, it is rigorously proven that certain "subcritical" Einstein-matter systems exhibit a monotone, generalized Kasner behaviour in the vicinity of a spacelike singularity. The D-dimensional coupled Einstein-dilaton-p-form system is subcritical if the dilaton couplings of the p-forms belong to some dimension-dependent open neighborhood of zero [1], while pure gravity is subcritical if $D \geq 11$ [13]. Our proof relies, like the recent Theorem [15] dealing with the (always subcritical [14]) Einstein-dilaton system, on the use of Fuchsian techniques, which enable one to construct local, analytic solutions to the full set of equations of motion. The solutions constructed are "general" in the sense that they depend on the maximal expected number of free functions.

1 Introduction

1.1 The problem

In recent papers [1, 2, 3], the dynamics of the coupled Einstein-dilaton-p-form system in D spacetime dimensions, with action (in units where $8\pi G = 1$),

$$S[g_{\alpha\beta}, \phi, A^{(j)}_{\gamma_1\cdots\gamma_{n_j}}] = S_E[g_{\alpha\beta}] + S_\phi[g_{\alpha\beta}, \phi] + \sum_{j=1}^{k} S_j[g_{\alpha\beta}, \phi, A^{(j)}_{\gamma_1\cdots\gamma_{n_j}}] + \text{ "more"}, \tag{1.1}$$

$$S_E[g_{\alpha\beta}] = \frac{1}{2}\int R\sqrt{-g}\, d^D x, \tag{1.2}$$

$$S_\phi[g_{\alpha\beta}, \phi] = -\frac{1}{2}\int \partial_\mu\phi\, \partial^\mu\phi \sqrt{-g}\, d^D x, \tag{1.3}$$

$$S_j[g_{\alpha\beta}, \phi, A^{(j)}_{\gamma_1\cdots\gamma_{n_j}}] = -\frac{1}{2(n_j+1)!}\int F^{(j)}_{\mu_1\cdots\mu_{n_j+1}} F^{(j)\,\mu_1\cdots\mu_{n_j+1}} e^{\lambda_j\phi}\sqrt{-g}\, d^D x, \tag{1.4}$$

was investigated in the vicinity of a spacelike ("cosmological") singularity along the lines initiated by Belinskii, Khalatnikov and Lifshitz (BKL) [4]. In (1.1), $g_{\alpha\beta}$ is the spacetime metric, ϕ is a massless scalar field known as the "dilaton", while the $A^{(j)}_{\gamma_1\cdots\gamma_{n_j}}$ are a collection of k exterior form gauge fields ($j = 1, \ldots, k$), with exponential couplings to the dilaton, each coupling being characterized by an individual constant λ_j ("dilaton coupling constant"). The $F^{(j)}$'s are the exterior derivatives $F^{(j)} = dA^{(j)}$, whereas "more" stands for possible coupling terms among the p-forms which can be either of the Yang-Mills type (1-forms), Chern-Simons type [5] or Chapline-Manton type [6, 7]. The degrees of the p-forms are restricted to be smaller than or equal to $D-2$ since a $(D-1)$-form (or D-form) gauge field carries no local degree of freedom. In particular, scalars ($n_j = 0$) are allowed among the $A^{(j)}$'s but we then require that the corresponding dilaton coupling λ_j be non-zero, so that there is only one "dilaton". Similarly, we require $\lambda_j \neq 0$ for the $(D-2)$-forms (if any), since these are "dual"[1] to scalars. This restriction to a single dilaton is mostly done for notational convenience: if there were other dilatons among the 0-forms, then, these must be explicitly treated on the same footing as ϕ and separated off from the p-forms because they play a distinct rôle. In particular, they would appear explicitly in the generalized Kasner conditions given below and in the determination of what we call the subcritical domain. The discussion would proceed otherwise in the same qualitative way.

The main motivation for studying actions of the class (1.1) is that these arise as bosonic sectors of supergravity theories related to superstring or M-theory. In fact, in view of various no-go theorems, p-form gauge fields appear to be the only massless, higher spin fields that can be consistently coupled to gravity. Furthermore, there can be only one type of graviton [8]. With this observation in mind, the Action (1.1) is actually quite general. The only restriction concerns the scalar sector: we assume the coupling to the dilaton to be exponential because this corresponds to the tree-level couplings of the dilaton field of string theory. Note, however, that string-loop effects are expected to generate more general couplings $\exp(\lambda\phi) \to B(\phi)$ which can exhibit interesting "attractor" behaviours [9]. We also restrict ourselves by not including scalar potentials; see, however, the end of the article for some remarks on the addition of a potential for the dilaton, which can be treated by our methods.

Two possible general, "competing" behaviours of the fields in the vicinity of the spacelike singularity have been identified[2]:

1. The simplest is the *"generalized Kasner behaviour"*, in which the spatial scale factors and the field $\exp(\phi)$ behave at each spatial point in a monotone, power-law fashion in terms of the proper time as one approaches the singularity, while the effect of the p-form fields $A^{(j)}$'s on the evolution of $g_{\mu\nu}$ and ϕ can be asymptotically neglected. In that regime the spatial curvature terms can be also neglected with respect to the leading order part of the extrinsic curvature terms. In other words, as emphasized by BKL, time derivatives asymptotically dominate over space derivatives so that one sometimes uses the terminology "velocity-dominated" behaviour [11], instead of "general-

[1] We recall that the Hodge duality between a (n_j+1)-form and a $(D-n_j-1)$-form allows one to replace (locally) a n_j-form potential $A^{(j)}$ by a $(D-n_j-2)$-form potential $A^{(j')}$ (with dilaton coupling $\lambda'_j = -\lambda_j$).

[2] For a recent extension of these ideas to the brane-worlds scenarios, see [10].

ized Kasner behaviour". We shall use both terminologies indifferently in this paper, recalling that in the presence of p-forms, which act as potentials for the evolution of the spatial metric and the dilaton (as do the spatial curvature terms), "velocity-dominance" means not only that the spatial curvature terms can be neglected, but also that the p-forms can be neglected in the Einstein-dilaton evolution equations.[3]

2. The second regime, known as *"oscillatory"* [4], or *"generalized mixmaster"* [12] behaviour, is more complicated. It can be described as the succession of an infinite number of increasingly shorter Kasner regimes as one goes to the singularity, one following the other according to a well-defined "collision" law. This asymptotic evolution is presumably strongly chaotic. It is expected that, at each spatial point, the scale factors of a general inhomogeneous solution essentially behave as in certain homogeneous models. For instance, for $D = 4$ pure gravity this guiding homogeneous model is the Bianchi IX model [4, 12], while for $D = 11$ supergravity it is its naive one-dimensional reduction involving space-independent metric and three-forms [2].

Whether it is the first or the second behaviour that is relevant depends on: (i) the spacetime dimension D, (ii) the field content (presence or absence of the dilaton, types of p-forms), and (iii) the values of the various dilaton couplings λ_j. Previous work reached the following conclusions:

- The oscillatory behaviour is general for pure gravity in spacetime dimension 4 [4], in fact, in all spacetime dimensions $4 \leq D \leq 10$, but is replaced by a Kasner-like behaviour in spacetime dimensions $D \geq 11$ [13]. (The sense in which we use "general" will be made precise below.)
- The Kasner-like behaviour is general for the gravity-dilaton system in all spacetime dimensions $D \geq 3$ (see [14, 15] for $D = 4$).
- The oscillatory behaviour is general for gravity coupled to p-forms, in absence of a dilaton or of a dual $(D-2)$-form $(0 < p < D-2)$ [2]. In particular, the bosonic sector of 11-dimensional supergravity is oscillatory [1]. Particular instances of this case have been studied in [16, 17, 18].
- The case of the gravity-dilaton-p-form system is more complicated to discuss because its behaviour depends on a combination of several factors, namely the dimension D, the menu of p-forms, and the numerical values of the dilaton couplings. For a given D and a given menu of p-forms there exists a "subcritical" domain $\mathcal{D}$ (an open neighborhood of the origin $\lambda_j = 0$ for all j's) such that: (i) when the λ_j belong to $\mathcal{D}$ the general behaviour is Kasner-

[3] The Kasner solution is generalized in two ways: first, the original Kasner exponents include a dilaton exponent (if there is a dilaton), which appears in the Kasner conditions; second, the exponents are not assumed to be constant in space. We shall shorten "exhibits generalized Kasner behaviour" to Kasner-like. We stress that we do not use this term to indicate that the solution becomes asymptotically homogeneous in space.

like, but (ii) when the λ_j do not belong to $\mathcal{D}$ the behaviour is oscillatory. Note that $\mathcal{D}$ is open. Indeed, the behaviour is oscillatory when the λ_j are on the boundary of $\mathcal{D}$, as happens for instance for the low-energy bosonic sectors of type I or heterotic superstrings [1]. For a single p-form, the subcritical domain $\mathcal{D}$ takes the simple form $|\lambda_j| < \lambda_j^c$, where λ_j^c depends on the form-degree and the spacetime dimension. (λ_j^c can be infinite.) For a collection of p-forms, $\mathcal{D}$ is more complicated and not just given by the Cartesian product of the subcritical intervals associated with each individual p-form.

The above statements were derived by adopting a line of thought analogous to that followed by BKL. Now, as understood by BKL themselves, these arguments, although quite convincing, are somewhat heuristic. It is true that the original arguments have received since then a considerable amount of both numerical and analytical support [19, 20, 21, 22, 23]. Yet, they still await a complete proof. One notable exception is the four-dimensional gravity-dilaton system, which has been rigorously demonstrated in [15] to be indeed Kasner-like, confirming the original analysis [14]. Using Fuchsian techniques, the authors of [15] have proven the existence of a local (analytic) Kasner-like solution to the Einstein-dilaton equations in four dimensions that contains as many arbitrary, physically relevant functions of space as there are local degrees of freedom, namely 6 (counting q and $\dot{q}$ independently). To our knowledge, this was the first construction, in a rigorous mathematical sense, of a general singular solution for a coupled Einstein-matter system. Note in this respect several previous works in which formal solutions had been constructed near (Kasner-like) cosmological singularities by explicit perturbative methods, to all orders of perturbation theory [24, 25].

The situation concerning the more complicated (and in some sense more interesting) generalized mixmaster regime is unfortunately – and perhaps not surprisingly – not so well developed. Rigorous results are scarce (note [26]) and even in the case of the spatially homogeneous Bianchi IX model only partial results exist in the literature [27].

The purpose of this paper is to extend the Fuchsian approach of [15] to the more complicated class of models described by the Action (1.1) and to prove that those among the above models that were predicted in [13, 1, 2] to be Kasner-like are indeed so. This provides many new instances where one can rigorously construct a general singular solution for a coupled Einstein-matter (or pure Einstein, in $D \geq 11$) system. In fact, our (Fuchsian-system-based) results prove that the formal perturbative solutions that can be explicitly built for these models do converge to exact solutions. This provides a further confirmation of the general validity of the BKL ideas. We shall also explicitly determine the subcritical domain $\mathcal{D}$ for a few illustrative models. For all the relevant systems, we construct local (near the singularity) analytic solutions, which are "general" in the sense that they contain the right number of freely adjustable arbitrary functions of space (in particular, these solutions have generically no isometries), and which exhibit the generalized (monotone) Kasner time dependence.

1.2 Strategy and outline of the paper

Our approach is the same as in [15], and results from that work will be used frequently here without restating the arguments. Here is an outline of the key steps.

A $d + 1$ decomposition is used, for d spatial dimensions, $d = D - 1$. A Gaussian time coordinate, t, is chosen such that the singularity occurs at $t = 0$. The first step in the argument consists of identifying the leading terms for all the variables. This is accomplished by writing down a set of evolution equations which is obtained by truncating the full evolution equations, and then solving this simpler set of evolution equations. This simpler evolution system is called the Kasner-like[4] evolution system (or, alternatively, the velocity-dominated system). It is a system of ordinary differential equations with respect to time (one at each spatial point) which coincides with the system that arises when investigating metric-dilaton solutions that depend only on time. The precise truncation rules are given in Subsection 2.2 below. The second step is to write down constraint equations for the Kasner-like system (called "velocity-dominated" constraints) and to show that these constraints propagate, i.e., that if they are satisfied by a solution to the Kasner-like evolution equations at some time $t_0 > 0$, then they are satisfied for all time $t > 0$. In the set of Kasner-like solutions, one expects that there is a subset, denoted by V, of solutions which have the property of being asymptotic to solutions of the complete Einstein-dilaton-p-form equations as $t \to 0$, i.e., as one goes to the singularity. This subset is characterized by inequalities on some of the initial data, which, however, are not always consistent. The existence of a non-empty V requires the dilaton couplings to belong to some range, the "subcritical range". When V is non-empty and open, the solutions in V involve as many arbitrary functions of space as a "general solution" of the full Einstein equations should. On the other hand, if V is empty the construction given in this paper breaks down and the dynamical system is expected to be not Kasner-like but rather oscillatory.

To show that indeed, the solutions in V (when it is non-empty) are asymptotic to true solutions, the third step is to identify decaying quantities such that these decaying quantities along with the leading terms mentioned above uniquely determine the variables, and to write down a *Fuchsian* system for the decaying quantities which is equivalent to the Einstein-matter evolution system. As the use of Fuchsian systems is central to our work let us briefly recall what a Fuchsian system is and how such a system is related to familiar iterative methods. For a more detailed introduction to Fuchsian techniques see [15, 28, 29, 30] and references therein. Note that we shall everywhere restrict ourselves to the analytic case. We expect that our results extend to the C^∞ case, but it is a non-trivial task to prove that they do.

[4]Note that we use the terms "Kasner-like solutions" to label both exact solutions of the truncated system and solutions of the full system that are asymptotic to such solutions. Which meaning is relevant should be clear from the context.

The general form of a Fuchsian system for a vector-valued unknown function u is

$$t\,\partial_t u + \mathcal{A}(x)\,u = f(t,x,u,u_x), \tag{1.5}$$

where the matrix $\mathcal{A}(x)$ is required to satisfy some positivity condition (see below), while the "source term" f on the right-hand side is required to be "regular." (See [15] for precise criteria allowing one to check when the positivity assumption on $\mathcal{A}(x)$ is satisfied and when f is regular.) A key point is that f is required to be bounded by terms of order $O(t^\delta)$ (with $t \to 0, \delta > 0$) as soon as u and their space derivatives u_x are in a bounded set (a simple, concrete example of a source term satisfying this condition is $f = t^{\delta_1} + t^{\delta_2}u + t^{\delta_3}u_x$, with δ_i's larger than δ). A convenient form of positivity condition to be satisfied by the matrix $\mathcal{A}(x)$ is that the operator norm of $\tau^{\mathcal{A}(x)}$ be bounded when $0 < \tau < 1$ (and when x varies in any open set). Essentially this condition restricts the eigenvalues of the matrix $\mathcal{A}(x)$ to have positive real parts. The basic property of Fuchsian systems that we shall use is that there is a unique solution to the Fuchsian equation which vanishes as t tends to zero [28]. One can understand this theorem as a mathematically rigorous version of the recursive method for solving the Equation (1.5). Indeed, when confronted with Equation (1.5), it is natural to construct a solution by an iterative process, starting with the zeroth order approximation $u_0 = 0$ (which is the unique solution of (1.5) with $f \equiv 0$ that tends to zero as $t \to 0$), and solving a sequence of equations of the form $t\partial_t u^{(n)} + \mathcal{A}(x)u^{(n)} = f(t,x,u^{(n-1)},u_x^{(n-1)})$. At each step in this iterative process the source term is a known function which essentially behaves (modulo logarithms) like a sum of powers of t (with space-dependent coefficients). The crucial step in the iteration is then to solve equations of the type $t\partial_t u + \mathcal{A}(x)u = C(x)t^{\delta(x)}$. The positivity condition on $\mathcal{A}(x)$ guarantees the absence of homogeneous solutions remaining bounded as $t \to 0$, and ensures the absence of "small denominators" in the (unique bounded) inhomogeneous solution generated by each partial source term: $u_{\rm inhom} = (\delta + \mathcal{A})^{-1}Ct^\delta$. (See, *e.g.,* [25] for a concrete iterative construction of a Kasner-like solution and the proof that it extends to all orders.) This link between Fuchsian systems and "good systems" that can be solved to all orders in a formal iteration makes it a priori probable that all cases which the heuristic approach à la BKL has shown to be asymptotic to a Kasner-like solution (by checking that the leading "post-Kasner" contribution is asymptotically sub-dominant) can be cast in a Fuchsian form. The main technical burden of the present work will indeed be to show in detail how this can be carried out for the evolution systems corresponding to all the sub-critical (i.e., non-oscillatory) Einstein-matter systems. Our Fuchsian formulation proves that (in the analytic case) the formal all-orders iterative solutions for the models we consider do actually converge to the unique, exact solution having a given leading Kasner asymptotic behaviour as $t \to 0$.

Finally, the fourth step of our strategy is to prove that the constructed solution does satisfy also all the Einstein and Gauss-like constraints so that it is a solution of the full set of Einstein-matter equations. We shall deal successively with the matter (Gauss-like) constraints, and the Einstein constraints.

Our paper is organized as follows. In Section 2, we first consider the paradigmatic example of gravity coupled to a massless scalar field and to a Maxwell field in 4 spacetime dimensions. The Action (1.1) reads in this case

$$S[g_{\alpha\beta}, \phi, A_\gamma] = \frac{1}{2} \int \{R - \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} e^{\lambda\phi}\} \sqrt{-g} \, d^4x. \tag{1.6}$$

For this simple example, we shall explicitly determine the subcritical domain $\mathcal{D}$, i.e., the critical value λ_c such that the system is Kasner-like when $-\lambda_c < \lambda < \lambda_c$. Because this case is exemplary of the general situation, while still being technically rather simple to handle, we shall describe in some detail the explicit steps of the Fuchsian approach.

In Section 3, vacuum solutions governed by the pure Einstein Action (1.2) with $D \geq 11$ are considered. This system was argued in [13] to be Kasner-like and we show here how this rigorously follows from the Fuchsian approach. Note that, contrary to what happens when a dilaton is present, Fuchsian techniques apply here even though not all Kasner exponents can be positive.

In Sections 4–8, the results of the previous sections are generalized to the wider class of systems (1.1). First, in Section 4, to solutions of Einstein's equation with spacetime dimension $D \geq 3$ and a matter source consisting of a massless scalar field, governed by the action $S_E[g_{\alpha\beta}] + S_\phi[g_{\alpha\beta}, \phi]$. This is the generalization to any $D \geq 3$ of the case $D = 4$ treated in [15]. In Section 5, we turn to the general situation described by the Action (1.1), without, however, including the additional terms represented there by "more". We then give some general rules for computing the subcritical domain of the dilaton couplings guaranteeing velocity-dominance (Section 6). The inclusion of interaction terms is considered in the last sections. It is shown that they do not affect the asymptotic analysis. This is done first for the Chern-Simons and Chapline-Manton interactions in Section 7, and next, in Section 8, for the Yang-Mills couplings (for some gauge group G), for which the action reads

$$S[g_{\alpha\beta}, \phi, A_\gamma] = \frac{1}{2} \int \{R - \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} F_{\mu\nu} \cdot F^{\mu\nu} e^{\lambda\phi}\} \sqrt{-g} \, d^Dx. \tag{1.7}$$

Here the dot product, $F \cdot F$, is a time-independent, Ad-invariant, non-degenerate scalar product on the Lie algebra of G (such a scalar product exists if the algebra is compact, or semi-simple). Contrary to what is done in Sections 2, 5 and 7, we must work now with the vector potential (and not just with the field strength), since it appears explicitly in the coupling terms.

In Section 9 we show that self-interactions of a rather general type for the scalar field can be included without changing the asymptotics of the solutions.

Explicitly, we add a (nonlinear) potential term,

$$S_{\rm NL}[g_{\alpha\beta}, \phi] = -\int V(\phi)\sqrt{-g}\, d^D x, \tag{1.8}$$

to the Action (1.1), where $V(\phi)$ must fulfill some assumptions given in Section 9. $V(\phi)$ may, for example, be an exponential function of ϕ, a constant, or a suitable power of ϕ. Similar forms for $V(\phi)$ were considered with $D = 4$ in [31].

Finally, in Section 10, we state two theorems that summarize the main results of the paper and give concluding remarks.

1.3 On the generality of our construction

As we shall see the number of arbitrary functions contained in solutions to the velocity-dominated constraint equations is equal to the number of arbitrary functions for solutions to the Einstein-matter constraints. In this function-counting sense, our construction describes what is customarily called a "general" solution of the system. Intuitively speaking, our construction concerns some "open set" of the set of all solutions (indeed, the Kasner-like behaviour of the solution is unchanged under arbitrary, small perturbations of the initial data, because this simply amounts to changing the integration functions). Note that, in the physics literature, such a "general" solution is often referred to as being a "generic" solution. However, in the mathematics literature the word "generic" is restricted to describing either an open dense subset of the set of all solutions, or (when this can be defined) a subset of measure unity of the set of all solutions. In this work we shall stick to the mathematical terminology. We shall have nothing rigorous to say about whether our general solution is also generic. However, we wish to emphasize the following points.

First, let us mention that the set V of solutions to the velocity-dominated equations that are asymptotic to solutions of the complete equations is not identical to the set U of all solutions to the velocity-dominated constraint equations. The subset $V \subset U$ is defined by imposing some inequalities on the free data. These inequalities do not change the number of free functions. Therefore the solutions in V are still "general". One can wonder whether there could be a co-existing general behaviour, corresponding to initial data that do not fulfill the inequalities. For instance, could such "bad" initial data lead to a generalized mixmaster regime? This is a difficult question and we shall only summarize here what is the existing evidence. There are heuristic arguments, supported by numerical study, [14, 32, 33, 34] that suggest that if one starts with initial data that do not fulfill the inequalities, one ends up, after a finite transient period (with a finite number of "collisions" with potential walls), with a solution that is asymptotically velocity-dominated, for which the inequalities are fulfilled almost everywhere. In that sense, the inequalities would not represent a real restriction since there is a dynamical mechanism that drives the solution to the regime where they are satisfied. For the subcritical values of the dilaton couplings that make the inequalities defining V consistent, there is thus no evidence for an alternative oscillatory regime corresponding to a different (open) region in the space of initial data[5]. It has indeed been shown that the inequalities defining V are no restriction in a large spatially homogeneous class [27]. Such rigorous results are, however, lacking in the inhomogeneous case. In fact, an interesting subtlety might take place in the inhomogeneous case. The heuristic arguments and numerical studies of [33, 34] suggest the possibility that the mechanism driving the system to V may be suppressed at exceptional spatial points in general spacetimes, with the result that the asymptotic data at the exceptional spatial points are not consistent with the inequalities we assume and lead to so-called "spikes". This picture has been given a firm basis in a scalar field model with symmetries [35] but the status of the spikes in a general context remains unclear.

Finally, since we only deal with spacelike singularities, the classes of solutions we consider do not contain all solutions governed by the Action (1.1). Other types of singularities (*e.g.* timelike or null ones) are known to exist. Whether these other types of singularities are general is, however, an open question.

1.4 Billiard picture

At each spatial point, the solution of the coupled Einstein-matter system can be pictured, in the vicinity of a spacelike singularity, as a billiard motion in a region of hyperbolic space [36, 37, 3, 38]. Hyperbolic billiards are chaotic when they have finite volume and non chaotic otherwise. In this latter case, the "billiard ball" generically escapes freely to infinity after a finite number of collisions with the bounding walls. Subcritical Einstein-matter systems define infinite-volume billiards. The velocity-dominated solutions correspond precisely to the last (as $t \to 0$) free motion (after all collisions have taken place), in which the billiard ball moves to infinity in hyperbolic space.

1.5 Conventions

We adopt a "mostly plus" signature $(- + + + \ldots)$. The spacetime dimension is $D \equiv d + 1$. Greek indices range from 0 to d, while Latin indices $\in \{1, \ldots, d\}$. The spatial Ricci tensor is labeled R and the spacetime Ricci tensor is labeled ${}^{(D)}R$. Our curvature conventions are such that the Ricci tensor of a sphere is positive definite. Einstein's equations read $G_{\alpha\beta} = T_{\alpha\beta}$, where $G_{\alpha\beta} = R_{\alpha\beta} - Rg_{\alpha\beta}/2$ denotes the Einstein tensor and $T_{\alpha\beta}$ denotes the matter stress-energy tensor, $T_{\alpha\beta} = -(2/\sqrt{-g})\delta S_{\mathrm{matter}}/\delta g^{\alpha\beta}$, and units such that $8\pi G = 1$. The spatial metric compatible covariant derivative is labeled ∇_a and the spacetime metric compatible covariant derivative is labeled ${}^{(D)}\nabla_\alpha$. The velocity-dominated metric compatible

[5]The oscillatory regime may however be present for peculiar initial data, presumably forming a set of zero measure. For instance, gravity + dilaton is generically Kasner-like, but exhibits an oscillatory behaviour for initial data with $\phi = 0$ (in $D < 11$ spacetime dimensions).

covariant derivative is labeled ${}^0\nabla_a$. According to the context, g denotes the (positive) determinant of g_{ab} in $d+1$-decomposed expressions, and the (negative) determinant of $g_{\mu\nu}$ in spacetime expressions. Whenever t^δ or $t^{-\delta}$ appears, δ denotes a strictly positive number, arbitrarily small. We use Einstein's summation convention on repeated tensor indices of different variances. (When the need arises to suspend the summation conventions for some non-tensorial indices, we shall explicitly mention it.) In expressions where there is a sum that the indices do not indicate, all sums in the expression are indicated explicitly by a summation symbol. Indices on the velocity-dominated metric and the velocity-dominated extrinsic curvature are raised and lowered with the velocity-dominated metric.

1.6 $d+1$ decomposition

Consider a solution to the Einstein's equations following from (1.1), consisting of a Lorentz metric and matter fields on a D-dimensional manifold M which is diffeomorphic to $(0,T)\times\Sigma$ for a d-dimensional manifold, Σ, such that the metric induced on each $t =$ constant hypersurface is Riemannian, for $t\in(0,T)$. Here D is an integer strictly greater than two. Furthermore, consider a $d+1$ decomposition of the Einstein tensor, $G_{\alpha\beta}$, and the stress-energy tensor, $T_{\alpha\beta}$, with a Gaussian time coordinate, $t\in(0,T)$, and a local frame $\{e_a\}$ on Σ. Note that the frame $e_a = e^i_a(x)\partial_i$ is time-independent. The spacetime metric reads $ds^2 = -dt^2 + g_{ab}(t,x)e^a e^b$, where $e^a = e^a_i(x)dx^i$ (with $e^a_i e^i_b = \delta^a_b$) is the co-frame. Let $\rho = T_{00}$, $j_a = -T_{0a}$ and $S_{ab} = T_{ab}$. Define

$$\begin{aligned} C &= 2G_{00} - 2T_{00} \\ &= -k^a{}_b\, k^b{}_a + (\operatorname{tr} k)^2 + R - 2\rho. \end{aligned} \tag{1.9}$$

$C = 0$ is the Hamiltonian constraint. Similarly, $C_a = 0$ is the momentum constraint, where

$$\begin{aligned} C_a &= -G_{0a} + T_{0a} \\ &= \nabla_b k^b{}_a - \nabla_a(\operatorname{tr} k) - j_a. \end{aligned} \tag{1.10}$$

In Gaussian coordinates, the relation between the metric and the extrinsic curvature is

$$\partial_t g_{ab} = -2k_{ab}. \tag{1.11}$$

The evolution equation for the extrinsic curvature is obtained by setting $E^a{}_b = 0$, with

$$E^a{}_b = {}^{(D)}R^a{}_b - T^a{}_b + \frac{1}{(D-2)}T\,\delta^a{}_b \tag{1.12}$$

$$\Rightarrow\ \partial_t k^a{}_b = R^a{}_b + (\operatorname{tr} k)\, k^a{}_b - M^a{}_b. \tag{1.13}$$

Here

$$M^a{}_b = S^a{}_b - \frac{1}{D-2}((\operatorname{tr} S) - \rho)\delta^a{}_b.$$

2 Scalar and Maxwell fields in four dimensions

2.1 Equations of motion

As said above, let us start by considering in detail, as archetypal system, the system defined by the Action (1.6), i.e., the spacetime dimension is $D = 4$ and the matter fields are a massless scalar field exponentially coupled to a Maxwell field, with the magnitude of the dilaton coupling constant smaller in magnitude than some positive real number determined below, $0 \leq |\lambda| < \lambda_c$. The stress-energy tensor of the matter fields is

$$T_{\mu\nu} = {}^{(4)}\nabla_\mu \phi \, {}^{(4)}\nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \, {}^{(4)}\nabla_\alpha \phi \, {}^{(4)}\nabla^\alpha \phi + [F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}] e^{\lambda\phi}.$$

The matter fields satisfy the following equations.

$$\begin{aligned} {}^{(4)}\nabla_\alpha \, {}^{(4)}\nabla^\alpha \phi &= \frac{\lambda}{4} F_{\alpha\beta} F^{\alpha\beta} e^{\lambda\phi}, \\ {}^{(4)}\nabla_\mu (F^{\mu\nu} e^{\lambda\phi}) &= 0, \\ {}^{(4)}\nabla_{[\alpha} F_{\beta\gamma]} &= 0. \end{aligned}$$

The 3+1 decomposition of the stress-energy tensor is best expressed in terms of the electric spatial vector density $\mathcal{E}^a = \sqrt{g}\, F^{0a} e^{\lambda\phi}$ and the magnetic antisymmetric spatial tensor F_{ab}.

$$\begin{aligned} \rho &= \frac{1}{2}\{(\partial_t \phi)^2 + g^{ab} e_a(\phi) e_b(\phi) + \frac{1}{g} g_{ab} \mathcal{E}^a \mathcal{E}^b e^{-\lambda\phi} + \frac{1}{2} g^{ab} g^{ch} F_{ac} F_{bh} e^{\lambda\phi}\}, \\ j_a &= -\partial_t \phi \, e_a(\phi) + \frac{1}{\sqrt{g}} \mathcal{E}^b \, F_{ab}, \\ M^a{}_b &= g^{ac} e_b(\phi) \, e_c(\phi) - \frac{1}{g}\{g_{bc} \mathcal{E}^a \mathcal{E}^c - \frac{1}{2} \delta^a{}_b g_{ch} \mathcal{E}^c \mathcal{E}^h\} e^{-\lambda\phi} \\ &\quad + \{g^{ac} g^{hi} F_{ch} F_{bi} - \frac{1}{4} \delta^a{}_b g^{ch} g^{ij} F_{ci} F_{hj}\} e^{\lambda\phi}. \end{aligned} \tag{2.1}$$

The matter constraint equations are

$$e_a(\mathcal{E}^a) + f^b_{ba} \, \mathcal{E}^a = 0 \tag{2.2}$$

$$e_{[a}(F_{bc]}) + f^h_{[ab} F_{c]h} = 0. \tag{2.3}$$

Here f^c_{ab} are the (time-independent) structure functions of the frame, $[e_a, e_b] = f^c_{ab} e_c$. The matter evolution equations are

$$\partial_t^2 \phi - (\mathrm{tr}k) \partial_t \phi = g^{ab} \nabla_a \nabla_b \phi + \frac{\lambda}{2g} g_{ab} \mathcal{E}^a \mathcal{E}^b e^{-\lambda\phi} - \frac{\lambda}{4} g^{ab} g^{ch} F_{ac} F_{bh} e^{\lambda\phi}, \tag{2.4}$$

$$\partial_t \mathcal{E}^a = e_b(\sqrt{g} g^{ac} g^{bh} F_{ch} e^{\lambda\phi}) + (f^i_{ib} g^{ac} + \frac{1}{2} f^a_{bi} g^{ic}) \sqrt{g} g^{bh} F_{ch} e^{\lambda\phi}, \tag{2.5}$$

$$\partial_t F_{ab} = -2 e_{[a}(\frac{1}{\sqrt{g}} g_{b]c} \mathcal{E}^c e^{-\lambda\phi}) + f^c_{ab} \frac{1}{\sqrt{g}} g_{ch} \mathcal{E}^h e^{-\lambda\phi}. \tag{2.6}$$

2.2 Velocity-dominated evolution equations and solution

The Kasner-like, or velocity-dominated, evolution equations are obtained from the full evolution equations by: (i) dropping the spatial derivatives from the right-hand sides of (1.13), (2.4), (2.5) and (2.6) (note that f^c_{ab}-terms count as derivatives and that we keep the time derivatives of the magnetic field in (2.6) even though $F_{ab} = \partial_a A_b - \partial_b A_a$); and (ii) dropping the p-form terms in both the Einstein and dilaton evolution equations. This is a general rule and yields in this case

$$\begin{aligned} \partial_t\, {}^0g_{ab} &= -2\, {}^0k_{ab}, & (2.7)\\ \partial_t\, {}^0k^a{}_b &= (\mathrm{tr}\, {}^0k)\, {}^0k^a{}_b, & (2.8)\\ \partial_t^2\, {}^0\phi - (\mathrm{tr}\, {}^0k)\, \partial_t\, {}^0\phi &= 0, & (2.9)\\ \partial_t\, {}^0\mathcal{E}^a &= 0, & (2.10)\\ \partial_t\, {}^0F_{ab} &= 0. & (2.11) \end{aligned}$$

(As we shall see below, interaction terms of Yang-Mills or other types – if any – should also be dropped.)

It is easy to find the general analytic solution of the evolution system (2.7)–(2.11) since the equations are the same as for “Bianchi type I” homogeneous models (one such set of equations per spatial point). Taking the trace of (2.8) shows that $-1/\mathrm{tr}\, {}^0k = t + C(x)$. By a suitable redefinition of the time variable one can set $C(x)$ to zero. Then (2.8) shows that $-t\, {}^0k^a{}_b \equiv K^a{}_b$ is a constant matrix (which must satisfy $\mathrm{tr}K = K^a{}_a = 1$, and be such that ${}^0g_{ac}(t_0)K^c{}_b$ is symmetric in a and b),

$${}^0k^a{}_b(t) = -t^{-1}\, K^a{}_b. \tag{2.12}$$

Injecting this information into (2.7) leads to a linear evolution system for ${}^0g_{ab}$: $t\, \partial_t\, {}^0g_{ab} = 2\, {}^0g_{ac}K^c{}_b$, which is solved by exponentiation,

$${}^0g_{ab}(t) = {}^0g_{ac}(t_0) \left[\left(\frac{t}{t_0} \right)^{2K} \right]^c{}_b. \tag{2.13}$$

The other evolution equations are also easy to solve,

$$\begin{aligned} {}^0\phi(t) &= A \ln t + B, & (2.14)\\ {}^0\mathcal{E}^a(t) &= {}^0\mathcal{E}^a & (2.15)\\ {}^0F_{ab}(t) &= {}^0F_{ab}. & (2.16) \end{aligned}$$

In (2.13) $(t/t_0)^{2K}$ denotes the exponentiation of a matrix. Quantities on the left-hand side of (2.12)–(2.16) may be functions of both time and space, while all the time dependence of the right-hand side is made explicit. For instance, (2.16) is saying that the spacetime dependence of the general magnetic field ${}^0F_{ab}(t, x)$ (solution of the velocity-dominated evolution system) is reduced to a simple space dependence ${}^0F_{ab}(x)$ (where ${}^0F_{ab}$ is an antisymmetric spatial tensor). Let p_a denote the eigenvalues of $K^a{}_b$, ordered such that $p_1 \leq p_2 \leq p_3$. Since $\mathrm{tr}K = 1$, we have the constraint

$$p_1 + p_2 + p_3 = 1. \tag{2.17}$$

In the works of BKL the matrix Solution (2.13) is simplified by using a special frame $\{e_a\}$ with respect to which the matrices ${}^0g_{ab}(t_0)$ and $K^a{}_b$ are diagonal. However, as emphasized in [15], this choice can not necessarily be made analytically on neighborhoods where the number of distinct eigenvalues of $K^a{}_b$ is not constant. To obtain an analytic solution, while still controlling the relation of the solution to the eigenvalues of $K^a{}_b$, a special construction was introduced in [15]. This construction is based on some (possibly small) neighborhood U_0 of an arbitrary spatial point $x_0 \in \Sigma$ and uses a set of auxiliary exponents $q_a(x)$. These auxiliary exponents remain numerically close to the exact "Kasner exponents" $p_a(x)$, are analytic and enable one to define an analytic frame (see below). To construct the auxiliary exponents $q_a(x)$ one distinguishes three cases:

Case I (near isotropic): If all three eigenvalues are equal at x_0, choose a number $\epsilon > 0$ so that for $x \in U_0$, $\max_{a,b} |p_a(x) - p_b(x)| < \epsilon/2$. In this case define $q_a = 1/3$ on U_0, $a = 1, 2, 3$.

Case II (near double eigenvalue): If the number of distinct eigenvalues at x_0 is two, choose $\epsilon > 0$ so that for $x \in U_0$, $\max_{a,b} |p_a - p_b| > \epsilon/2$, and $|p_{a'} - p_{b'}| < \epsilon/2$ for some pair, a', b', $a' \neq b'$, shrinking U_0 if necessary. Denote by $p_\perp$ the distinguished exponent not equal to $p_{a'}$, $p_{b'}$. In this case define $q_\perp = p_\perp$ and $q_{a'} = q_{b'} = (1 - q_\perp)/2$ on U_0.

Case III (near diagonalizable): If all eigenvalues are distinct at x_0, choose $\epsilon > 0$ so that for $x \in U_0$, $\min_{\substack{a,b \\ a\neq b}} |p_a(x) - p_b(x)| > \epsilon/2$, shrinking U_0 if necessary. In this case define $q_a = p_a$ on U_0.

The frame $\{e_a\}$, called the adapted frame, is required to be such that the related (time-dependent) frame $\{\tilde{e}_a(t) \equiv t^{-q_a} e_a\}$ is orthonormal with respect to the velocity-dominated metric at some time $t_0 > 0$, i.e., such that ${}^0g_{ab}(t_0) = t_0^{2q_a} \delta_{ab}$. (Here and in the rest of the paper, the Einstein summation convention does not apply to indices on q_a and p_a. These indices should be ignored when determining sums. Furthermore, quantities with a tilde will refer to the frame $\{\tilde{e}_a(t)\}$.)

In addition, in Case II it is required that $e_\perp$ be an eigenvector of K corresponding to $q_\perp$ and that $e_{a'}$, $e_{b'}$ span the eigenspace of K corresponding to the eigenvalues $p_{a'}$, $p_{b'}$. In case III it is required that the e_a be eigenvectors of K corresponding to the eigenvalues $q_a (\equiv p_a)$. In all cases it is required that $\{e_a\}$ be analytic. The auxiliary exponents, q_a, are analytic, satisfy the Kasner relation $\sum q_a = 1$, are ordered ($q_1 \leq q_2 \leq q_3$), and satisfy $q_1 \geq p_1$, $q_3 \leq p_3$ and $\max_a |q_a - p_a| < \epsilon/2$. If $q_a \neq q_b$, then ${}^0g_{ab}$, ${}^0g^{ab}$, ${}^0\tilde{g}_{ab}$ and ${}^0\tilde{g}^{ab}$ all vanish, and the same is true with g replaced by k.

Equations (2.12)–(2.16), with the form of $g_{ab}(t_0)$ and $K^a{}_b$ specialized as given just above, are the general analytic solution to the velocity-dominated evolution equations in the sense that any analytic solution to the velocity-dominated evolution equations takes this form near any $x_0 \in \Sigma$ by choice of (global) time coordinate and (local) spatial frame.

2.3 Velocity-dominated constraint equations

When written in terms of the velocity-dominated variables, the velocity-dominated constraints take the same form as the full constraint equations, except the Hamiltonian constraint, which is obtained by dropping spatial gradients and electromagnetic contributions to the energy-density. This is a general rule, valid also for the more general models considered below. Thus, if we define

$$\begin{aligned} {}^0\rho &= \frac{1}{2}(\partial_t {}^0\phi)^2, \\ {}^0j_a &= -\partial_t {}^0\phi \, e_a({}^0\phi) + \frac{1}{\sqrt{{}^0g}} {}^0\mathcal{E}^b \, {}^0F_{ab}, \end{aligned}$$

we get ${}^0C = 0$ and ${}^0C_a = 0$ for the velocity-dominated constraints corresponding to the Hamiltonian and momentum constraints, with

$$ {}^0C = -{}^0k^a{}_b \, {}^0k^b{}_a + (\mathrm{tr}\, {}^0k)^2 - 2\, {}^0\rho, \tag{2.18}$$

$$ {}^0C_a = {}^0\nabla_b \, {}^0k^b{}_a - e_a(\mathrm{tr}\, {}^0k) - {}^0j_a. \tag{2.19}$$

The velocity-dominated matter constraint equations read

$$\begin{aligned} e_a({}^0\mathcal{E}^a) + f^b_{ba} \, {}^0\mathcal{E}^a &= 0, \\ e_{[a}({}^0F_{bc]}) + f^h_{[ab} \, {}^0F_{c]h} &= 0. \end{aligned}$$

For the Solution (2.12)–(2.14) the velocity-dominated Hamiltonian constraint equation is equivalent to

$$\sum {p_a}^2 + A^2 = 1. \tag{2.20}$$

The conditions (2.17) and (2.20) are the famous Kasner conditions when the dilaton is present. While p_1 is necessarily non-positive when $A = 0$, this is no longer the case when the dilaton is nontrivial ($A \neq 0$): all p_a's can then be positive. This is the major feature associated with the presence of the dilaton, which turns the mixmaster behaviour of (4-dimensional) vacuum gravity into the velocity-dominated behaviour. We shall call (p_a, A) the Kasner exponents (because they are the exponents of the proper time in the solution for the scale factors and $\exp\phi$) and refer to (2.17) and (2.20) as the Kasner conditions (note that A is often denoted p_ϕ to emphasize its relation to the kinetic energy of ϕ, and its similarity with the other exponents).

A straightforward calculation shows that

$$ \partial_t {}^0C - 2(\mathrm{tr}\, {}^0k)\, {}^0C = 0, \tag{2.21}$$

$$ \partial_t {}^0C_a - (\mathrm{tr}\, {}^0k)\, {}^0C_a = -\frac{1}{2} e_a({}^0C). \tag{2.22}$$

Thus if the velocity-dominated Hamiltonian and momentum constraints are satisfied at some $t_0 > 0$, then they are satisfied for all $t > 0$. Similarly, since ${}^0\mathcal{E}^a$ and ${}^0F_{ab}$ are independent of time, if the matter constraints are satisfied at some time $t_0 > 0$, then they are clearly satisfied for all time.

2.4 Critical value of dilaton coupling λ_c

Our ultimate goal is to show that the velocity-dominated solutions asymptotically approach (as $t \to 0$) solutions of the original system of equations. We shall prove that this is the case provided the Kasner exponents p_i, A, subject to the Kasner conditions

$$\sum {p_a}^2 - \left(\sum p_a\right)^2 + A^2 = 0, \quad \sum p_a = 1 \tag{2.23}$$

obey additional restrictions. These restrictions are inequalities on the Kasner exponents and read explicitly

$$2p_1 - \lambda A > 0, \qquad p_1 > 0, \qquad 2p_1 + \lambda A > 0. \tag{2.24}$$

As explained in [1], and rigorously checked below, these restrictions are necessary and sufficient to ensure that the terms that are dropped when replacing the full Einstein-dilaton-Maxwell equations by the velocity-dominated equations become indeed negligible as $t \to 0$. More precisely, the first condition (respectively the third) among (2.24) guarantees that one can neglect the electric (respectively, magnetic) part of the energy-momentum tensor of the electromagnetic field in the Einstein equations, whereas the condition $p_1 > 0$ is necessary for the spatial curvature terms to be asymptotically negligible. The conditions (2.24) define the set V of velocity-dominated solutions referred to in the introduction.

It is clear that if $|\lambda|$ is small enough – in particular, if $\lambda = 0$ – the Inequalities (2.24) can be fulfilled since the Kasner exponents can be all positive when the dilaton is included. But if $|\lambda|$ is greater that some critical value λ_c, it is impossible to fulfill simultaneously the Kasner conditions (2.23) and the Inequalities (2.24), because one of the terms $\pm\lambda A$ becomes more negative than $2p_1$ is positive. In that case, the set V is empty and our construction breaks down. For $|\lambda| < \lambda_c$, however, the set V is non-empty and, in fact, stable under small perturbations of the Kasner exponents since (2.24) defines an open region on the Kasner sphere. We determine in this subsection the critical value λ_c such that (2.23) and (2.24) are compatible whenever $|\lambda| < \lambda_c$.

To that end, we follow the geometric approach of [3, 39]. In the 4-dimensional space of the Kasner exponents (p_a, A), we consider the "wall chamber" $\mathcal{W}$ defined to be the conical domain where

$$p_1 \le p_2 \le p_3, \;\; 2p_a - \lambda A \ge 0, \;\; p_a \ge 0, \;\; 2p_a + \lambda A \ge 0. \tag{2.25}$$

These inequalities are not all independent since the four conditions

$$p_1 \le p_2 \le p_3, \;\; 2p_1 - \lambda A \ge 0, \;\; 2p_1 + \lambda A \ge 0 \tag{2.26}$$

imply all others. The quadratic Kasner condition (2.23) can be rewritten

$$G_{\mu\nu} p^\mu p^\nu = 0, \qquad (p^\mu) \equiv (p_a, A) \tag{2.27}$$

where $G_{\mu\nu}$ defines a metric in "Kasner-exponent space"

$$dS^2 = G_{\mu\nu} dp^\mu dp^\nu = \sum {dp_a}^2 - \left(\sum dp_a\right)^2 + (dA)^2 \tag{2.28}$$

The metric (2.28) has Minkowskian signature $(-,+,+,+)$. An example of timelike direction is given by $p_1 = p_2 = p_3$, $A = 0$. Inside or on the light cone, the function $\sum p_a$ does not vanish. The upper light cone (in the space of the Kasner exponents) is conventionally defined by (2.27) and the extra condition $\sum p_a > 0$. It is clear from our discussion that the Kasner conditions (2.23) and the Inequalities (2.24) are compatible if and only if there are light like directions in the interior of the wall chamber $\mathcal{W}$ (by rescaling $p^\mu \to \alpha p^\mu$, $\alpha > 0$, one can always make $\sum p_a = 1$ for any point in the interior of the wall chamber so that this condition does not bring a restriction). The problem amounts accordingly to determining the relative position of the light cone (2.27) and the wall chamber (2.26).

This is most easily done by computing the edges of (2.26), i.e., the one-dimensional intersections of three faces among the four faces (2.26) of $\mathcal{W}$. There are four of them: (i) $p_1 = p_2 = A = 0$, $p_3 = \alpha$; (ii) $p_1 = A = 0; p_2 = p_3 = \alpha$; (iii) $2p_1 = 2p_2 = 2p_3 = \lambda A = \alpha$; and (iv) $2p_1 = 2p_2 = 2p_3 = -\lambda A = \alpha$, where in each case, $\alpha \geq 0$ is a parameter along the edge ($\alpha = 0$ being the origin). The vectors e^μ_A $(A = 1, 2, 3, 4)$ along the edges corresponding to $\alpha = 1$, namely $(0, 0, 1, 0)$, $(0, 1, 1, 0)$, $(1/2, 1/2, 1/2, 1/\lambda)$ and $(1/2, 1/2, 1/2, -1/\lambda)$ form a basis in Kasner-exponent space. Any vector v^μ can thus be expanded along the e^μ_A, $v^\mu = v^A e^\mu_A$. A point P in Kasner-exponent space is on or inside the wall chamber $\mathcal{W}$ if and only if its coordinates p^A in this basis fulfill $p^A \geq 0$ with P inside when $p^A > 0$ for all $A's$. Thus, if all the edge vectors e^μ_A are timelike or lightlike, the Kasner conditions are incompatible with the Inequalities (2.24) since any linear combination of causal vectors with non-negative coefficients is on or inside the forward light cone (the e^μ_A's are future-directed since $p_1 + p_2 + p_3 > 0$ for all of them). If, however, one (or more) of the edge vectors lies outside the light cone, then, the Kasner conditions and the Inequalities (2.24) are compatible. The nature of some of the edge vectors depends on the value of the dilaton coupling λ: while the first one is always lightlike and the second one always timelike, the squared norm of the last two is $-3/2 + 1/\lambda^2 = (2 - 3\lambda^2)/(2\lambda^2)$. This determines the critical value

$$\lambda_c = \sqrt{\frac{2}{3}} \tag{2.29}$$

such that the edge vectors are timelike or null (incompatible inequalities) if $|\lambda| \geq \lambda_c$, but spacelike (compatible inequalities) if $|\lambda| < \lambda_c$. Note that the value of λ that arises from dimensionally reducing 5-dimensional vacuum gravity down to 4 dimensions is $\lambda = \sqrt{6}$ and exceeds the critical value. This "explains" the conclusion reached in [14] that the gravity-dilaton-Maxwell system obtained by Kaluza-Klein reduction of 5-dimensional gravity is oscillatory.

We shall assume from now on that $|\lambda| < \lambda_c$ and that the Kasner exponents fulfill the above inequalities. For later use, we choose a number $\sigma > 0$ so that, for all $x \in U_0$, $\sigma < 2p_1 - \lambda A$, $\sigma < 2p_1 + \lambda A$ and $\sigma < p_1/2$. Reduce ϵ if necessary so that $\epsilon < \sigma/7$. If ϵ is reduced, it may be necessary to shrink U_0 so that the conditions imposed in Section 2.2 remain satisfied. In Section 2.5 it is assumed that ϵ and U_0

are such that the conditions imposed in Section 2.2 and the conditions imposed in this paragraph are all satisfied.

2.5 Fuchsian system which is equivalent to the Einstein-matter evolution equations

2.5.1 Rewriting of equations

Theorem 3 in [15] (Theorem 4.2 in preprint version), on which we rely for our result, states that a Fuchsian equation (i.e., as we mentioned above, an equation of the form (1.5) where $\mathcal{A}$ satisfies a positivity condition and f is regular, which includes a boundedness property) has a unique solution u that vanishes as $t \downarrow 0$, and furthermore spatial derivatives of u of any order vanish as $t \downarrow 0$, as shown in [28]. Our goal is to recast the Einstein-matter evolution equations as a Fuchsian equation for the deviations from the velocity-dominated solutions. Thus, we denote the unknown vector u as

$$u = (\gamma^a{}_b, \lambda^a{}_{bc}, \kappa^a{}_b, \psi, \omega_a, \chi, \xi^a, \varphi_{ab}) \tag{2.30}$$

where the variables $\gamma^a{}_b$ *etc.* are related to the Einstein-matter variables by

$$\begin{aligned}
g_{ab} &= {}^0g_{ab} + {}^0g_{ac}t^{\alpha^c{}_b}\gamma^c{}_b, && (2.31)\\
e_c(\gamma^a{}_b) &= t^{-\zeta}\lambda^a{}_{bc}, && (2.32)\\
k_{ab} &= g_{ac}({}^0k^c{}_b + t^{-1+\alpha^c{}_b}\kappa^c{}_b), && (2.33)\\
\phi &= {}^0\phi + t^\beta\psi, && (2.34)\\
e_a(\psi) &= t^{-\zeta}\omega_a, && (2.35)\\
t\,\partial_t\psi + \beta\,\psi &= \chi, && (2.36)\\
\mathcal{E}^a &= {}^0\mathcal{E}^a + t^\beta\xi^a, && (2.37)\\
F_{ab} &= {}^0F_{ab} + t^\beta\varphi_{ab}. && (2.38)
\end{aligned}$$

In the first of these equations $t^{\alpha^c{}_b}$ is *not* the exponentiation of a matrix with components $\alpha^c{}_b$ such as occurs in (2.13). The expression $t^{\alpha^c{}_b}$ is for each fixed value of c and b the number which is t raised to the power given by the number $\alpha^c{}_b$ (defined below). In Equations (2.31) and (2.33) there is no summation on the index b (but there is a summation on c). In (2.38) φ_{ab} is a totally antisymmetric spatial tensor, which contributes three independent components to u. This assumption is consistent with the form of the evolution equation for φ_{ab}, Equation (2.46) below. The exponents appearing in (2.31)–(2.38) are as follows. Define $\alpha_0 = 4\epsilon$, $\beta = \epsilon/100$ and $\zeta = \epsilon/200$ (where ϵ is the same (small) quantity which entered the definition of the auxiliary exponents q_a in Section 2.2 and which was further restricted at the end of Section 2.4). All of these quantities are independent of t and x. Finally define

$$\alpha^a{}_b = 2\max(q_b - q_a, 0) + \alpha_0 = 2q_{\max\{a,b\}} - 2q_a + \alpha_0.$$

Note that the numbers $\alpha^a{}_b$ are all strictly positive. In the second definition of $\alpha^a{}_b$ we have used the fact that the q_a's are ordered. The role of $\alpha^a{}_b$ is to shift the spectrum of the Fuchsian-system matrix $\mathcal{A}$, in Equation (1.5), to be positive. It is not clear to what extent the choice of $\alpha^a{}_b$ is fixed by the requirement of getting a Fuchsian system. It seems that the (triangle-like) Inequality (42) of [15] (Inequality (5.9) in preprint version) is a key property of these coefficients. We shall further comment below on the specific choice of $\alpha^a{}_b$ and its link with the BKL-type approach to the cosmological behaviour near $t = 0$.

When writing the first-order evolution system for u we momentarily abandon the restriction that g_{ab} and k_{ab} be symmetric, as in [15]. Thus we need to define g^{ab}, and we do so by requiring that $g_{ab}g^{bc} = \delta_a{}^c$. This implies that $g^{ab}g_{bc} = \delta^a{}_c$. We lower indices on tensors by contraction with the second index of g_{ab} and also raise indices on tensors by contraction with the second index of g^{ab}. This choice is so that raising and then lowering a given index results in the original tensor, and the same for lowering and then raising an index. The position of the indices on quantities appearing in u and other such quantities is fixed. Repeated indices on these quantities imply a summation. On the other hand, as we already mentioned above, one qualifies the summation convention by insisting that indices repeated only because of their occurrence on p_a, q_a, $\alpha^a{}_b$ and other such non-tensorial quantities should be ignored when determining sums.

Substituting (2.31)–(2.38) in the evolution equations yields equations of motion for u of the form (1.5)

$$t\,\partial_t\gamma^a{}_b + \alpha^a{}_b\gamma^a{}_b + 2\kappa^a{}_b - 2(t\,{}^0k^a{}_c)\gamma^c{}_b + 2\gamma^a{}_c(t\,{}^0k^c{}_b) = -2\,t^{\alpha^a{}_c+\alpha^c{}_b-\alpha^a{}_b}\gamma^a{}_c\kappa^c{}_b \tag{2.39}$$

$$t\,\partial_t\lambda^a{}_{bc} = t^\zeta e_c(t\,\partial_t\gamma^a{}_b) + \zeta\, t^\zeta e_c(\gamma^a{}_b) \tag{2.40}$$

$$t\,\partial_t\kappa^a{}_b + \alpha^a{}_b\kappa^a{}_b - (t\,{}^0k^a{}_b)(\mathrm{tr}\kappa) = t^{\alpha_0}(\mathrm{tr}\kappa)\kappa^a{}_b + t^{2-\alpha^a{}_b}({}^SR^a{}_b - M^a{}_b) \tag{2.41}$$

$$t\,\partial_t\psi + \beta\psi - \chi = 0 \tag{2.42}$$

$$t\,\partial_t\omega_a = t^\zeta\{e_a(\chi) + (\zeta-\beta)e_a(\psi)\} \tag{2.43}$$

$$t\,\partial_t\chi + \beta\chi = t^{\alpha_0-\beta}(\mathrm{tr}\,\kappa)(A + t^\beta\chi) + t^{2-\beta}g^{ab}\nabla_a\nabla_b{}^0\phi + t^{2-\zeta}\nabla^a\omega_a + t^{2-\beta}\{\frac{\lambda}{2g}g_{ab}\mathcal{E}^a\mathcal{E}^b e^{-\lambda\phi} - \frac{\lambda}{4}g^{ab}g^{ch}F_{ac}F_{bh}e^{\lambda\phi}\} \tag{2.44}$$

$$t\,\partial_t\xi^a + \beta\xi^a = t^{1-\beta}\{e_b(\sqrt{g}g^{ac}g^{bh}F_{ch}e^{\lambda\phi}) + (f^i_{ib}g^{ac} + \frac{1}{2}f^a_{bi}g^{ic})\sqrt{g}g^{bh}F_{ch}e^{\lambda\phi}\} \tag{2.45}$$

$$t\,\partial_t\varphi_{ab} + \beta\varphi_{ab} = t^{1-\beta}\{-2e_{[a}(\frac{1}{\sqrt{g}}g_{b]c}\mathcal{E}^c e^{-\lambda\phi}) + f^c_{ab}\frac{1}{\sqrt{g}}g_{ch}\mathcal{E}^h e^{-\lambda\phi}\} \tag{2.46}$$

All the quantities entering these equations have been defined, except ${}^SR^a{}_b$. This is done by taking the Ricci tensor of the symmetric part $g_{(ab)}$ of g_{ab} [15]. More explicitly, ${}^S R^a{}_b = g^{ac}\,{}^S R_{cb}$, with

$$\begin{aligned} {}^S R_{ab} &= t^{q_a+q_b}\,{}^S\tilde{R}_{ahb}{}^h && (2.47)\\ &= t^{q_a+q_b}\{\tilde{e}_h({}^S\tilde{\Gamma}^h_{ab}) - \tilde{e}_a({}^S\tilde{\Gamma}^h_{hb}) + {}^S\tilde{\Gamma}^i_{ab}\,{}^S\tilde{\Gamma}^h_{hi} - {}^S\tilde{\Gamma}^i_{hb}\,{}^S\tilde{\Gamma}^h_{ai} + \tilde{f}^i_{ah}\,{}^S\tilde{\Gamma}^h_{ib}\}, \end{aligned}$$

and the connection coefficients in the frame $\{\tilde{e}_a\}$,

$${}^S\tilde{\Gamma}^c_{ab} = \frac{1}{2}\,{}^S\tilde{g}^{ch}\left\{\tilde{e}_a(\tilde{g}_{(bh)}) + \tilde{e}_b(\tilde{g}_{(ha)}) - \tilde{e}_h(\tilde{g}_{(ab)}) - \tilde{g}_{(ia)}\tilde{f}^i_{bh} - \tilde{g}_{(bi)}\tilde{f}^i_{ah}\right\} + \frac{1}{2}\tilde{f}^c_{ab}. \tag{2.48}$$

Here, ${}^S\tilde{g}^{ab}$ is defined as the inverse of $\tilde{g}_{(ab)}$. Once it is shown that the tensor g_{ab} in Equation (2.31) is symmetric, then it follows that ${}^S R^a{}_b = R^a{}_b$ and that Equations (2.39)–(2.46) are equivalent to the Einstein-matter equations.

2.5.2 The system (2.39)–(2.46) is Fuchsian

A good deal of the work needed to show that Equation (1.5) (as written out in Equations (2.39)–(2.46)) is Fuchsian was done in [15], in the massless scalar field case considered there. The form of the velocity-dominated evolution and the form of the function u are the same in the two cases except for the crucial addition of new source terms and new evolution equations involving the Maxwell field. The presence of the new components does not alter already existing parts of the matrix $\mathcal{A}$, nor already existing terms in f. The difference between $\mathcal{A}$ here and $\mathcal{A}$ in the massless scalar field case considered in [15] is that here there are additional rows and columns, such that the only non-vanishing new entries are on the diagonal and strictly positive. Therefore the argument in [15] that their $\mathcal{A}$ satisfies the appropriate positivity condition implies that our $\mathcal{A}$ satisfies the appropriate positivity condition.

On the other hand, it is crucial to control in detail the new source terms in f, connected to the Maxwell field, which were absent in [15]. It is for the study of these terms that the results of [1], and in particular the Inequalities (2.24) which were shown there to guarantee that Maxwell source terms become asymptotically subdominant near the singularity, become important. Recall that the crucial criterion for the source $f(t, x, u, u_x)$ is that it be $O(t^\delta)$, for some strictly positive δ. In regard to this estimate, we use the notation "big O," "$\preceq$" and "small o" as follows. Given two functions $F(t, x, u, u_x)$ and $G(t, x, u, u_x)$ we use the notation $F \preceq G$, to denote that, for every compact set K, there exists a constant C and a number $t_0 > 0$ such that $|F(t, x, u, u_x)| \le C|G(t, x, u, u_x)|$ when $(x, u, u_x) \in K$ and $0 < t \le t_0$ (see Definition 1 in [15]). If G is a function only of t (e.g. a power of t), then we replace $F \preceq G$ with $F = O(G)$. If $f(t, x, u, u_x) = O(t^\delta)$, then by reducing the value of δ (keeping it positive) we have that $f(t, x, u, u_x) = o(t^\delta)$ with a "small o" which denotes that f/t^δ tends to zero uniformly on compact sets K as $t \to 0$.

The new source terms involving the Maxwell field are: the last four terms in $M^a{}_b$ (see Equation (2.1)), the last two terms on the right-hand side of Equation (2.44) and the terms of the right-hand sides of Equations (2.45) and (2.46).

The calculation of the estimates starts in the frame, $\{\tilde{e}_a\}$, defined in Section 2.2. For more details concerning the basic estimates, we refer the reader to [15]. In the frame $\{\tilde{e}_a\}$ the Kasner-like metric is (*cf.* (2.13))

$$
\begin{aligned}
{}^0\tilde{g}_{ab} &= {}^0\tilde{g}_{ac}(t_0)\left[\left(\frac{t}{t_0}\right)^{2(K-Q)}\right]^c{}_b, &(2.49)\\
{}^0\tilde{g}^{ab} &= \left[\left(\frac{t}{t_0}\right)^{-2(K-Q)}\right]^a{}_c\,{}^0\tilde{g}^{cb}(t_0), &(2.50)
\end{aligned}
$$

where the matrix Q is the diagonal matrix $Q^a{}_b \equiv q_a\delta^a{}_b$ which commutes with K. With our choice of frame, ${}^0\tilde{g}_{ab}(t_0) = \delta_{ab}$ and ${}^0\tilde{g}^{ab}(t_0) = \delta^{ab}$. In Lemma 2 in [15] (Lemma 5.1 in preprint version), the form of (2.49) and (2.50) is considered and it is shown that ${}^0\tilde{g}_{ab} = O(t^{-\epsilon})$ and ${}^0\tilde{g}^{ab} = O(t^{-\epsilon})$. It is useful to write down expressions for the proposed metric and extrinsic curvature in the frame $\{\tilde{e}_a\}$. The components in terms of this frame are

$$
\begin{aligned}
\tilde{g}_{ab} &= {}^0\tilde{g}_{ab} + {}^0\tilde{g}_{ac}t^{\tilde{\alpha}^c{}_b}\gamma^c{}_b,\\
\tilde{k}_{ab} &= \tilde{g}_{ac}({}^0\tilde{k}^c{}_b + t^{-1+\tilde{\alpha}^c{}_b}\kappa^c{}_b).
\end{aligned}
$$

Here, $\tilde{\alpha}^a{}_b = \alpha^a{}_b + q_a - q_b = |q_a - q_b| + \alpha_0$ is symmetric in a, b, $\tilde{\alpha}^a{}_b = \tilde{\alpha}^b{}_a$. To get an estimate for the inverse metric, we note first that the inverse of $g_{ac}{}^0g^{cb}$ is given by $g^{ca0}g_{cb}$. Thus it is possible to express the latter quantity algebraically in terms of ${}^0g_{ab}$ and $\gamma^a{}_b$. Now define

$$
\bar{\gamma}^a{}_b = -t^{-\tilde{\alpha}^a_b}(\delta^a_b - \tilde{g}^{ac0}\tilde{g}_{cb}), \tag{2.51}
$$

which, from what we just observed, can be expressed algebraically in terms of known quantities and $\gamma^a{}_b$. Then one has

$$
\tilde{g}^{ab} ={}^0 \tilde{g}^{ab} + t^{\tilde{\alpha}^a{}_c}\bar{\gamma}^a{}_c\,{}^0\tilde{g}^{cb}. \tag{2.52}
$$

As a consequence of an argument given in [15] which uses the (triangle-like) Inequality (42) of that paper ((5.9) in preprint version) and the matrix identity preceding it, this exhibits $\bar{\gamma}^a{}_b$ as a regular function of $\gamma^a{}_b$. In particular, if it is known that $\gamma^a{}_b$ is $o(1)$ then the same is true of $\bar{\gamma}^a{}_b$.

To better grasp the usefulness of the introduction of the exponents $\alpha^a{}_b$ and $\tilde{\alpha}^a{}_b$, and the link of the Fuchsian estimates with the approximate estimates used in the BKL-like works, let us consider more closely the simple case where all the Kasner exponents are distinct (Case III). In this case $p_a = q_a$ and one can diagonalize the Kasner-metric, so that, in the rescaled frame $\tilde{e}_a$, we have simply (for all $t \leq t_0$) ${}^0\tilde{g}_{ab}(t) = \delta_{ab}$. In such a case, the BKL-type estimates would be obtained (in the time-dependent rescaled frame $\tilde{e}_a$) by approximating the exact metric by its Kasner limit, i.e., simply $\tilde{g}^{\mathrm{BKL}}_{ab}(t) = \delta_{ab}$. By contrast, the estimates of the Fuchsian analysis are made with the exact metric, $\tilde{g}_{ab}(t) = \delta_{ab} + t^{\tilde{\alpha}^a{}_b}\gamma^a{}_b$, in which $\gamma^a{}_b$, being part of u, is considered to be in a compact set and hence is bounded. As the diagonal $\tilde{\alpha}^a{}_a = \alpha_0 > 0$, we see that (in the frame $\tilde{e}_a$) the diagonal components of the "Fuchsian" metric asymptote those of the "BKL" metric, and that both are close to one. Concerning the non-diagonal components (in the frame $\tilde{e}_a$) of the "Fuchsian" metric we see that they are constrained, by construction (i.e., by the choice $\tilde{\alpha}^a{}_b = |q_a - q_b| + \alpha_0$), to tend to zero faster than $t^{|q_a-q_b|}$. This closeness between the metrics used in the two types of estimates explains the parallelism between the rigorous results derived here and the heuristic estimates used in BKL-type works. If we come back to the general case where the Kasner metric cannot be diagonalized in an analytic fashion, the optimal estimates become worse by a negative power of t (coming from the estimate of the matrix difference $2(K-Q)$ in Equations (2.49), (2.50) above). The proposed metric in the frame $\{\tilde{e}_a\}$ satisfies then

$$\tilde{g}_{ab} \preceq t^{|q_a-q_b|-\epsilon} \qquad \text{and} \qquad \tilde{g}^{ab} \preceq t^{|q_a-q_b|-\epsilon}.$$

The proposed inverse metric in the adapted frame is

$$g^{ab} = {}^0g^{ab} + t^{\alpha^a{}_c}\bar{\gamma}^a{}_c\, {}^0g^{cb}.$$

The proposed metric in the adapted frame satisfies

$$g_{ab} \preceq t^{2q_{\max\{a,b\}}-\epsilon} \qquad \text{and} \qquad g^{ab} \preceq t^{-2q_{\min\{a,b\}}-\epsilon}. \tag{2.53}$$

Estimates of spatial derivatives of the proposed metric are also needed.

$$\begin{aligned} e_c(\tilde{g}_{ab}) &\preceq t^{|q_a-q_b|-\delta-\epsilon} &\qquad \text{and} \qquad & e_c(\tilde{g}^{ab}) \preceq t^{|q_a-q_b|-\delta-\epsilon}, \\ e_c(g_{ab}) &\preceq t^{2q_{\max\{a,b\}}-\delta-\epsilon} &\qquad \text{and} \qquad & e_c(g^{ab}) \preceq t^{-2q_{\min\{a,b\}}-\delta-\epsilon} \end{aligned} \tag{2.54}$$

for some strictly positive δ.

The determinant of the proposed metric also appears in some of the new source terms. From (2.13), the form of ${}^0g_{ab}(t_0)$ and $\operatorname{tr} K = 1$, one gets ${}^0g = t^2$. From (2.49) and ${}^0\tilde{g}_{ab}(t_0) = \delta_{ab}$ one gets ${}^0\tilde{g} = 1$. The expression for the determinant is a sum of terms of the form $g_{ab}g_{cd}g_{ef}$, such that in each term, each index, 1, 2, 3, occurs exactly twice. From the Kasner relation for the q_a's and the relation between the two frames, it follows that $g = t^2\tilde{g}$. Considering the form of the various expressions, one then obtains $1/g = O(t^{-2})$, $\sqrt{g} = O(t)$, $1/\sqrt{g} = O(t^{-1})$, and $1/\sqrt{g} - 1/\sqrt{{}^0g} = O(t^{-1+\alpha_0-3\epsilon}) = O(t^{-1+\epsilon})$. Spatial derivatives of the determinant also appear in f. Considering the form of $\tilde{g} - {}^0\tilde{g}$ and that $e_a({}^0\tilde{g}) = 0$, it follows that $e_a(\tilde{g}) = O(t^{\alpha_0-\delta-3\epsilon})$, and

$$e_a(g) = O(t^{2+\alpha_0-\delta-3\epsilon}). \tag{2.55}$$

Finally,

$$e_a(g^{-1/2}) = -\frac{e_a(g)}{2g^{3/2}} = O(t^{-1+\alpha_0-\delta-3\epsilon}).$$

Let us now consider the new source terms in f, beginning with the last four terms of $t^{2-\alpha^a{}_b}M^a{}_b$. To estimate the contributions of $\mathcal{E}^a$ and F_{ab} it is sufficient to note from (2.37) and (2.38) that $\mathcal{E}^a = O(1)$ and $F_{ab} = O(1)$. Then we get, using the definition of $\alpha^a{}_b$ and (2.53),

$$\begin{aligned}
&t^{2-\alpha^a{}_b}\frac{1}{g}\{g_{bc}\mathcal{E}^a\mathcal{E}^c - \frac{1}{2}\delta^a{}_b g_{ch}\mathcal{E}^c\mathcal{E}^h\}e^{-\lambda\phi} \\
&\quad\preceq \sum_c t^{-2q_{\max\{a,b\}}+2q_a+2q_{\max\{b,c\}}-\lambda A-\alpha_0-\epsilon} + \sum_{c,h} t^{+2q_{\max\{c,h\}}-\lambda A-\alpha_0-\epsilon} \\
&\quad\preceq t^{2q_1-\lambda A-\alpha_0-\epsilon} = O(t^{-\alpha_0-\epsilon+\sigma}) = O(t^\delta), \\
&t^{2-\alpha^a{}_b}\{g^{ac}g^{hi}F_{ch}F_{bi} - \frac{1}{4}\delta^a{}_b g^{ch}g^{ij}F_{ci}F_{hj}\}e^{\lambda\phi} \\
&\quad\preceq \sum_{c,h\neq c,i\neq b} t^{2-2q_{\max\{a,b\}}+2q_a-2q_{\min\{a,c\}}-2q_{\min\{h,i\}}+\lambda A-\alpha_0-2\epsilon} \\
&\quad+ \sum_{c,h,i\neq c,j\neq h} t^{2-2q_{\min\{c,h\}}-2q_{\min\{i,j\}}+\lambda A-\alpha_0-2\epsilon} \\
&\quad\preceq t^{2q_1+\lambda A-\alpha_0-2\epsilon} = O(t^{-\alpha_0-2\epsilon+\sigma}) = O(t^\delta)
\end{aligned}$$

for some strictly positive δ. The crucial inputs in getting these estimates are the Inequalities (2.24). We recall also that the quantity σ (linked to (2.24) being satisfied) was introduced at the end of Subsection 2.4. The estimate of the last two terms on the right-hand side of (2.44) is

$$\begin{aligned}
t^{2-\beta}\frac{1}{g}g_{ab}\mathcal{E}^a\mathcal{E}^b e^{-\lambda\phi} &= O(t^{-\beta-\epsilon+\sigma}) = O(t^\delta), \\
t^{2-\beta}g^{ab}g^{ch}F_{ac}F_{bh}e^{\lambda\phi} &= O(t^{-\beta-2\epsilon+\sigma}) = O(t^\delta)
\end{aligned}$$

The right-hand side of (2.45) is $O(t^{\alpha_0-\beta-\delta-5\epsilon+\sigma}) = O(t^\delta)$. The right-hand side of (2.46) is $O(t^{\alpha_0-\beta-\delta-4\epsilon+\sigma}) = O(t^\delta)$. The other terms which occur in f were estimated in [15], resulting in $f = O(t^\delta)$.

To show that we indeed have a Fuchsian equation, we need to check not only that $f = O(t^\delta)$, but also that $\partial_u f = O(t^\delta)$ and $\partial_{u_x} f = O(t^\delta)$, along with other regularity conditions [15, 28]. In [15] it is shown that f is regular with Equation (31) in that paper and the remarks following Equation (31). In our case there is a factor involving the determinant of the metric in various of the terms in f which are not present in the case considered in [15]. The discussion surrounding Equation (31) in [15] applies to our case as well, even for terms in f containing $g^{\pm 1/2}$. The Kasner-like contribution is the leading term, and this function of t and x can be factored out. What is left is of the form $w(t, x, u, u_x)(1 + h(t, x, u))^{\pm 1/2}$, which is analytic in h at $h = 0$. The conditions listed following Equation (31) hold. Thus we conclude that (1.5) as written out in (2.39)–(2.46) is a Fuchsian equation.

2.5.3 Symmetry of metric

It remains to show that g_{ab} is symmetric, so that Equation (1.5) as written out in (2.39)–(2.46) is equivalent to the Einstein-matter evolution equations. The structure of the argument is the same in any dimension and so it will be written down for general d[6]. The number of distinct eigenvalues of $K^a{}_b$ is maximal almost everywhere. Thus it is enough to show that $g_{[ab]}$ and $k_{[ab]}$ vanish in the case that the Kasner-like metric is diagonal, since then by analytic continuation they vanish on the entire domain. We therefore consider the case that the Kasner-like metric is diagonal.

The redefinitions (2.31), (2.33) from the variables g_{ab}, k_{ab} to the variables $\gamma^a{}_b$, $\kappa^a{}_b$ were viewed in the previous subsections as a change between variables with no particular symmetry properties in their indices (18 on each side). One can, however enforce $g_{[ab]} = 0$ by assuming that $\gamma^a{}_b$ is symmetric and vice versa. Indeed, under our diagonality assumption for the Kasner-like metric, $\tilde{g}_{ab}(t) = \delta_{ab} + t^{\tilde{\alpha}^a{}_b}\gamma^a{}_b$ where $\tilde{\alpha}^a{}_b = |q_a - q_b| + \alpha_0$ is *symmetric* in (a, b). Accordingly, imposing the symmetry $\gamma^a{}_b = \gamma^b{}_a$ algebraically ensures the symmetry of g_{ab}. Similarly, one can enforce $k_{[ab]}$ to vanish by imposing consistent constraints on $\kappa^a{}_b$: inserting (2.31) into (2.33) (with the velocity-dominated solution diagonal) and writing out the constraint $k_{ab} - k_{ba} = 0$ gives the following condition on $\kappa^a{}_b$

$$\kappa^a{}_b - \kappa^b{}_a - \gamma^a{}_b p_b + \gamma^b{}_a p_a + t^{\alpha_{(ab)c}}(\gamma^a{}_c \kappa^c{}_b - \gamma^b{}_c \kappa^c{}_a) = 0, \tag{2.56}$$

with $\alpha_{(ab)c} = 2p_{\max\{a,c\}} + 2p_{\max\{b,c\}} - 2p_{\max\{a,b\}} - 2p_c + \alpha_0$. These conditions show that there are only six independent components among the $\kappa^a{}_b$, which can be taken to be those with $a \leq b$. This is because, the relation (2.56) can be solved uniquely for the components $\kappa^a{}_b$ with $a > b$, given the other ones, at least for t small. That this is true can be seen as follows. Rearrange the Equations (2.56) so that the terms containing $\kappa^a{}_b$ with $a > b$ are on the left-hand side and all other terms are on the right-hand side. The result is an inhomogeneous linear system of the form $A(t, x)v(t, x) = w(t, x)$ where $A(t, x)$ and $w(t, x)$ are known quantities and v denotes the components $\kappa^a{}_b$ with $a > b$ which we want to determine. Furthermore $A(t, x) = I + o(1)$, where I denotes the identity matrix. It follows that $A(t)$ is invertible for t small, which is what we wanted to show. The solution $\kappa^a{}_b$ $(a > b)$ remains moreover bounded when $\gamma^b{}_a$ and $\kappa^a{}_b$ are in a compact set. We shall assume from now on that $\gamma^a{}_b$ is symmetric and $\kappa^a{}_b$ constrained by (2.56), so that symmetry of the metric is automatic. The redefinitions (2.31), (2.33) from g_{ab}, k_{ab} to $\gamma^a{}_b$, $\kappa^a{}_b$ can now be viewed as an invertible change of variables, from 12

[6] The argument for the symmetry of the metric in [15] is not valid as written since some terms were omitted in the evolution equation for the antisymmetric part of the extrinsic curvature. The correct equation is

$$\partial_t(k_{ab} - k_{ba}) = (\mathrm{tr}k)(k_{ab} - k_{ba}) - 2(k_{ac}k^c{}_b - k_{bc}k^c{}_a).$$

In the following a proof of the symmetry of the tensors g_{ab} and k_{ab} is supplied with the help of a different method.

independent variables to 12 independent variables. We can also clearly assume $\lambda^a{}_{bc}$ in (2.32) to be symmetric in a, b.

With these conventions, there are less components in u than in the previous subsections. The independent components can be taken to be $\gamma^a{}_b$, $\kappa^a{}_b$ and $\lambda^a{}_{bc}$ with $a \leq b$, together with the matter variables. An independent system of evolution equations is given by (2.39)–(2.41) with $a \leq b$ for the gravitational variables, and the same evolution equations as before for the matter variables. These evolution equations are equivalent to all the original evolution equations, since the Equations (2.39)–(2.41) with $a > b$ are then automatically fulfilled, as can be shown using the fact that the Einstein tensor and the stress-energy tensor are symmetric for symmetric metrics. To see this it must be shown that given a symmetric tensor S_{ab}, the vanishing of $S^a{}_b = g^{ac}S_{cb}$ for $a \leq b$ implies that $S_{ab} = 0$. Consider the linear map which takes a symmetric tensor S_{ab}, raises an index, and keeps the components of the result with $a \leq b$. This is a mapping between vector spaces of dimension $d(d+1)/2$ and can be shown to be an isomorphism by elementary linear algebra. This proves the desired result.

Now, this reduced evolution system is also Fuchsian. This follows from the same reasoning as above, which still holds because all components of u, including the non-independent ones, can still be assumed to be bounded. Therefore, there is a unique u that goes to zero, which must be equal to the one considered in the previous subsections. The metric considered previously is thus indeed symmetric.

2.5.4 Unique solution on a neighborhood of the singularity

Given an analytic solution to the velocity-dominated evolution equations on $(0, \infty) \times \Sigma$, such that Inequalities (2.24) are satisfied, we now have a solution u to a Fuchsian equation (and a corresponding solution to the Einstein-matter evolution equations) in the intersection of a neighborhood of the singularity with $(0, \infty) \times U_0$ where U_0 is a neighborhood of an arbitrary point on Σ. These local solutions can be patched together to get a solution to the Einstein-matter evolution equations everywhere in space near the singularity. It may seem like there could be a problem patching together the solutions obtained on distinct neighborhoods with non-empty intersection because the Fuchsian equation is not the same for different allowed choices of ϵ and adapted local frame. The construction is possible because different allowed choices of ϵ and local frame result in a well-defined relationship between the different solutions u which are obtained, such that the corresponding Einstein-matter variables agree on the intersection (up to change of basis). It therefore follows that given an analytic solution to the velocity-dominated evolution equations on $(0, \infty) \times \Sigma$, such that Inequalities (2.24) are satisfied, our construction uniquely determines a solution to the Einstein-matter evolution equations everywhere in space, near the singularity.

2.6 Einstein-matter constraints

2.6.1 Matter constraints

The time derivative of the matter constraint quantities (the left-hand side of Equations (2.2) and (2.3)) vanishes. If the velocity-dominated matter constraints are satisfied, the matter constraint quantities are $o(1)$. A quantity which is constant in time and $o(1)$ must vanish. Therefore the matter constraints are satisfied.

2.6.2 Diagonal Kasner metrics

It remains to show that the Hamiltonian and momentum constraints are satisfied, that C and C_a, defined in (1.9) and (1.10), vanish. Since we now have a metric, $g_{\mu\nu}$, it follows that $\nabla_\mu G^{\mu\nu} = 0$. Since the matter evolution and constraint equations are satisfied, it follows that $\nabla_\mu T^{\mu\nu} = 0$. From the vanishing of the right-hand side of (1.12) and the vanishing of the covariant divergence of both the Einstein tensor and the stress-energy tensor, it follows that

$$\partial_t C = 2(\mathrm{tr}k)C - 2\nabla^a C_a \tag{2.57}$$

$$\partial_t C_a = (\mathrm{tr}k)\, C_a - \frac{1}{2}\nabla_a C. \tag{2.58}$$

Now define $\bar{C} = t^{2-\eta_1} C$ and $\bar{C}_a = t^{1-\eta_2} C_a$, with $0 < \eta_2 < \eta_1 < \beta$.

$$t\,\partial_t \bar{C} + \eta_1 \bar{C} = 2(1 + t\,\mathrm{tr}\,k)\bar{C} - 2t^{2-\eta_1+\eta_2}\nabla^a \bar{C}_a \tag{2.59}$$

$$t\,\partial_t \bar{C}_a + \eta_2 \bar{C}_a = (1 + t\,\mathrm{tr}\,k)\bar{C}_a - \frac{1}{2}t^{\eta_1-\eta_2}\nabla_a \bar{C} \tag{2.60}$$

On the right-hand side of (2.59) and (2.60) $\bar{C}$ and $\bar{C}_a$ are to be considered as components of $u = (\bar{C}, \bar{C}_a)$. If it is shown that (2.59) and (2.60) is a Fuchsian system, then there is a unique solution u such that $u = o(1)$. It is clear that $u = 0$ is a solution to (2.59) and (2.60). If it is shown that $\bar{C} = o(1)$ and $\bar{C}_a = o(1)$, (i.e., that $C = o(t^{-2+\eta_1})$ and $C_a = o(t^{-1+\eta_2})$), then they must be this unique solution. Furthermore, it is sufficient to consider the case that the Kasner-like metric is diagonal, since the number of distinct eigenvalues of $K^a{}_b$ is maximal on an open set of Σ. If the constraints vanish on an open set of their domain, then by analytic continuation they vanish everywhere on their domain.

Therefore we consider the case that the Kasner-like metric is diagonal and show first that

$$1 + t\,\mathrm{tr}\,k = O(t^\delta) \tag{2.61}$$

$$\nabla^a \bar{C}_a = O(t^{-2+\delta+\eta_1-\eta_2}) \tag{2.62}$$

(when $\bar{C}_a$ is bounded) so that the system (2.59), (2.60) is Fuchsian (the complete regularity of $f(t, x, u, u_x)$ defined by (2.59) and (2.60) can be easily verified); and second, that

$$C = o(t^{-2+\eta_1}) \quad (2.63)$$
$$C_a = o(t^{-1+\eta_2}). \quad (2.64)$$

Some facts which will be used to show this follow. Consider indices $a \in \{1,2,3\}$. The following inequalities hold for some positive integer n and for real numbers, q_a, ordered such that if $a < b$, then $q_a \le q_b$. (In later sections we define ordered auxiliary exponents, q_a, for $a \in \{1,\dots,d\}$, for arbitrary fixed $d \ge 2$. Then (2.65)–(2.67) hold more generally for indices in $\{1,\dots,d\}$.)

$$q_{a_0} + \sum_{i=1}^{n} |q_{a_{i-1}} - q_{a_i}| - q_{a_n} \ge 0 \quad (2.65)$$

$$q_{a_0} + \sum_{i=1}^{n} |q_{a_{i-1}} - q_{a_i}| + q_{a_n} \ge 2q_{\max\{a_k,a_j\}} \quad (2.66)$$

$$-q_{a_0} + \sum_{i=1}^{n} |q_{a_{i-1}} - q_{a_i}| - q_{a_n} \ge -2q_{\min\{a_k,a_j\}} \quad (2.67)$$

The latter two inequalities hold for any k, j in $\{0,\dots,n\}$.

In the case that the Kasner-like metric is diagonal, $q_a = p_a$. The metric in the frame $\{\tilde{e}_a\}$ is ${}^0\tilde{g}_{ab} = \delta_{ab}$,

$$\tilde{g}_{ab} = \delta_{ab} + t^{\tilde{\alpha}^a{}_b}\gamma^a{}_b \preceq t^{|p_a - p_b|},$$
$$\tilde{g}^{ab} = \delta^{ab} + t^{\tilde{\alpha}^a{}_b}\bar{\gamma}^a{}_b \preceq t^{|p_a - p_b|}.$$

The extrinsic curvature satisfies $t\,{}^0k^a{}_b = -\delta^a{}_b\, p_b$,

$$t\, k^a{}_b = -\delta^a{}_b\, p_b + t^{\alpha^a{}_b}\kappa^a{}_b,$$
$$t\,(\tilde{k}^a{}_b - {}^0\tilde{k}^a{}_b) = t^{\tilde{\alpha}^a{}_b}\kappa^a{}_b,$$

and $t\,\mathrm{tr}\,{}^0k = -1$,

$$t\,\mathrm{tr}\,k = -1 + t^{\alpha_0}\,\mathrm{tr}\kappa, \quad (2.68)$$
$$t^2\{(\mathrm{tr}\,k)^2 - (\mathrm{tr}\,{}^0k)^2\} = O(t^{\alpha_0}). \quad (2.69)$$

The following estimates will also be useful.

$$-k^a{}_b\, k^b{}_a + {}^0k^a{}_b\, {}^0k^b{}_a = -2t^{-2+\alpha_0}\kappa^a{}_a\, p_a - t^{-2+\alpha^a{}_b+\alpha^b{}_a}\kappa^a{}_b\,\kappa^b{}_a$$
$$= O(t^{-2+\alpha_0}), \quad (2.70)$$

and

$$e_a(\mathrm{tr}\,k - \mathrm{tr}\,{}^0k) = e_a(t^{-1+\alpha_0}\mathrm{tr}\,\kappa) = O(t^{-1+\alpha_0}). \quad (2.71)$$

The structure functions of the frame $\{\tilde{e}_a\}$ are

$$\begin{aligned}\tilde{f}^c_{ab} &= t^{p_c-p_a-p_b} f^c_{ab} - \ln t\, e_a(p_b)\, t^{-p_a}\delta^c{}_b + \ln t\, e_b(p_a)\, t^{-p_b}\delta^c{}_a \\ &\preceq t^{p_c-p_a-p_b-\delta}.\end{aligned}$$

It is convenient to have an estimate of $\tilde{\Gamma}^c_{ab}$, the connection coefficients (2.48) in the frame $\{\tilde{e}_a\}$, term by term.

$$\text{Term A:} \qquad \tilde{g}^{ch}\tilde{e}_a(\tilde{g}_{bh}) \preceq \sum_h t^{|p_c-p_h|-p_a+|p_b-p_h|-\delta}, \tag{2.72}$$

$$\text{Term B:} \qquad \tilde{g}^{ch}\tilde{e}_b(\tilde{g}_{ha}) \preceq \sum_h t^{|p_c-p_h|-p_b+|p_h-p_a|-\delta}, \tag{2.73}$$

$$\text{Term C:} \qquad \tilde{g}^{ch}\tilde{e}_h(\tilde{g}_{ab}) \preceq \sum_h t^{|p_c-p_h|-p_h+|p_a-p_b|-\delta}, \tag{2.74}$$

$$\text{Term D:} \qquad \tilde{g}^{ci}\tilde{g}_{ha}\tilde{f}^h_{bi} \preceq \sum_{h,i\neq b} t^{|p_c-p_i|+|p_h-p_a|+p_h-p_b-p_i-\delta}, \tag{2.75}$$

$$\text{Term E:} \qquad \tilde{g}^{ci}\tilde{g}_{bh}\tilde{f}^h_{ai} \preceq \sum_{h,i\neq a} t^{|p_c-p_i|+|p_b-p_h|+p_h-p_a-p_i-\delta}, \tag{2.76}$$

$$\text{Term F:} \qquad \tilde{f}^c_{ab} \preceq t^{p_c-p_a-p_b-\delta}. \tag{2.77}$$

The difference between the connection coefficients for the metric $\tilde{g}_{ab}$ and those for the Kasner-like metric is $\Delta\tilde{\Gamma}^c_{ab} = \tilde{\Gamma}^c_{ab} - {}^0\tilde{\Gamma}^c_{ab}$. It is useful to have the estimates

$$\tilde{\Gamma}^a_{ac} = \frac{1}{2}\tilde{g}^{ab}\tilde{e}_c(\tilde{g}_{ab}) + \tilde{f}^a_{ac} \preceq t^{-p_c-\delta},$$

and

$$\Delta\tilde{\Gamma}^a_{ac} = \frac{1}{2}\tilde{g}^{ab}\tilde{e}_c(\tilde{g}_{ab}) \preceq t^{-p_c+\alpha_0-\delta}. \tag{2.78}$$

2.6.3 Momentum and Hamiltonian constraints

First, we show (2.61) and (2.62). From equation (2.68), $1 + t\,\mathrm{tr}\,k = O(t^{\alpha_0})$. Similarly, we can estimate $\nabla^a\bar{C}_a$,

$$\begin{aligned}\nabla^a\bar{C}_a &= \tilde{g}^{ab}\,\tilde{\nabla}_a\tilde{\bar{C}}_b \\ &= \tilde{g}^{ab}\{t^{-p_a}\, e_a(\bar{C}_b\, t^{-p_b}) - \tilde{\Gamma}^c_{ab}\,\bar{C}_c\, t^{-p_c}\}\end{aligned}$$

The first term is

$$\tilde{g}^{ab}t^{-p_a}\, e_a(\bar{C}_b\, t^{-p_b}) \preceq t^{|p_a-p_b|-p_a-p_b-\delta} \preceq t^{-2p_{\min\{a,b\}}-\delta}. \tag{2.79}$$

From (2.72)–(2.76) the second term is

$$\tilde{g}^{ab}\tilde{\Gamma}^c_{ab}\,\bar{C}_c\, t^{-p_c} \preceq t^{-2p_3-\delta}. \tag{2.80}$$

From (2.79) and (2.80), the desired estimate, $\nabla^a \bar{C}_a = O(t^{-2+\eta_1-\eta_2})$ is obtained. Thus, the system (2.59), (2.60) is Fuchsian.

Next we turn to (2.63) and (2.64). A term that appears in the momentum constraint is $\nabla_a k^a{}_b$. The estimate is needed in the adapted frame, and the covariant derivative is calculated in the frame $\{\tilde{e}_a\}$. This adds a factor of t^{p_b},

$$\nabla_a k^a{}_b = \{\tilde{e}_a(\tilde{k}^a{}_b) + \tilde{\Gamma}^a_{ac}\tilde{k}^c{}_b - \tilde{\Gamma}^c_{ab}\tilde{k}^a{}_c\}\, t^{p_b}$$

Furthermore, the quantity whose estimate will be required is the difference between this term and the corresponding term in the velocity-dominated constraint,

$$\nabla_a k^a{}_b - {}^0\nabla_a {}^0k^a{}_b = \{\tilde{e}_a(\tilde{k}^a{}_b - {}^0\tilde{k}^a{}_b) + \Delta\tilde{\Gamma}^a_{ac}\,{}^0\tilde{k}^c{}_b + \tilde{\Gamma}^a_{ac}(\tilde{k}^c{}_b - {}^0\tilde{k}^c{}_b)\}t^{p_b} \quad (2.81)$$

$$- \tilde{\Gamma}^c_{ab}\tilde{k}^a{}_c t^{p_b} + {}^0\tilde{\Gamma}^c_{ab}\,{}^0\tilde{k}^a{}_c\, t^{p_b} \quad (2.82)$$

The right-hand side of (2.81) is $O(t^{-1+\alpha_0-\delta})$. The terms in line (2.82) originating from Term E of the connection coefficients (see (2.76)) are cancelled in the sum, due to the antisymmetry of $\tilde{f}^h_{ai}$ and the symmetry of $\tilde{k}^{ai}$ and ${}^0\tilde{k}^{ai}$. For estimating the rest of the terms in line (2.82), it is convenient to rewrite this line as,

$$-\tilde{\Gamma}^c_{ab}\tilde{k}^a{}_c t^{p_b} + {}^0\tilde{\Gamma}^c_{ab}\,{}^0\tilde{k}^a{}_c\, t^{p_b} = -\Delta\tilde{\Gamma}^c_{ab}\,{}^0\tilde{k}^a{}_c\, t^{p_b} - \tilde{\Gamma}^c_{ab}(\tilde{k}^a{}_c - {}^0\tilde{k}^a{}_c)\, t^{p_b}, \quad (2.83)$$

with

$$\Delta\tilde{\Gamma}^c_{ab} = \frac{1}{2}\{\tilde{e}_a(t^{\tilde{\alpha}^b{}_c}\gamma^b{}_c) + \tilde{e}_b(t^{\tilde{\alpha}^c{}_a}\gamma^c{}_a) - \tilde{e}_c(t^{\tilde{\alpha}^a{}_b}\gamma^a{}_b) \quad (2.84)$$

$$+ \sum_h t^{\tilde{\alpha}^c{}_h}\bar{\gamma}^c{}_h[\tilde{e}_a(t^{\tilde{\alpha}^b{}_h}\gamma^b{}_h) + \tilde{e}_b(t^{\tilde{\alpha}^h{}_a}\gamma^h{}_a) - \tilde{e}_h(t^{\tilde{\alpha}^a{}_b}\gamma^a{}_b)] \quad (2.85)$$

$$- \sum_i t^{\tilde{\alpha}^i{}_a}\gamma^i{}_a\tilde{f}^i_{bc} - \sum_h t^{\tilde{\alpha}^c{}_h}\bar{\gamma}^c{}_h\tilde{f}^a_{bh} - \sum_{hi} t^{\tilde{\alpha}^c{}_h}\bar{\gamma}^c{}_h t^{\tilde{\alpha}^i{}_a}\gamma^i{}_a\tilde{f}^i_{bh} \quad (2.86)$$

$$- t^{\tilde{\alpha}^b{}_i}\gamma^b{}_i\tilde{f}^i_{ac} - \sum_h t^{\tilde{\alpha}^c{}_h}\bar{\gamma}^c{}_h\tilde{f}^b_{ah} - \sum_{hi} t^{\tilde{\alpha}^c{}_h}\bar{\gamma}^c{}_h t^{\tilde{\alpha}^b{}_i}\gamma^b{}_i\tilde{f}^i_{ah}\}. \quad (2.87)$$

The terms in line (2.87) need not be considered since they originate from Term E of the connection coefficients and as stated above the contribution from this term is cancelled by terms in $\Lambda = \tilde{\Gamma}^c_{ab}\,{}^0\tilde{k}^a{}_c\, t^{p_b}$. So considering only lines (2.84)–(2.86), the first term on the right-hand side of (2.83) is

$$\Delta\tilde{\Gamma}^a_{ab}\, p_a\, t^{-1+p_b} = O(t^{-1+\alpha_0-\delta}) + \text{terms which are cancelled by } \Lambda. \quad (2.88)$$

Since the terms in the sum come with different weights, p_a, (2.78) cannot be used in (2.88). But the estimate is straightforward. For example, the term in (2.88) originating from the 3rd term in line (2.86) $\preceq \sum_{a,h,i} t^{-1+|p_a-p_h|+|p_i-p_a|+p_i-p_h+2\alpha_0-\delta} = O(t^{-1+\alpha_0-\delta})$. Finally consider the rest of the right-hand side of (2.83),

$$-\frac{1}{2}\left[\tilde{g}^{ch}\{\tilde{e}_a(\tilde{g}_{bh}) + \tilde{e}_b(\tilde{g}_{ha}) - \tilde{e}_h(\tilde{g}_{ab}) - \tilde{g}_{ia}\tilde{f}^i_{bh} - \tilde{g}_{bi}\tilde{f}^i_{ah}\} + \tilde{f}^c_{ab}\right] t^{\tilde{\alpha}^a{}_c}\kappa^a{}_c\, t^{-1+p_b}. \quad (2.89)$$

For all terms except the 5th term in (2.89), the estimate, $O(t^{-1+\alpha_0-\delta})$ can be obtained from (2.72)–(2.77). The fifth term originates from Term E of the connection coefficients, and was already considered above. Therefore

$$\nabla_a k^a{}_b - {}^0\nabla_a {}^0k^a{}_b = O(t^{-1+\alpha_0-\delta}). \tag{2.90}$$

Next the matter terms are estimated. For the Hamiltonian constraint, an estimate of $\rho - {}^0\rho$ is needed.

$$\begin{aligned}
(\partial_t\phi)^2 - (\partial_t {}^0\phi)^2 &= \{2\,\partial_t {}^0\phi + t^{-1+\beta}(\beta\psi + t\partial_t\psi)\}\, t^{-1+\beta}(\beta\psi + t\partial_t\psi) \\
&= \{2A + t^{\beta}(\beta\psi + t\partial_t\psi)\}\, t^{-2+\beta}(\beta\psi + t\partial_t\psi) \\
&= o(t^{-2+\eta_1}), \\
g^{ab}e_a(\phi)e_b(\phi) &\preceq t^{-2p_3-\delta-\epsilon} = o(t^{-2+\eta_1}), \\
\frac{1}{g} g_{ab}\mathcal{E}^a\mathcal{E}^b e^{-\lambda_j\phi} &= O(t^{-2-\epsilon+\sigma}) = o(t^{-2+\eta_1}), \\
g^{ab}g^{ch}F_{ac}F_{bh}e^{\lambda_j\phi} &= O(t^{-2-2\epsilon+\sigma}) = o(t^{-2+\eta_1}).
\end{aligned}$$

Therefore,

$$\rho - {}^0\rho = o(t^{-2+\eta_1}). \tag{2.91}$$

The difference between the matter terms in the momentum constraint and in ${}^0C_a$ is

$$\begin{aligned}
-\partial_t\phi\, e_a(\phi) + \partial_t {}^0\phi\, e_a({}^0\phi) &= -\partial_t {}^0\phi\, e_a(t^\beta\psi) - \partial_t(t^\beta\psi)e_a(\phi) = O(t^{-1+\beta-\delta}), \\
\left(\frac{1}{\sqrt{g}} - \frac{1}{\sqrt{{}^0g}}\right){}^0\mathcal{E}^b\, {}^0F_{ab} &+ \frac{1}{\sqrt{g}}(\mathcal{E}^b F_{ab} - {}^0\mathcal{E}^b\, {}^0F_{ab}) = o(t^{-1+\eta_2}).
\end{aligned} \tag{2.92}$$

Estimates related to the determinant which are relevant to (2.92) can be found immediately preceding Equation (2.55). From the estimates just obtained,

$$j_a - {}^0j_a = o(t^{-1+\eta_2}). \tag{2.93}$$

From $R = O(t^{-2+\alpha_0})$ (shown in [15]) and from ${}^0C = 0$, (2.70), (2.69), (2.91) and the relative magnitude of the various exponents, it follows that $C = o(t^{-2+\eta_1})$. From ${}^0C_a = 0$, (2.90), (2.71), (2.93) and the relative magnitude of the various exponents, it follows that $C_a = o(t^{-1+\eta_2})$.

Since (2.63)–(2.64) are satisfied, the Hamiltonian and momentum constraints are satisfied.

2.7 Counting the number of arbitrary functions

The number of degrees of freedom of the Einstein-Maxwell-dilaton system in 4 spacetime dimensions is 5 : 2 for the gravitational field, 2 for the electromagnetic field and 1 for the dilaton. Hence, a general solution to the equations of motion should contain 10 freely adjustable, physically relevant, functions of space (each degree of freedom needs two initial data, q and $\dot{q}$). This is exactly the number that appears in the above Kasner-like solutions.

- The metric carries four, physically relevant, distinct functions of space. This is the standard calculation [4].
- The scalar field carries two functions of space, A and B.
- The electromagnetic field carries six functions of space, ${}^0\mathcal{E}^a$ and ${}^0F_{ab}$. These are physically relevant because they are gauge invariant, but they are subject to two constraints, leaving four independent functions.

A different way to arrive at the same conclusions is to observe that the respective number of fields, dynamical equations and (first class) constraints are the same for the velocity-dominated system and the full system. Hence, a general solution of the velocity-dominated system (in the sense of function counting) will contain the same number of physically distinct, arbitrary functions as a general solution of the full system. This general argument applies to all systems considered below and hence will not be repeated.

In [15] a different way of assessing the generality of the solutions constructed was used. This involved exhibiting a correspondence between solutions of the velocity-dominated constraints and solutions of the full constraints using the conformal method. That method starts with certain free data and shows the existence of a unique solution of the constraints corresponding to each set of free data. It is a standard method for exploring the solution space of the full Einstein constraints [42] and in [15] it was shown how to modify it to apply to the velocity-dominated constraints. While it is likely that the conformal method can be applied in some way to all the matter models considered in this paper, the details will only be worked out in two cases which suffice to illustrate the main aspects of the procedure. These are the Einstein-Maxwell-dilaton system with $D = 4$ (this section) and the Einstein vacuum equations with arbitrary $D \geq 4$ (next section). Even in those cases no attempt will be made to give an exhaustive treatment of all issues arising. It will, however, be shown that the strategies presented for solving the velocity-dominated constraints are successful in some important situations.

The procedure presented in the following is slightly different from that used in [15]. Even for the case of the Einstein-scalar field system with $D = 4$ it gives results which are in principle stronger than those in [15] since they are not confined to solutions which are close to isotropic ones. In the presence of exponential dilaton couplings a change of method seems unavoidable. One part of the conformal method concerns the construction of symmetric second rank tensors which are traceless and have prescribed divergence from the truly free data. In this step there is no difference between the full constraints and the velocity-dominated ones. An account of the methods applied to the full constraints in the case $D = 4$ can be found in [42]. (These arguments generalize in a straightforward way to other D. It is merely necessary to find the correct conformal rescalings. For $D \geq 4$ and vacuum these are written down in the next section.) In view of this we say, with a slight abuse of terminology, that the free data consists of a collection $\tilde{g}_{ab}$, $\tilde{k}_{ab}$, H, ϕ, $\tilde{\phi}_t$, $\mathcal{E}^a$, F_{ab} where $\tilde{g}_{ab}$ is a Riemannian metric, $\tilde{k}_{ab}$ is a symmetric tensor with vanishing trace and prescribed divergence with respect to $\tilde{g}_{ab}$, H is a non-zero constant, ϕ and $\tilde{\phi}_t$ are scalar functions and $\mathcal{E}^a$ and F_{ab} are objects of the same kind as elsewhere in this section. All these objects are defined on a three-dimensional manifold. Next we introduce a positive real-valued function ω which is used to construct solutions of the constraints from the free data. Define $g_{ab} = \omega^4\tilde{g}_{ab}$, $k_{ab} = \omega^{-2}\tilde{k}_{ab} + Hg_{ab}$, $\phi_t = \omega^{-6}\tilde{\phi}_t$. The objects g_{ab}, k_{ab}, ϕ, ϕ_t, $\mathcal{E}^a$ and F_{ab} satisfy the constraints provided the divergence of $\tilde{k}_{ab}$ is prescribed as $\omega^6 j_a$ and ω satisfies a nonlinear equation which in the case of the full Einstein equations is known as the Lichnerowicz equation. In the case of the velocity-dominated constraints it is an algebraic equation. The Lichnerowicz equation is of the form

$$\Delta_{\tilde{g}}\omega - \frac{1}{8}R_{\tilde{g}}\omega + \sum_{i=1}^{3} a_i\omega^{\alpha_i} - \frac{3}{4}H^2\omega^5 = 0 \tag{2.94}$$

Here $\alpha_1 = 1$, $\alpha_2 = -3$ and $\alpha_3 = -7$. The functions a_i depend on the free data and their exact form is unimportant. All that is of interest are that each a_i is non-negative and that at any point of space $a_1 = 0$ iff $\nabla_a\phi = 0$, $a_2 = 0$ iff the electromagnetic data vanish and $a_3 = 0$ iff ϕ_t and $\tilde{k}_{ab}$ vanish. Next consider the velocity-dominated constraints for $d = 3$. The analogue of the elliptic Equation (2.94) is the algebraic equation

$$b\omega^{-7} - \frac{3}{4}H^2\omega^5 = 0 \tag{2.95}$$

Here b is a non-negative function which vanishes at a point of space iff ϕ_t and $\tilde{k}_{ab}$ vanish. This can be solved trivially for $\omega > 0$ provided b does not vanish at any point since the mean curvature H is non-zero. For each choice of free data satisfying this non-vanishing condition there is a unique solution ω of (2.95).

In order to compare the sets of solutions of the full and velocity-dominated constraints in these two cases it remains to investigate the solvability of the elliptic Equation (2.94) for ω. A discussion of this type of problem in any dimension can be found in [43]. We would like to show that for suitable metrics on a compact manifold the equation for ω always has a unique solution, i.e., the situation is exactly as in the case of the velocity-dominated equations. The problem can be simplified by the use of the Yamabe theorem, which says that any metric can be conformally transformed to a metric of constant scalar curvature -1, 0 or 1. In the following only the cases of negative and vanishing scalar curvature of the metric supplied by the Yamabe theorem will be considered. A key role in the existence and uniqueness theorems for Equation (2.94) is played by the positive zeros of the algebraic expressions $x + 8\sum_{i=1}^{3} a_i x^{\alpha_i} - 6H^2x^5$ and $8\sum_{i=1}^{3} a_i x^{\alpha_i} - 6H^2x^5$. Provided $\sum_{a=1}^{3} a_i$ does not vanish anywhere it is possible to show that each of the algebraic expressions has a unique positive zero for each value of the parameters.

The significance of the information which has been obtained concerning the zeros of certain algebraic expressions is that it guarantees the existence of a positive solution of the corresponding elliptic equations for any set of free data satisfying the inequalities already stated using the method of sub- and supersolutions (*cf.* [43]). In the case of Equation (2.94) uniqueness also holds. For in that case the equation has a form considered in [44] for which uniqueness is demonstrated in that paper. The advantage of the three-dimensional case is that there the problem reduces to the analysis of the roots of a cubic equation, a relatively simple task compared to the analysis of the zeros of the more complicated algebraic expressions occurring in higher dimensions.

For the purpose of analyzing Kasner-like (monotone) singularities it is not enough to know about producing just any solutions of the constraints. What we have shown is that (i) if the Kasner constraints are satisfied at time t_0, then they are propagated at all times by the velocity-dominated evolution equations; and (ii) if the Kasner constraints are satisfied, the exact constraints are also satisfied. It is also necessary to verify, however, that one can satisfy simultaneously the Kasner constraints *and* the inequalities necessary for applying the Fuchsian arguments, i.e., we must make sure that we can produce a sufficiently large class of solutions which satisfy the inequalities necessary to make them consistent with Kasner behaviour. Because of the indirect nature of the way of solving the momentum constraint (which has not been explained here) it is not easy to control the generalized Kasner exponents of the resulting spacetime. There is however, one practical possibility. Choose a spatially homogeneous solution with Abelian isometry group (for $d = 3$ this means Bianchi type I) which satisfies the necessary inequalities. Take the free data from that solution and deform it slightly. Then the generalized Kasner exponents of the final solution of the velocity-dominated equations will also only be changed slightly. If the homogeneous solution is defined on the torus T^3 then it is known that any other metric of constant scalar curvature has non-positive scalar curvature. Therefore we are in the case for which existence and uniqueness is discussed above. We could also start with a negatively curved Friedmann model.

3 Vacuum solutions with $D \geq 11$

The second class of solutions we consider is governed by the Action (1.2), with $D \geq 11$. The $d + 1$ decomposition is as in Section 1.6, with the matter terms vanishing. The Kasner-like evolution equations are (2.7) and (2.8). The general analytic solution of these equations is given by (2.12) and (2.13). To obtain this form, we again adapt a global time coordinate such that the singularity is at $t = 0$. We label the eigenvalues of K, $p_1, \ldots, p_d$, such that $p_a \leq p_b$ if $a < b$. The eigenvalues again satisfy $\sum_{i=1}^{d} p_i = 1$, coming from $\operatorname{tr} K = 1$. As in the $D = 4$ case, in order to preserve analyticity even near the points where some of the eigenvalues coincide, while retaining control of the solution in terms of the eigenvalues, we introduce a special construction involving auxiliary exponents and an adapted frame.

In higher dimensions, there are more possibilities to take care of, but the idea is the same as in the $D = 4$ case. Consider an arbitrary point $x_0 \in \Sigma$. Let

n be the number of distinct eigenvalues of K at x_0. Let m_i be the multiplicity of p_{A_i}, $i \in \{1, \dots, n\}$, with p_{A_i} such that p_b is strictly less than p_{A_i} if $b < A_i$. Thus $p_{A_i}, \dots, p_{A_i+m_i-1}$ are equal at x_0. For each integer $a \in \{A_i, \dots, A_i + m_i - 1\}$, define

$$q_a = \frac{1}{m_i} \sum_{j=A_i}^{A_i+m_i-1} p_j$$

on a neighborhood of x_0, U_0. Note that if $m_i = 1$, then $q_{A_i} = p_{A_i}$. Shrinking U_0 if necessary, choose $\epsilon > 0$ such that for $x \in U_0$, for $a \in \{A_i, \dots, A_i + m_i - 1\}$ and for $b \in \{A_j, \dots, A_j + m_j - 1\}$, if $i = j$, then $|p_a - p_b| < \epsilon/2$, while if $i \neq j$, $|p_a - p_b| > \epsilon/2$.

The adapted frame $\{e_a\}$ is again required to be analytic and such that the related frame $\{\tilde{e}_a\}$ is orthonormal with respect to the Kasner-like metric at some time $t_0 > 0$, with $\tilde{e}_a = t^{-q_a} e_a$. In addition, it is required that $e_{A_i}, \dots, e_{A_i+m_i-1}$ span the eigenspace of K corresponding to the eigenvalues $p_{A_i}, \dots, p_{A_i+m_i-1}$. Note that if $m_i = 1$ then e_{A_i} is an eigenvector of K corresponding to the eigenvalue q_{A_i}. The auxiliary exponents, q_a, are analytic, satisfy the Kasner relation ($\sum q_a = 1$), are ordered ($q_a \leq q_b$ for $a < b$), and satisfy $q_1 \geq p_1$, $q_d \leq p_d$ and $\max_a |q_a - p_a| < \epsilon/2$. If $q_a \neq q_b$, then ${}^0g_{ab}$, ${}^0g^{ab}$, ${}^0\tilde{g}_{ab}$ and ${}^0\tilde{g}^{ab}$ all vanish, and the same is true with g replaced by k.

The velocity-dominated constraints corresponding to the Hamiltonian and momentum constraints are ${}^0C = 0$ and ${}^0C_a = 0$, with ${}^0C$ and ${}^0C_a$ as in Equations (2.18) and (2.19), with the matter terms vanishing. For the Solution (2.12)–(2.13) the velocity-dominated Hamiltonian constraint equation is equivalent to $\sum p_a^2 = 1$. Equations (2.21)–(2.22) are satisfied, so if the velocity-dominated constraints are satisfied at some t_0, then they are satisfied for all $t > 0$.

For this class of solutions, the Inequality [13], $2p_1 + p_2 + \cdots + p_{d-2} > 0$, or equivalently,

$$1 + p_1 - p_d - p_{d-1} > 0, \tag{3.1}$$

defines the set V which was referred to in the introduction. As shown in [13], this inequality can be realized when the spacetime dimension D is equal to or greater than 11. As in our Maxwell archetypal example above, we expect that this inequality will be crucial to control the effect of the source terms (here linked to the spatial curvature) near the singularity. It is again convenient to introduce a number $\sigma > 0$ so that, for all $x \in U_0$, $4\sigma < 1 + p_1 - p_d - p_{d-1}$. Reduce ϵ if necessary so that $\epsilon < \sigma/(2d+1)$. If ϵ is reduced, it may be necessary to shrink U_0 so that the conditions imposed above remain satisfied. It is assumed that ϵ and U_0 are such that the conditions imposed above and the condition imposed in this paragraph are all satisfied.

We again recast the evolution equations in the form (1.5) and show, for $D \geq 11$, that (1.5) is Fuchsian and equivalent to the vacuum Einstein equation, with quantities u, $\mathcal{A}$ and f as follows. Let $u = (\gamma^a{}_b, \lambda^c{}_{ef}, \kappa^h{}_i)$ be related to the Einstein variables by (2.31)–(2.33). For general d define $\alpha_0 = (d+1)\epsilon$ and define $\alpha^a{}_b$

in terms of α_0 as in Section 2. Let $\mathcal{A}$ and f, be given by Equations (2.39)–(2.41), with $M^a{}_b = 0$. The argument that $\mathcal{A}$ in Equation (1.5) satisfies the appropriate positivity condition is analogous to the part of the argument concerning the submatrix of $\mathcal{A}$ corresponding to (γ, κ) in [15]. A transformation to a frame in which ${}^0g_{ab}$ is diagonal induces a similarity transformation of $\mathcal{A}$. The eigenvalues of the submatrix are calculated in this representation in [15], and the generalization of the calculation to integer $d \geq 2$ is straightforward.

To obtain $f = O(t^\delta)$ requires the estimate $t^{2-\alpha^a{}_b}\,{}^SR^a{}_b = O(t^\delta)$. The strategy used here is different from that used to estimate the curvature in [15]. The general problem is one of organization. There are many terms to be estimated, each of which on its own is not too difficult to handle. The difficulty is to maintain an overview of the different terms. The procedure in [15] made essential use of the fact that the indices only take three distinct values and in the case of higher dimensions, where this simplification is not available, an alternative approach had to be developed.

First ${}^SR^a{}_b$ is estimated by considering each of the five terms in the Expression (2.47). These five terms are expanded by considering each of the six terms in (2.48) if the indices on ${}^S\Gamma^c_{ab}$ are distinct, but carrying out the summation before estimating ${}^S\Gamma^a_{ab}$. There are thus 55 terms to estimate. While many of these terms are actually identical up to numerical factors, the ease with which each term can be estimated, using the Inequalities (2.65)–(2.67), led to estimation of all 55 terms, rather than first combining terms. We do however, take into account that $f^i_{jk} = 0$ if $j = k$ for obtaining the estimates.

Once an equation such as (1.5) is shown to be Fuchsian, then it follows that spatial derivatives of u of any order are $o(1)$. At the stage of the argument we are at here, we cannot assume $u_{xx} = O(1)$. This means that $t^{-\zeta}\lambda^a{}_{bc}$ must be used for $e_b(\gamma^a{}_c)$ in places where a spatial derivative of $e_b(\gamma^a{}_c)$ occurs. This makes a slight difference, compared to Section 2.6, in what estimate of the terms in the connection coefficients is used for the first and second terms of (2.47) ($t^{-\delta}$ is replaced by $t^{-\zeta}$). There are additional differences from (2.72)–(2.77), because there it is assumed that the Kasner-like metric is diagonal. The estimates ${}^0\tilde{g}_{ab} = O(t^{-\epsilon})$ and ${}^0\tilde{g}^{ab} = O(t^{-\epsilon})$, obtained in Lemma 2 of [15], hold in the case we are considering, so that $\tilde{g}_{(ab)} \preceq t^{|q_a - q_b| - \epsilon}$ and (see [15]) ${}^S\tilde{g}^{ab} \preceq t^{|q_a - q_b| - \epsilon}$. This adds factors of $t^{-\epsilon}$ to the estimate of terms in the connection coefficients.

With these considerations, from (2.48),

$$ {}^S\tilde{\Gamma}^a_{ac} = \frac{1}{2}\,{}^S\tilde{g}^{ab}\tilde{e}_c(\tilde{g}_{(ab)}) + \tilde{f}^a_{ac} \preceq t^{-q_c - 2\epsilon - \zeta}. \qquad (3.2) $$

Here we do not write out the estimates of all 55 terms, but instead give some examples, with a number designating which term of (2.47) is being considered (1–5), and a letter designating which term of (2.48) is being considered (A–F).

Thus, for example, term 1C is

$$\begin{aligned}
&t^{-q_a+q_b}\tilde{g}^{ac}\tilde{e}_h({}^S\tilde{g}^{hi}\tilde{e}_i(\tilde{g}_{(cb)})) \\
&\quad\preceq \sum_{c,h,i} t^{-q_a+q_b+|q_a-q_c|-q_h+|q_h-q_i|-q_i+|q_c-q_b|-\delta-3\epsilon-\zeta} \\
&\quad\preceq t^{-2q_a+2q_{\max\{a,b\}}-2q_d-\delta-3\epsilon-\zeta}. \qquad (3.3)
\end{aligned}$$

Term 3E is

$$\begin{aligned}
&t^{-q_a+q_b}\tilde{g}^{ac}\,{}^S\tilde{g}^{ik}\tilde{g}_{(bj)}\tilde{f}^j_{ck}\,{}^S\tilde{\Gamma}^h_{hi} \\
&\quad\preceq \sum_{c,i,j,k\neq c} t^{-q_a+q_b+|q_a-q_c|+|q_i-q_k|+|q_b-q_j|+q_j-q_c-q_k-q_i-\delta-5\epsilon} \\
&\quad\preceq \sum_{c,k\neq c} t^{-2q_{\min\{a,c\}}+2q_b-2q_{\min\{d,k\}}-\delta-5\epsilon}. \qquad (3.4)
\end{aligned}$$

In term 4, $t^{-q_a+q_b}\tilde{g}^{ac}\,{}^S\tilde{\Gamma}^h_{ib}\,{}^S\tilde{\Gamma}^i_{ch}$, the terms resulting from expanding ${}^S\tilde{\Gamma}^h_{ib}$ are designated by small letters a–f, and those from ${}^S\tilde{\Gamma}^i_{ch}$ are designated by capital letters A–F. Term 4dA is

$$\begin{aligned}
&t^{-q_a+q_b}\tilde{g}^{ac}\,{}^S\tilde{g}^{hl}\tilde{g}_{(ji)}\tilde{f}^j_{bl}\,{}^S\tilde{g}^{ik}\tilde{e}_c(\tilde{g}_{(hk)}) \\
\preceq\ &\sum_{c,h,i,j,k,l} t^{-q_a+q_b+|q_a-q_c|+|q_h-q_l|+|q_j-q_i|+q_j-q_b-q_l+|q_i-q_k|-q_c+|q_h-q_k|-\delta-5\epsilon} \\
\preceq\ &t^{-2q_a-\delta-5\epsilon}. \qquad (3.5)
\end{aligned}$$

Term 4eD is

$$\begin{aligned}
&t^{-q_a+q_b}\tilde{g}^{ac}\,{}^S\tilde{g}^{hn}\tilde{g}_{(bj)}\tilde{f}^j_{in}\,{}^S\tilde{g}^{il}\tilde{g}_{(kc)}\tilde{f}^k_{hl} \\
&\preceq \sum_{c,h,i,j,k,l\neq h,n\neq i} \\
&\qquad t^{-q_a+q_b+|q_a-q_c|+|q_h-q_n|+|q_b-q_j|+q_j-q_i-q_n+|q_i-q_l|+|q_k-q_c|+q_k-q_h-q_l-\delta-5\epsilon} \\
&\preceq t^{2q_b-2q_d-2q_{d-1}-\delta-5\epsilon} \qquad (3.6)
\end{aligned}$$

Term 5D is

$$\begin{aligned}
&t^{-q_a+q_b}\tilde{g}^{ac}\tilde{f}^i_{cj}\,{}^S\tilde{g}^{jk}\tilde{g}_{(hi)}\tilde{f}^h_{bk} \\
&\quad\preceq \sum_{c,h,i,j\neq c,k\neq b} t^{-q_a+q_b+|q_a-q_c|+q_i-q_c-q_j+|q_j-q_k|+|q_h-q_i|+q_h-q_b-q_k-\delta-3\epsilon} \\
&\quad\preceq \sum_{c,j\neq c,k\neq b} t^{-2q_{\min\{a,c\}}+2q_1-2q_{\min\{j,k\}}-\delta-3\epsilon}. \qquad (3.7)
\end{aligned}$$

The estimates of the remaining terms are obtained as these. The examples include one of the terms which limits the estimate for each possible choice of indices

a and b. The result is,

$$ {}^S R^a{}_b \preceq t^{2q_b - 2q_d - 2q_{d-1} - \delta - 5\epsilon} + \sum_{c, j \neq c, k \neq b} t^{-2q_{\min\{a,c\}} + 2q_1 - 2q_{\min\{j,k\}} - \delta - 5\epsilon}. \tag{3.8} $$

And

$$ \begin{aligned} t^{2-\alpha^a{}_b}\, {}^S R^a{}_b \quad &\preceq \quad \{ t^{2q_{\min\{a,b\}} - 2q_d - 2q_{d-1}} \\ &\quad + \sum_{c \geq a, j \neq c, k \neq b} t^{-2q_{\max\{a,b\}} + 2q_1 - 2q_{\min\{j,k\}}} \} t^{2 - \alpha_0 - \delta - 5\epsilon} \\ &\preceq \quad t^{2 - 2q_d - 2q_{d-1} + 2q_1 - (d+7)\epsilon} \preceq t^{8\sigma - (d+7)\epsilon} = O(t^\delta). \end{aligned} \tag{3.9} $$

The estimate of the rest of the terms in f is obtained straightforwardly by checking that the exponent of t in each case is strictly positive. The other regularity conditions that f should satisfy are shown to hold by Equation (31) in [15] and the remarks following Equation (31). The symmetry of g_{ab} is shown for all d's in Subsection 2.5.3. That the Hamiltonian and momentum constraints are satisfied is shown by the direct analogue of argument made in Section 2.6 and the estimate $R = o(t^{-2+\eta_1})$ obtained from Equation (3.8). The only change is that Equation (2.80) is replaced by

$$ \tilde{g}^{ab} \tilde{\Gamma}^c_{ab} \bar{C}_c t^{-p_c} \preceq t^{-2p_d - \delta}. \tag{3.10} $$

To conclude this section we discuss the solution of the velocity-dominated constraints for the vacuum equations and $D \geq 4$. The case $D = 3$ could be discussed in a similar way but the analogue of the Lichnerowicz equation has a different form and so for brevity that case will be omitted. The discussion proceeds in a way which is parallel to that of the last section. As already indicated there, the essential task is the analysis of the Lichnerowicz equation. In the present case we start with free data $\tilde{g}_{ab}$, $\tilde{k}_{ab}$ and H where $\tilde{k}_{ab}$ has zero divergence. The actual data are defined by $g_{ab} = \omega^{4/(d-2)} \tilde{g}_{ab}$ and $k_{ab} = \omega^{-2} \tilde{k}_{ab} + H g_{ab}$. The constraints will be satisfied is ω satisfies the following analogue of the Lichnerowicz equation:

$$ \Delta_{\tilde{g}} \omega + \frac{d-2}{4(d-1)} (-R_{\tilde{g}} \omega + \tilde{k}^{ab} \tilde{k}_{ab} \omega^{\frac{3d-2}{d-2}}) - \frac{d(d-2)}{4} H^2 \omega^{\frac{d+2}{d-2}} = 0 \tag{3.11} $$

The corresponding equation in the velocity-dominated case is

$$ \frac{d-2}{4(d-1)} \tilde{k}^{ab} \tilde{k}_{ab} \omega^{\frac{3d-2}{d-2}} - \frac{d(d-2)}{4} H^2 \omega^{\frac{d+2}{d-2}} = 0 \tag{3.12} $$

As in the case of (2.95) it is trivial to solve (3.12) provided $\tilde{k}_{ab}$ does not vanish at any point. To determine the solvability of Equation (3.11) it is necessary to study the positive zeros of the algebraic expressions $x + bx^{\frac{3d-2}{d-2}} - ax^{\frac{d+2}{d-2}}$ and $bx^{\frac{3d-2}{d-2}} - ax^{\frac{d+2}{d-2}}$ where $a > 0$ and $b > 0$. The second expression is very close to what we had in the velocity-dominated case and clearly has a unique positive zero for any values of a and b satisfying the inequalities assumed. Looking for positive zeros of the first algebraic expression is equivalent to looking for positive solutions of $x^{-\frac{d+2}{d-2}} + ax^{-2} - b = 0$. Note that the function on the left-hand side of this equation is evidently decreasing for all positive x, tends to infinity as $x \to 0$ and tends to $-b$ as $x \to \infty$. Hence as long as the constant b is non-zero this function has exactly one positive zero, as desired. This is what is needed to obtain an existence theorem. It would be desirable to also obtain a uniqueness theorem for the solution of (3.11). To obtain solutions of the velocity-dominated constraints of the right kind to be consistent with Kasner-like behaviour we can use the same approach as in the last section, starting with Kasner solutions with an appropriate set of Kasner exponents.

4 Massless scalar field, $D \geq 3$

Consider Einstein's equations, $D \geq 3$, with a massless scalar field as source, the action given by $S_E[g_{\alpha\beta}] + S_\phi[g_{\alpha\beta}, \phi]$, and $d+1$ decomposition as in Section 1.6. The stress-energy tensor is

$$T_{\mu\nu} = {}^{(D)}\nabla_\mu \phi \, {}^{(D)}\nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \, {}^{(D)}\nabla_\alpha \phi \, {}^{(D)}\nabla^\alpha \phi. \tag{4.1}$$

Thus $\rho = \frac{1}{2}\{(\partial_t \phi)^2 + g^{ab} e_a(\phi) e_b(\phi)\}$, $j_a = -\partial_t \phi \, e_a(\phi)$, and $M^a{}_b = g^{ac} e_b(\phi) \, e_c(\phi)$. A crucial step in the generalization to arbitrary $D \geq 3$ is that the cancellation of terms involving $\partial_t \phi$ in the expression for $M^a{}_b$ is not particular to $D = 4$. The scalar field satisfies ${}^{(D)}\nabla_\alpha \, {}^{(D)}\nabla^\alpha \phi = 0$, which has $d+1$ decomposition

$$\partial_t^2 \phi - (\mathrm{tr} k) \partial_t \phi = g^{ab} \nabla_a \nabla_b \, \phi. \tag{4.2}$$

Let the Kasner-like evolution equations be Equations (2.7)–(2.9), with Solutions (2.12)–(2.14) for time coordinate as in Section 3. Given a point $x_0 \in \Sigma$, let the neighborhood U_0, the (local) adapted frame and the constant ϵ be as in Section 3. Define ${}^0\rho = \frac{1}{2}(\partial_t \, {}^0\phi)^2$ and ${}^0j_a = -\partial_t \, {}^0\phi \, e_a({}^0\phi)$. The velocity-dominated constraints corresponding to the Hamiltonian and momentum constraints are ${}^0C = 0$ and ${}^0C_a = 0$, with ${}^0C$ and ${}^0C_a$ given by Equations (2.18) and (2.19). For the Solution (2.12)–(2.14) the velocity-dominated Hamiltonian constraint is equivalent to $\sum {p_a}^2 + A^2 = 1$. Equations (2.21) and (2.22) are satisfied so if the velocity-dominated constraints are satisfied at some t_0, then they are satisfied for all $t > 0$. The restriction defining the set V is the Inequality (3.1). (If $D < 11$, then satisfying simultaneously (3.1), $\sum p_a = 1$ and $\sum {p_a}^2 + A^2 = 1$ requires $A \neq 0$. Note that conversely, for $D = 3$, the restrictions defining V are simply equivalent to $A \neq 0$, since (3.1) is in this case a consequence of $p_1 + p_2 = 1$ and $p_1^2 + p_2^2 < 1$). The constant $\sigma > 0$ is chosen so that, for all $x \in U_0$, $4\sigma < 1 + p_1 - p_d - p_{d-1}$ from which it follows that

$$\sigma < 2 - 2p_d \; . \tag{4.3}$$

Now reduce ϵ if necessary so that $\epsilon < \sigma/(2d+1)$. As before, this may in turn require shrinking U_0.

The unknown $u = (\gamma^a{}_b, \lambda^a{}_{bc}, \kappa^a{}_b, \psi, \omega_a, \chi)$ is related to the Einstein-matter variables by (2.31)–(2.36). The quantities $\mathcal{A}$ and f appearing in Equation (1.5) are given by the evolution Equations (2.39)–(2.43) and

$$t\,\partial_t\chi + \beta\chi = t^{\alpha_0-\beta}(\mathrm{tr}\,\kappa)(A + t^\beta\chi) + t^{2-\beta}\,{}^S g^{ab}\,{}^S\nabla_a\,{}^S\nabla_b\,{}^0\phi + t^{2-\zeta}\,{}^S\nabla^a\omega_a. \quad (4.4)$$

The argument that the matrix $\mathcal{A}$ satisfies the appropriate positivity condition is analogous to the argument in [15]. Regarding the estimate $f = O(t^\delta)$, the estimate $t^{2-\alpha^a{}_b}\,{}^S R^a{}_b = O(t^\delta)$ was obtained in Equation (3.9). The estimate $t^{2-\alpha^a{}_b} M^a{}_b = O(t^\delta)$ follows from the Inequality (4.3) and from $q_d < p_d$. The only other terms in f whose estimates are not immediate from the estimates made in [15] are the last two terms on the right-hand side of Equation (4.4). The covariant derivative compatible with the symmetrized metric is used in Equation (4.4) for convenience. From the estimate ${}^S\tilde{g}^{ab} \preceq t^{|q_a-q_b|-\epsilon}$ [15], Equations (2.53) and (2.54),

$${}^S g^{ab} \preceq t^{-2q_{\min\{a,b\}}-\epsilon} \qquad \text{and} \qquad e_c(g_{(ab)}) \preceq t^{2q_{\max\{a,b\}}-\delta-\epsilon}.$$

Therefore,

$$\begin{aligned} {}^S g^{ab}\,{}^S\Gamma^c_{ab} &= {}^S g^{ab}\,{}^S g^{ch}\Big(e_a(g_{(bh)}) - \frac{1}{2}e_h(g_{(ab)})\Big) - {}^S g^{ch} f^a_{ah} \\ &\preceq t^{-2q_d-\delta-3\epsilon} \end{aligned}$$

and

$$\begin{aligned} t^{2-\beta}\,{}^S g^{ab}\,{}^S\nabla_a\,{}^S\nabla_b\,{}^0\phi &= t^{2-\beta}\,{}^S g^{ab}\{e_a(e_b({}^0\phi)) - {}^S\Gamma^c_{ab}e_c({}^0\phi)\} \\ &\preceq t^{2-2q_d-\beta-\delta-3\epsilon} = O(t^\delta), \\ t^{2-\zeta}\,{}^S\nabla^a\omega_a &= t^{2-\zeta}\,{}^S g^{ab}\{e_a(w_b) - {}^S\Gamma^c_{ab}w_c\} \\ &\preceq t^{2-2q_d-\zeta-\delta-3\epsilon} = O(t^\delta) \end{aligned}$$

The other regularity conditions that f should satisfy are again shown to hold by Equation (31) in [15] and the remarks following Equation (31). That g_{ab} is symmetric (so that Equation (4.4) and Equation (4.2) are equivalent) is shown as in Subsection 2.5.3. That the Hamiltonian and momentum constraints are satisfied is shown by the analogue of the argument made in Section 2.6 and the estimate $R = o(t^{-2+\eta_1})$ obtained from Equation (3.8).

Note that the case $D = 3$ of this result has an interesting connection to the Einstein vacuum equations in $D = 4$. As it follows from standard Kaluza-Klein lines, the solutions of the latter with polarized $U(1)$ symmetry are equivalent to the Einstein-scalar field system in $D = 3$ (see e.g. [40] and [41], Section 5). Hence the result of this section implies that we have constructed the most general known class of singular solutions of the Einstein vacuum equations in four spacetime dimensions. These spacetimes have one spacelike Killing vector.

5 Matter fields derived from n-form potentials

5.1 Equations of motion

We now turn to the general system (1.1), but without the interaction terms "more". These are considered in Section 7 below. The action is the sum of (1.2), (1.3) and k additional terms, each of the form (1.4). The argument is based on that of Section 4. It is enough here to note the differences. Furthermore, since there is no coupling between additional matter fields, the differences from the argument made in Section 4 can be noted for each additional matter field independently of the others. Therefore consider the jth additional matter field,

$$F_{\mu_0\cdots\mu_{n_j}} = (n_j+1)\nabla_{[\mu_0}A_{\mu_1\cdots\mu_{n_j}]},$$

with A an n_j-form. This matter field contributes the following additional terms to the stress-energy tensor, Equation (4.1),

$$T_{\mu\nu} = \cdots + \left\{\frac{1}{n_j!}F_{\mu\alpha_1\cdots\alpha_{n_j}}F_\nu{}^{\alpha_1\cdots\alpha_{n_j}} - \frac{1}{2(n_j+1)!}g_{\mu\nu}F_{\alpha_0\cdots\alpha_{n_j}}F^{\alpha_0\cdots\alpha_{n_j}}\right\}e^{\lambda_j\phi}.$$

Define $\mathcal{E}^{a_1\cdots a_{n_j}} = \sqrt{g}\,F^{0a_1\cdots a_{n_j}}\ e^{\lambda_j\phi}$. If $n_j = 0$, $\mathcal{E}$ is a spatial scalar density. Throughout this section and the next we use the following conventions. If $n_j = 0$, then $P_{a_1\cdots a_{n_j}}$ is a scalar, $g_{a_1b_1}\cdots g_{a_{n_j}b_{n_j}} = 1$, etc. The $d+1$ decomposition of the contribution of this matter field to the stress-energy tensor is

$$\begin{aligned}
\rho &= \cdots + \frac{1}{2\,g\,n_j!}g_{a_1b_1}\cdots g_{a_{n_j}b_{n_j}}\mathcal{E}^{a_1\cdots a_{n_j}}\mathcal{E}^{b_1\cdots b_{n_j}}e^{-\lambda_j\phi} \\
&\quad + \frac{1}{2(n_j+1)!}\,g^{a_0b_0}\cdots g^{a_{n_j}b_{n_j}}F_{a_0\cdots a_{n_j}}F_{b_0\cdots b_{n_j}}e^{\lambda_j\phi}, \qquad (5.1)\\
j_a &= \cdots + \frac{1}{\sqrt{g}\,n_j!}\,\mathcal{E}^{b_1\cdots b_{n_j}}\,F_{ab_1\cdots b_{n_j}}, \qquad (5.2)\\
M^a{}_b &= \cdots - \frac{1}{g}\Big(\frac{n_j}{n_j!}\,g_{bh_1}g_{c_2h_2}\cdots g_{c_{n_j}h_{n_j}}\mathcal{E}^{ac_2\cdots c_{n_j}}\mathcal{E}^{h_1\cdots h_{n_j}} \\
&\quad - \frac{n_j}{(d-1)n_j!}\,\delta^a{}_b\,g_{c_1h_1}\cdots g_{c_{n_j}h_{n_j}}\mathcal{E}^{c_1\cdots c_{n_j}}\mathcal{E}^{h_1\cdots h_{n_j}}\Big)e^{-\lambda_j\phi} \\
&\quad + \Big(\frac{1}{n_j!}\,g^{ac}g^{h_1i_1}\cdots g^{h_{n_j}i_{n_j}}F_{ch_1\cdots h_{n_j}}F_{bi_1\cdots i_{n_j}} \\
&\quad - \frac{n_j}{(d-1)(n_j+1)!}\,\delta^a{}_b\,g^{c_0h_0}\cdots g^{c_{n_j}h_{n_j}}F_{c_0\cdots c_{n_j}}F_{h_0\cdots h_{n_j}}\Big)e^{\lambda_j\phi}.
\end{aligned}$$

The jth matter field satisfies

$$^{(D)}\nabla_\mu(F^{\mu\nu_1\cdots\nu_{n_j}}e^{\lambda_j\phi}) = 0, \qquad (5.3)$$

$$^{(D)}\nabla_{[\mu}F_{\nu_0\cdots\nu_{n_j}]} = 0, \qquad (5.4)$$

with $d+1$ decomposition into constraint equations,

$$e_a(\mathcal{E}^{ab_2\cdots b_{n_j}}) + f^c_{ca}\,\mathcal{E}^{ab_2\cdots b_{n_j}} + \frac{1}{2}\sum_{i=2}^{n_j} f^{b_i}_{ac}\mathcal{E}^{ab_2\cdots c\cdots b_{n_j}} = 0, \tag{5.5}$$

$$e_{[a}(F_{b_0\cdots b_{n_j}]}) - \frac{(n_j+1)}{2} f^c_{[ab_0}F_{|c|b_1\cdots b_{n_j}]} = 0, \tag{5.6}$$

and evolution equations,

$$\partial_t\mathcal{E}^{a_1\cdots a_{n_j}} = -e_b(\sqrt{g}g^{bc_0}g^{a_1c_1}\cdots g^{a_{n_j}c_{n_j}}F_{c_0\cdots c_{n_j}}e^{\lambda_j\phi}) - \{f^h_{hb}g^{bc_0}g^{a_1c_1}\cdots g^{a_{n_j}c_{n_j}} + \frac{1}{2}\sum_{i=1}^{n_j} f^{a_i}_{bh}g^{bc_0}g^{a_1c_1}\cdots g^{hc_i}\cdots g^{a_{n_j}c_{n_j}}\}\sqrt{g}F_{c_0\cdots c_{n_j}}e^{\lambda_j\phi}, \tag{5.7}$$

$$\partial_t F_{a_0\cdots a_{n_j}} = -(n_j+1)e_{[a_0}(\frac{1}{\sqrt{g}}g_{a_1|b_1|}\cdots g_{a_{n_j}]b_{n_j}}\mathcal{E}^{b_1\cdots b_{n_j}}e^{-\lambda_j\phi}) + \frac{(n_j+1)n_j}{2\sqrt{g}} f^c_{[a_0a_1}g_{|c||b_1|}g_{a_2|b_2|}\cdots g_{a_{n_j}]b_{n_j}}\mathcal{E}^{b_1\cdots b_{n_j}}e^{-\lambda_j\phi}. \tag{5.8}$$

The jth matter field contributes the following terms to the evolution Equation (4.2) for ϕ.

$$\partial_t^2\phi - (\mathrm{tr}k)\partial_t\phi = \cdots + \frac{\lambda_j}{2\,g\,n_j!}g_{a_1b_1}\cdots g_{a_{n_j}b_{n_j}}\mathcal{E}^{a_1\cdots a_{n_j}}\mathcal{E}^{b_1\cdots b_{n_j}}e^{-\lambda_j\phi} - \frac{\lambda_j}{2(n_j+1)!}g^{a_0b_0}\cdots g^{a_{n_j}b_{n_j}}F_{a_0\cdots a_{n_j}}F_{b_0\cdots b_{n_j}}e^{\lambda_j\phi} \tag{5.9}$$

5.2 Velocity-dominated system

The Kasner-like evolution equations corresponding to this matter field are $\partial_t\,{}^0\mathcal{E}^{a_1\cdots a_{n_j}} = 0$ and $\partial_t\,{}^0F_{a_0\cdots a_{n_j}} = 0$. The quantities ${}^0\mathcal{E}^{a_1\cdots a_{n_j}}$ and ${}^0F_{a_0\cdots a_{n_j}}$ are constant in time with analytic spatial dependence and both are totally antisymmetric.

The velocity-dominated matter constraint equations are Equations (5.5) and (5.6) with ${}^0\mathcal{E}$ and ${}^0F$ substituted for $\mathcal{E}$ and F. Since all quantities in the velocity-dominated matter constraints are independent of time, if the matter constraints are satisfied at some time $t_0 > 0$, then they are satisfied for all $t > 0$. This matter field does not contribute to ${}^0\rho$. Its contribution to ${}^0j_a$ is the term shown on the right-hand side of Equation (5.2) with ${}^0g$, ${}^0\mathcal{E}$ and ${}^0F$ substituted for g, $\mathcal{E}$ and F. The velocity-dominated constraints corresponding to the Hamiltonian and momentum constraints are ${}^0C = 0$ and ${}^0C_a = 0$, with ${}^0C$ and ${}^0C_a$ given by Equations (2.18) and (2.19). Equations (2.21) and (2.22) are satisfied, so as before, if the velocity-dominated constraints are satisfied at some $t_0 > 0$, then they are satisfied for all $t > 0$.

The presence of the matter field $A^{(j)}$ puts the following restrictions on the set V [1].

$$2p_1 + \cdots + 2p_{n_j} - \lambda_j A > 0 \qquad \text{and} \qquad 2p_1 + \cdots + 2p_{d-n_j-1} + \lambda_j A > 0. \quad (5.10)$$

The restrictions generalize the Inequalities (2.24) found for a Maxwell field in 4 dimensions and, like them, guarantee that one can asymptotically neglect the p-form $A^{(j)}$ in the Einstein-dilaton dynamical equations. (For $n_j = 0$, the inequality on the left of (5.10) is $-\lambda_j A > 0$ while for $n_j = 1$ it is $2p_1 - \lambda_j A > 0$. For $n_j = d-1$, the inequality on the right is $\lambda_j A > 0$, while for $n_j = d-2$ it is $2p_1 + \lambda_j A > 0$.)

The constant σ is reduced from its value in Section 4, if necessary, so that, for all $x \in U_0$, $\sigma < 2p_1 + \cdots + 2p_{n_j} - \lambda_j A$ and $\sigma < 2p_1 + \cdots + 2p_{d-n_j-1} + \lambda_j A$. If σ is reduced, it may be necessary to reduce ϵ, and in turn shrink U_0, so that the conditions imposed in Section 4 are still all satisfied.

5.3 Fuchsian property – estimates

The jth matter field contributes the following components to the unknown u in the Fuchsian Equation (1.5).

$$\mathcal{E}^{a_1\cdots a_{n_j}} = {}^0\mathcal{E}^{a_1\cdots a_{n_j}} + t^\beta \xi^{a_1\cdots a_{n_j}}, \quad (5.11)$$

$$F_{a_0\cdots a_{n_j}} = {}^0F_{a_0\cdots a_{n_j}} + t^\beta \varphi_{a_0\cdots a_{n_j}}. \quad (5.12)$$

Here, $\beta = \epsilon/100$ as above, $\xi^{a_1\cdots a_{n_j}}$ is a totally antisymmetric spatial tensor density, so contributes $\binom{d}{n_j}$ independent components to u, and $\varphi_{a_0\cdots a_{n_j}}$ is a totally antisymmetric spatial tensor, so contributes $\binom{d}{n_j+1}$ components to u. This is consistent with the form of the evolution equations. Note that $\mathcal{E}^{a_1\cdots a_{n_j}} = O(1)$ and $F_{a_0\cdots a_{n_j}} = O(1)$.

This matter field contributes additional rows and columns to the matrix $\mathcal{A}$ such that the only non-vanishing new entries are on the diagonal and strictly positive. Therefore, the presence of this matter field does not alter that $\mathcal{A}$ satisfies the appropriate positivity condition.

The terms in the source f which must be estimated on account of the jth matter field are the following. It contributes terms to the components of f corresponding to κ through its contribution to $M^a{}_b$.

$$\begin{aligned}
&t^{2-\alpha^a{}_b}\frac{1}{g} g_{bh_1} g_{c_2h_2}\cdots g_{c_{n_j}h_{n_j}} \mathcal{E}^{ac_2\cdots c_{n_j}} \mathcal{E}^{h_1\cdots h_{n_j}} e^{-\lambda_j\phi} \\
&\quad \preceq \sum t^{-2q_{\max\{a,b\}}+2q_a+2q_{\max\{b,h_1\}}+\cdots+2q_{\max\{c_{n_j},h_{n_j}\}}-\lambda_j A-\alpha_0-n_j\epsilon} \\
&\quad \preceq t^{2q_1+\cdots+2q_{n_j}-\lambda_j A-\alpha_0-n_j\epsilon} = O(t^{-\alpha_0-n_j\epsilon+\sigma}) = O(t^\delta) \quad (5.13) \\
&t^{2-\alpha^a{}_b} g^{ac} g^{h_1i_1}\cdots g^{h_{n_j}i_{n_j}} F_{ch_1\cdots h_{n_j}} F_{bi_1\cdots i_{n_j}} e^{\lambda_j\phi} \\
&\quad \preceq \sum t^{2-2q_{\max\{a,b\}}+2q_a-2q_{\min\{a,c\}}-2q_{\min\{h_1,i_1\}}-\cdots-2q_{\min\{h_{n_j},i_{n_j}\}}+\lambda_j A-\alpha_0-(n_j+1)\epsilon} \\
&\quad \preceq t^{2q_1+\cdots+2q_{d-n_j-1}+\lambda_j A-\alpha_0-(n_j+1)\epsilon} = O(t^{-\alpha_0-(n_j+1)\epsilon+\sigma}) = O(t^\delta) \quad (5.14)
\end{aligned}$$

Here it is used that both $\mathcal{E}^{a_1\cdots a_{n_j}}$ and $F_{a_0\cdots a_{n_j}}$ are totally antisymmetric, so that the sums indicated by a summation symbol are not over all indices. Note that the Inequalities (5.10) have been crucially used in getting the Estimates (5.13) and (5.14). The desired estimates for the other two terms are obtained similarly.

The terms contributed to the component of f corresponding to χ by the jth matter field are obtained by multiplying the right-hand side of Equation (5.9) by $t^{2-\beta}$.

$$t^{2-\beta}\frac{1}{g}g_{a_1b_1}\cdots g_{a_{n_j}b_{n_j}}\mathcal{E}^{a_1\cdots a_{n_j}}\mathcal{E}^{b_1\cdots b_{n_j}}e^{-\lambda_j\phi}=O(t^{-\beta-n_j\epsilon+\sigma})=O(t^{\delta}) \quad (5.15)$$

$$t^{2-\beta}g^{a_0b_0}\cdots g^{a_{n_j}b_{n_j}}F_{a_0\cdots a_{n_j}}F_{b_0\cdots b_{n_j}}e^{\lambda_j\phi}=O(t^{-\beta-(n_j+1)\epsilon+\sigma})=O(t^{\delta}). \quad (5.16)$$

The terms in f corresponding to $\xi^{a_1\cdots a_{n_j}}$ for the jth matter field are obtained by multiplying the right-hand side of Equation (5.7) by $t^{1-\beta}$. These terms are $O(t^{-\beta-\delta-(n_j+1)\epsilon+\sigma})=O(t^{\delta})$. The terms in f corresponding to $\varphi_{a_0\cdots a_{n_j}}$ for the jth matter field are obtained by multiplying the right-hand side of Equation (5.8) by $t^{1-\beta}$. These terms are $O(t^{-\beta-\delta-n_j\epsilon+\sigma})=O(t^{\delta})$. Thus the terms which occur in f due to the jth matter field are $O(t^{\delta})$.

The time derivative of the matter constraint quantities for the jth field (the left-hand side of Equations (5.5) and (5.6)) vanishes. If the velocity-dominated matter constraints are satisfied, the matter constraint quantities are $o(1)$. A quantity which is both constant in time and $o(1)$ must vanish. Therefore the matter constraints for the jth field are satisfied.

Next the matter terms due to the jth field in the Einstein constraints are estimated, in order to verify that they are consistent with Equations (2.63) and (2.64). The contribution to the Hamiltonian constraint is, from Equation (5.1),

$$\frac{1}{g}g_{a_1b_1}\cdots g_{a_{n_j}b_{n_j}}\mathcal{E}^{a_1\cdots a_{n_j}}\mathcal{E}^{b_1\cdots b_{n_j}}e^{-\lambda_j\phi}=O(t^{-2-n_j\epsilon+\sigma})=o(t^{-2+\eta_1}), \quad (5.17)$$

$$g^{a_0b_0}\cdots g^{a_{n_j}b_{n_j}}F_{a_0\cdots a_{n_j}}F_{b_0\cdots b_{n_j}}e^{\lambda_j\phi}=O(t^{-2-(n_j+1)\epsilon+\sigma})=o(t^{-2+\eta_1}). \quad (5.18)$$

The contribution to the momentum constraint is

$$\begin{aligned} j_a-{}^0j_a &= \cdots+\left(\frac{1}{\sqrt{g}}-\frac{1}{\sqrt{{}^0g}}\right){}^0\mathcal{E}^{b_1\cdots b_{n_j}}\,{}^0F_{ab_1\cdots b_{n_j}} \\ &\quad +\frac{1}{\sqrt{g}}\left(\mathcal{E}^{b_1\cdots b_{n_j}}F_{ab_1\cdots b_{n_j}}-{}^0\mathcal{E}^{b_1\cdots b_{n_j}}\,{}^0F_{ab_1\cdots b_{n_j}}\right)=o(t^{-1+\eta_2}). \end{aligned} \quad (5.19)$$

Estimates related to the determinant which are relevant to (5.19) are analogues of the estimates for $d=3$ immediately preceding Equation (2.55). The form of these estimates for general d will now be presented. These are $1/\sqrt{g}-1/\sqrt{{}^0g}=O(t^{-1+\alpha_0-d\epsilon})$, $e_a(\tilde{g})=O(t^{\alpha_0-\delta-d\epsilon})$,

$$e_a(g)=O(t^{2+\alpha_0-\delta-d\epsilon}), \quad (5.20)$$

and

$$e_a(g^{-1/2})=-\frac{e_a(g)}{2g^{3/2}}=O(t^{-1+\alpha_0-\delta-d\epsilon}).$$

6 Determination of subcritical domain

The explicit determination of the subcritical range of the dilaton couplings for which the inequalities on the Kasner exponents are consistent so that V exists may be a complicated matter. We consider in this section a few cases and give some general rules. As in Subsection 2.4, we introduce the metric

$$dS^2 = G_{\mu\nu} dp^\mu dp^\nu = \sum {dp_a}^2 - (\sum dp_a)^2 + (dA)^2 \qquad (6.1)$$

in the D-dimensional space of the Kasner exponents $(p_a, A) \equiv (p^\mu)$. This metric has again Minkowskian signature $(-, +, +, \cdots, +)$. The forward light cone is defined by

$$G_{\mu\nu} p^\mu p^\nu = 0, \quad \sum p_a > 0. \qquad (6.2)$$

The Kasner conditions met in the previous section are equivalent to the conditions that the Kasner exponents be on the forward light cone (since $\sum p_a = 1$ can always be achieved by positive rescalings).

The wall chamber $\mathcal{W}$ is now defined by

$$p_1 \leq p_2 \leq \cdots \leq p_d \qquad (6.3)$$

$$2p_1 + p_2 + \cdots + p_{d-2} \geq 0 \qquad (6.4)$$

and, for each p-form,

$$p_1 + p_2 + \cdots + p_{n_j} - \frac{\lambda_j}{2} A \geq 0 \qquad (6.5)$$

$$p_1 + p_2 + \cdots + p_{d-n_j-1} + \frac{\lambda_j}{2} A \geq 0. \qquad (6.6)$$

These inequalities may not be all independent. The question is to determine the "allowed" values of the dilaton couplings for which the wall chamber contains in its interior future-directed lightlike vectors. It is clear that this set is non-empty since the inequalities can be all fulfilled when the couplings are zero (the p_a's can be chosen to be positive in the presence of a dilaton).

6.1 Einstein-dilaton-Maxwell system in D dimensions

We consider first the case of a single 1-form in $D \geq 4$ dimensions. This case is simple because the Inequalities (6.4) are then consequences of (6.5) and (6.6), which read

$$p_1 - \frac{\lambda}{2} A \geq 0, \qquad p_1 + p_2 + \cdots + p_{d-2} + \frac{\lambda}{2} A \geq 0. \qquad (6.7)$$

Furthermore, the number of faces of the wall chamber (defined by these inequalities and (6.3)) is exactly D and the edge vectors form a basis. Thus, the analysis of Subsection 2.4 can be repeated.

A basis of edge vectors can be taken to be

$$(0,0,\cdots,0,1,0) \tag{6.8}$$

$$\left(-\frac{d-k-2}{k+1},\cdots,-\frac{d-k-2}{k+1},1,\cdots,1,-\frac{2(d-k-2)}{\lambda(k+1)}\right),\ k=1,2,\cdots,d-2 \tag{6.9}$$

$$(1,1,\cdots,1,\frac{2}{\lambda}) \tag{6.10}$$

$$(1,1,\cdots,1,-\frac{2(d-2)}{\lambda}) \tag{6.11}$$

In (6.9), the first k components are equal to $-\frac{d-k-2}{k+1}$ and the next $d-k$ components are equal to 1.

The first vector is lightlike. The kth vector in the group (6.9) has squared norm

$$-\frac{(d-1)[k^2-k(d-3)+d]}{(k+1)^2}+\frac{4(d-k-2)^2}{\lambda^2(k+1)^2}, \qquad k=1,2,\cdots,d-2 \tag{6.12}$$

while (6.10) and (6.11) have norm squared equal to

$$-d(d-1)+\frac{4}{\lambda^2} \tag{6.13}$$

and

$$-d(d-1)+\frac{4(d-2)^2}{\lambda^2}, \tag{6.14}$$

respectively. The subcritical values of λ must (by definition) be such that at least one of the Expressions (6.12), (6.13) or (6.14) is positive. To determine the boundaries $\pm\lambda_c$ of the subcritical interval, we first note that (6.13) is positive whenever $|\lambda| < \Lambda_1$, with $\Lambda_1 = 2/\sqrt{d(d-1)}$. Similarly, (6.14) is positive whenever $|\lambda| < \Lambda_2$ with $\Lambda_2 = 2(d-2)/\sqrt{d(d-1)}$. To analyze the sign of (6.12), we must consider two cases, according to whether $k^2-k(d-3)+d$ is positive or negative.

If $d < 9$, the factor $k^2-k(d-3)+d$ is always positive (for any choice of k, $k=1,2,\cdots,d-2$) and the Expression (6.12) is positive provided $|\lambda| < \Pi_k$, with

$$\Pi_k = \frac{2(d-k-2)}{\sqrt{(d-1)[k^2-k(d-3)+d]}}. \tag{6.15}$$

The critical value λ_c is equal to the largest number among Λ_1, Λ_2 and Π_k. This largest number is Λ_2 for $d=3,4,5,6$, Π_1 for $d=7$ and Π_2 for $d=8$. We thus have the following list of critical couplings:

$$\lambda_c = \sqrt{\frac{2}{3}}, \qquad d=3$$

$$\lambda_c = \frac{2}{\sqrt{3}}, \qquad d=4$$

$$\begin{aligned}
\lambda_c &= \frac{3}{\sqrt{5}}, \qquad d = 5 \\
\lambda_c &= \frac{4\sqrt{2}}{\sqrt{15}}, \qquad d = 6 \\
\lambda_c &= \frac{2\sqrt{2}}{\sqrt{3}}, \qquad d = 7 \\
\lambda_c &= \frac{4\sqrt{2}}{\sqrt{7}}, \qquad d = 8.
\end{aligned} \tag{6.16}$$

Note that the value of the dilaton coupling that comes from dimensional reduction of vacuum gravity in one dimension higher

$$\lambda_{KK} = 2\sqrt{\frac{d}{d-1}} \tag{6.17}$$

is always strictly greater than the critical value, except for $d = 8$, where $\lambda_{KK} = \lambda_c$. (The corresponding values of the Kasner exponents are those of the point on the Kasner sphere exhibited in [13] for $D = 10$, where all gravitational inequalities are marginally fulfilled.)

If $d \geq 9$, the factor $k^2 - k(d-3) + d$ is non-positive for

$$\frac{d-3-\sqrt{(d-9)(d-1)}}{2} \leq k \leq \frac{d-3+\sqrt{(d-9)(d-1)}}{2} \tag{6.18}$$

(this always occurs for $k = 3$). Thus, the Expression (6.12) is positive for such k's no matter what λ is. This implies that the critical value of λ is infinite,

$$\lambda_c = \infty, \qquad d \geq 9. \tag{6.19}$$

The fact that $D = 10$ appears as a critical dimension for the Einstein-dilaton-Maxwell system, above which the system is velocity-dominated no matter what the value of the dilaton coupling is in the line of the findings of [13], since the edges (6.9) differ from those of the pure gravity wall chambers only by an additional component along the spacelike dilaton direction.

6.2 Einstein-dilaton system with one p-form ($p \neq 0$, $p \neq D-2$)

The same geometrical procedure for determining the critical values of the dilaton couplings can be followed when there is only one p-form in the system ($p \neq 0$, $p \neq D-2$), because in that case the wall chamber has exactly D faces and the edge vectors form a basis. Indeed, the gravitational Inequalities (6.4) are always consequences of the symmetry Inequalities (6.3) and the form Inequalities (6.5) and (6.6) (for $n_j \neq 0$ and $n_j \neq D-2$),

$$2p_1+p_2+\cdots+p_{d-2} = (p_1+\cdots+p_{n_j}-\frac{\lambda_j}{2}A)+(p_1+p_{n_j+1}+\cdots+p_{d-2}+\frac{\lambda_j}{2}A). \tag{6.20}$$

So, if there is only one p-form (with $p \neq 0$ and $p \neq D-2$), the $D-2$ symmetry Inequalities (6.3) together with the two form Inequalities (6.5) and (6.6) completely define the wall chamber, which has D faces. We shall not provide an explicit example of a calculation of λ_c for such a system, since it proceeds as for a 1-form.

When there is more than one exterior form, one can still drop the gravitational inequalities (if there is at least one p-form with $p \neq 0$ and $p \neq D-2$), but the situation is more involved because the inequalities corresponding to different forms are usually independent, so that the wall chamber has more than D faces (its intersection with the hyperplane $\sum p_a = 1$ is not a simplex). The calculation is then more laborious. The same feature arises for a 0-form, which we now examine.

6.3 0-form in 4 dimensions

We consider the case of a 0-form in 4 spacetime dimensions. As explained above, we impose the condition $\lambda \neq 0$ to the corresponding dilaton coupling[7]. Without loss of generality (in view of the $\phi \to -\phi$ symmetry), we can assume $\lambda > 0$. The inequalities defining the subcritical domain relevant to the 0-form case can be brought to the form

$$p_1 > 0 \tag{6.21}$$
$$A > 0 \tag{6.22}$$
$$p_1 + p_2 - \frac{\lambda}{2} A > 0 \tag{6.23}$$
$$p_2 - p_1 > 0 \tag{6.24}$$
$$p_3 - p_2 > 0 \tag{6.25}$$

We denote by α, β, γ, δ and ϵ the corresponding border hyperplanes (i.e., $\alpha : p_1 = 0$, $\beta : A = 0$ etc). The Inequalities (6.21)–(6.25) guarantee that all potential walls are negligible asymptotically. They are independent. The five faces α, β, γ, δ and ϵ intersect along the 7 one-dimensional edges generated by the vectors:

$$e_1 = (0,0,1,0) \in \alpha \cap \beta \cap \gamma = \alpha \cap \beta \cap \delta = \alpha \cap \gamma \cap \delta = \beta \cap \gamma \cap \delta \tag{6.26}$$
$$e_2 = (0,1,1,0) \in \alpha \cap \beta \cap \epsilon \tag{6.27}$$
$$e_3 = (0,1,1,\frac{2}{\lambda}) \in \alpha \cap \gamma \cap \epsilon \tag{6.28}$$
$$e_4 = (0,0,0,1) \in \alpha \cap \delta \cap \epsilon \tag{6.29}$$
$$e_5 = (-1,1,1,0) \in \beta \cap \gamma \cap \epsilon \tag{6.30}$$
$$e_6 = (1,1,1,0) \in \beta \cap \delta \cap \epsilon \tag{6.31}$$
$$e_7 = (1,1,1,\frac{4}{\lambda}) \in \gamma \cap \delta \cap \epsilon \tag{6.32}$$

[7] The case $\lambda = 0$ is clearly in the subcritical region but must be treated separately because there are then two dilatons. The Kasner conditions read $p_1+p_2+\cdots+p_d = 1$ and $p_1^2+\cdots p_d^2+A_1^2+A_2^2 = 1$, where the scalar fields behave as $\phi_1 \sim A_1 \ln t$, $\phi_2 \sim A_2 \ln t$. This allows positive p_i's, which enables one to drop spatial derivatives as $t \to 0$. The system is velocity-dominated.

Among these vectors, neither e_4 nor e_5 bound the subcritical domain since e_4 is such that $p_1 + p_2 - (\lambda/2)A < 0$ (changing its sign would make $A < 0$), while e_5 is such that $p_1 < 0$ (changing its sign would make $p_2 - p_1 < 0$).

The edge-vectors $\{e_1, e_2, e_3, e_6, e_7\}$ form a complete (but not linearly independent) set. Any vector can be expanded as

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 + v_6 e_6 + v_7 e_7 \tag{6.33}$$

The coefficients v_1, v_2, v_3, v_6, v_7 are not independent but can be changed as

$$v_2 \to v_2 + 2k,\ v_3 \to v_3 - 2k,\ v_6 \to v_6 - k,\ v_7 \to v_7 + k \tag{6.34}$$

For (6.33) to be interior to the wall chamber, the coefficients v_1, v_2, v_3, v_6 and v_7 must fulfill

$$v_1 > 0,\ \ v_2 + v_3 > 0,\ \ v_2 + 2v_6 > 0,\ \ v_3 + 2v_7 > 0,\ \ v_6 + v_7 > 0. \tag{6.35}$$

Using the above redefinitions, which leave the inequalities invariant, we can make $v_A \geq 0$, $A = 1, 2, 3, 6, 7$, with at most two v_A's equal to zero. Indeed, let $s = \min(v_2, v_3, 2v_6, 2v_7)$. Assume for definiteness that $s = v_2$ (the other cases are treated in exactly the same way). One has then $v_2 \leq 2v_7$. Take $2k = -s$ in the redefinitions (6.34). This makes v_2 equal to zero and makes v_7 equal to $v_7 - (v_2/2) \geq 0$. Because of (6.35), the new v_3 and v_6 are strictly positive, as claimed. Thus, one sees that any vector in the wall chamber can be expanded as in (6.33) with non-negative coefficients. But the vectors e_1, e_2, e_3, e_6 and e_7 are all future-pointing and timelike or null when $\lambda \geq \sqrt{8/3}$. It follows that for such λ's, there is no lightlike direction in the interior of the wall chamber. Conversely, if $\lambda < \sqrt{8/3}$, the vector e_7 is spacelike and one can find an interior vector $\alpha e_1 + \beta e_2 + e_7$ ($\alpha, \beta > 0$) that is lightlike. We can thus conclude:

$$\lambda_c = \sqrt{\frac{8}{3}} \quad \text{for a 0-form in 4 dimensions,} \tag{6.36}$$

i.e., the system is velocity-dominated for $|\lambda| < \sqrt{8/3}$.

The action for the matter fields in the case of a 0-form A coupled to a dilaton ϕ is

$$S_\phi[g_{\alpha\beta}, \phi, A] = -\frac{1}{2}\int (\partial_\mu \phi\, \partial^\mu \phi + e^{\lambda\phi} \partial_\mu A\, \partial^\mu A)\sqrt{-g}\, d^4x \tag{6.37}$$

Note that this is the action for a wave map (also known as a nonlinear σ-model or hyperbolic harmonic map) with values in a two-dimensional Riemannian manifold of constant negative curvature. Its curvature is proportional to λ^2. Thus we obtain an interesting statement on velocity-dominated behaviour for the Einstein equations coupled to certain wave maps. Note for comparison that wave maps in flat space occurring naturally in the context of solutions of the vacuum Einstein equations with symmetry, for instance in Gowdy spacetimes (*cf.* [34]), are defined by a Lagrangian of the above type (using the flat metric) with $\lambda = 2$.

6.4 Collection of 1-forms

We now turn to a system of several 1-forms. It is clear that if these have all the same dilaton coupling, as in the Yang-Mills Action (1.7), then, the critical value of λ is just that computed in (6.16) and (6.19) since each form brings in the same walls. The situation is more complicated if the dilaton couplings are different. One could naively think that the subcritical domain is then just the Cartesian product of the individual subcritical intervals $[-\lambda_c^{(j)}, \lambda_c^{(j)}]$, but this is not true because the intersection of the wall chambers associated with each 1-form may have no interior lightlike direction, even if each wall chamber has some.

This is best seen on the example of two 1-forms in D spacetime dimensions with opposite dilaton couplings. The relevant inequalities, from which all others follow, are in this case

$$p_1 - \frac{\lambda}{2}A > 0, \quad p_1 + \frac{\lambda}{2}A > 0 \tag{6.38}$$

$$p_1 < p_2 < \cdots < p_d \tag{6.39}$$

and can be easily analyzed because they determine, in this particular instance, a simplex in the hyperplane $\sum p_a = 1$. It follows from (6.38) that $p_1 > 0$. The edge-vectors can be taken to be $(0, \ldots, 0, 1, \ldots, 1, 0)$ (k zeros, $d-k$ ones, $k = 1, \ldots, d-1$) and $(1, 1, \ldots, 1, \pm 2/\lambda)$. The first $d-1$ edge-vectors are timelike or null, while the last two are spacelike provided $-d(d-1)\lambda^2 + 4 > 0$. This yields

$$\lambda_c = \frac{2}{\sqrt{d(d-1)}} \quad \text{for two 1-forms with opposite dilaton couplings} \tag{6.40}$$

Accordingly, λ_c is finite for any spacetime dimension (and in fact, tends to zero as $d \to \infty$), even though $\lambda_c = \infty$ for a single 1-form whenever $d > 8$.

7 Coupling between the matter fields

The actions for the bosonic sectors of the low-energy limits of superstring theories or M-theory contain coupling terms between the p-forms, indicated by "more" in (1.1). These coupling terms are of the Chern-Simons or the Chapline-Manton type. In this section, we show that these terms are consistent with the results obtained in Section 5, in that they are also asymptotically negligible in the dynamical equations of motion when the Kasner exponents are subject to the above Inequalities (6.3)–(6.6).

More precisely, the form of the velocity-dominated evolution equations and solutions are in each case exactly as in Section 5. The velocity-dominated matter constraints have additional terms, but as before, the velocity-dominated matter variables (besides the dilaton) are constant in time, so if the constraints are satisfied at some $t > 0$ they are satisfied for all $t > 0$. The quantities ${}^0\rho$ and ${}^0j_a$ are defined exactly as in Section 5. Since the velocity-dominated evolution equations are also the same, there is nothing additional to check concerning the velocity-dominated Hamiltonian and momentum constraints.

Turning now to the exact equations, the restrictions defining the set V are unchanged from Section 5. The form of the evolution equation for the dilaton is unchanged. The form of the stress-energy tensor is also unchanged, and so the form of the Einstein evolution equations and the Einstein constraints is unchanged. The additional matter field variables considered in Section 5 are still all $O(1)$, so estimates of terms involving the matter fields do not need to be reconsidered, as long as their form has not changed, for instance, in the argument that the Einstein constraints are satisfied. That the matter constraints are satisfied follows as in the other cases, once it is verified that their time derivative vanishes and that they are $o(1)$. Since so much of the argument is identical to that of Section 5, we only point out the few places where there are differences.

7.1 Chern-Simons terms

First we consider the coupling of i of the additional matter fields via a Chern-Simons term in the action. These additional matter fields should be such that

$$i - 1 + \sum_{j=1}^{i} n_j = D. \tag{7.1}$$

The Chern-Simons term which is added to the action is

$$S_{\mathrm{CS}}[A^{(1)}_{\gamma_1 \cdots \gamma_{n_1}}, \cdots, A^{(i)}_{\gamma_1 \cdots \gamma_{n_i}}] = \int A^{(1)} \wedge dA^{(2)} \wedge \cdots \wedge dA^{(i)}. \tag{7.2}$$

The variation of this term with respect to both the metric and the dilaton field, ϕ, vanishes. The matter Equation (5.4) is unchanged, since it is still the case that $F^{(j)} = dA^{(j)}$ for all j. But Equation (5.3) for each of the i coupled matter fields acquires a non-vanishing right-hand side.

$$\begin{aligned} &{}^{(D)}\nabla_\mu (F^{(j)\mu\nu_1\cdots\nu_{n_j}} e^{\lambda_j \phi})\sqrt{-g} \\ &\qquad = C_j\, \epsilon^{\cdots\nu_1\cdots\nu_{n_j}\cdots} F^{(1)}_{\cdots} \cdots F^{(j-1)}_{\cdots} F^{(j+1)}_{\cdots} \cdots F^{(i)}_{\cdots} \end{aligned} \tag{7.3}$$

Here $\epsilon^{0\cdots d} = 1$ and C_j is a numerical factor. Next, considering the $d+1$ decomposition of Equation (7.3), the constraint Equation (5.5), for the jth coupled matter field, acquires the following term on its right-hand side,

$$-C_j \epsilon^{\cdots 0 b_1 \cdots} F^{(1)}_{\cdots} \cdots F^{(j-1)}_{\cdots} F^{(j+1)}_{\cdots} \cdots F^{(i)}_{\cdots} \tag{7.4}$$

Here all indices which are not explicit are spatial. So, only magnetic fields appear in (7.4). The following term is added to the right-hand side of the evolution

Equation (5.7) for the jth coupled matter field.

$$
\begin{aligned}
&-C_j\Big\{\sum_{m=1}^{j-1}(n_m+1)\epsilon^{\cdots 0c_1\cdots c_{n_m}\cdots a_1\cdots a_{n_j}\cdots}F^{(1)}_{\cdots}\cdots\\
&\times\Big(\frac{1}{\sqrt{g}}g_{c_1h_1}\cdots g_{c_{n_m}h_{n_m}}\mathcal{E}^{(m)h_1\cdots h_{n_m}}e^{-\lambda_m\phi}\Big)\cdots F^{(j-1)}_{\cdots}F^{(j+1)}_{\cdots}\cdots F^{(i)}_{\cdots}\Big\}\\
&-C_j\Big\{\sum_{m=j+1}^{i}(n_m+1)\epsilon^{\cdots a_1\cdots a_{n_j}\cdots 0c_1\cdots c_m\cdots}F^{(1)}_{\cdots}\cdots F^{(j-1)}_{\cdots}F^{(j+1)}_{\cdots}\cdots \qquad (7.5)\\
&\times\Big(\frac{1}{\sqrt{g}}g_{c_1h_1}\cdots g_{c_{n_m}h_{n_m}}\mathcal{E}^{(m)h_1\cdots h_{n_m}}e^{-\lambda_m\phi}\Big)\cdots F^{(i)}_{\cdots}\Big\}.
\end{aligned}
$$

Again, all indices which are not explicit are spatial. There is in each term only one electric field.

The velocity-dominated matter constraint equations for the jth coupled matter field can be obtained from the "full" matter constraint equations for the same field by substituting the velocity-dominated quantities for all variables.

The only additional terms occurring in f are due to Equation (7.5). The form of the mth term on the right-hand side of Equation (7.5) is just like the form of the terms on the right-hand side of Equation (5.8) for the mth coupled field. The factors which differ, comparing the mth term of (7.5) to Equation (5.8) for the mth field, are 0(1). Since in both cases a factor of $t^{1-\beta}$ is added in order to obtain the terms appearing in f, the estimate that the additional terms in f due to the Chern-Simons coupling are $O(t^\delta)$ is obtained just as the corresponding previously obtained estimates.

7.2 Chapline-Manton couplings

Next we consider Chapline-Manton couplings. For definiteness, we treat two explicit examples, leaving to the reader the task of checking that the general case works in exactly the same way. The first coupling is between an n-form A and an $(n+1)$-form B and is equivalent to making B massive. Let $F = dA + B$ and $H = dB$. The gauge transformations are $B \to B + d\eta$, for arbitrary n-form η, and $A \to A - \eta + d\gamma$, for arbitrary $(n-1)$-form γ. (If $n = 0$, then $d\gamma$ is replaced by a constant scalar and we require that the corresponding constant, λ_A, in the coupling to the dilaton be nonzero.) The form of the action is the same as in Section 5, but since F now depends on B and not just on A, the variation of the action with respect to B acquires an additional term. Also, it is now the case that $dF = H$.

The matter Equation (5.3) is unchanged for F and Equation (5.4) is unchanged for H. Equation (5.3) for H and Equation (5.4) for F are now as follows.

$$
{}^{(D)}\nabla_\mu(H^{\mu\nu_0\cdots\nu_n}e^{\lambda_B\phi}) = F^{\nu_0\cdots\nu_n}e^{\lambda_A\phi}, \qquad (7.6)
$$

$$
{}^{(D)}\nabla_{[\mu}F_{\nu_0\cdots\nu_n]} = \frac{1}{(n+2)}H_{\mu\nu_0\cdots\nu_n}. \qquad (7.7)
$$

Define $\mathcal{E}^{a_1\cdots a_n} = \sqrt{g}\, F^{0a_1\cdots a_n} e^{\lambda_A\phi}$ and $\mathcal{D}^{a_0\cdots a_n} = \sqrt{g}\, H^{0a_0\cdots a_n} e^{\lambda_B\phi}$. The matter constraint equations which are affected are

$$e_a(\mathcal{D}^{ab_1\cdots b_n}) + f^c_{ca}\,\mathcal{D}^{ab_1\cdots b_n} + \frac{1}{2}\sum_{i=1}^{n} f^{b_i}_{ac}\mathcal{D}^{ab_1\cdots c\cdots b_n} = -\mathcal{E}^{b_1\cdots b_n}, \tag{7.8}$$

$$e_{[a}(F_{b_0\cdots b_n]}) - \frac{(n+1)}{2} f^c_{[ab_0}F_{|c|b_1\cdots b_n]} = \frac{1}{n+2}H_{ab_0\cdots b_n}, \tag{7.9}$$

The additional term which appears on the right-hand side of Equation (5.7) for $\mathcal{D}^{a_0\cdots a_n}$ is

$$\sqrt{g}g^{a_0b_0}\cdots g^{a_nb_n}F_{b_0\cdots b_n}e^{\lambda_A\phi}. \tag{7.10}$$

The additional term which appears on the right-hand side of Equation (5.8) for $F_{a_0\cdots a_n}$ is

$$\frac{-1}{\sqrt{g}}g_{a_0b_0}\cdots g_{a_nb_n}\mathcal{D}^{b_0\cdots b_n}e^{-\lambda_B\phi}. \tag{7.11}$$

The velocity-dominated matter constraint equations which are affected can be obtained from Equation (7.8) and (7.9) by substituting the corresponding velocity-dominated quantities for all variables. The only additional terms occurring in f are due to Equations (7.10) for $\mathcal{D}$ and (7.11) for F. The form of the additional terms in these equations is just like the form of the terms which appear in Equations (5.7) for $\mathcal{E}$ and in (5.8) for H. Therefore the estimate that the additional terms are $O(t^\delta)$ is obtained just as the corresponding previously obtained estimates.

The second Chapline-Manton type coupling is between an n-form A and a $(2n)$-form B. Let $F = dA$ and $H = dB + A\wedge F$. The gauge transformations are $A \to A + d\gamma$, for arbitrary $(n-1)$-form γ, and $B \to B + d\eta - \gamma\wedge F$, for arbitrary $(2n-1)$-form η. (If $n=0$ the gauge transformations are $A\to A+C$ and $B\to B+D-CA$ for constant scalars C and D and we require both $\lambda_A \neq 0$ and also $\lambda_B \neq 0$.) The form of the action is again the same as in Section 5. Define $\mathcal{E}^{a_1\cdots a_n} = \sqrt{g}\, F^{0a_1\cdots a_n}e^{\lambda_A\phi}$ and $\mathcal{D}^{a_1\cdots a_{2n}} = \sqrt{g}\, H^{0a_1\cdots a_{2n}}e^{\lambda_B\phi}$. The matter Equations (5.3) for F and (5.4) for H are affected, only if n is odd. The equation for F which is affected (if n is odd) and its $d+1$ decomposition are

$$^{(D)}\nabla_\mu(F^{\mu\nu_1\cdots\nu_n}e^{\lambda_A\phi}) = \frac{2}{(n+1)!}H^{\mu\nu_1\cdots\nu_n\sigma_1\cdots\sigma_n}F_{\mu\sigma_1\cdots\sigma_n}e^{\lambda_B\phi}, \tag{7.12}$$

$$e_a(\mathcal{E}^{ab_2\cdots b_n}) + f^c_{ca}\,\mathcal{E}^{ab_2\cdots b_n} + \frac{1}{2}\sum_{i=2}^{n} f^{b_i}_{ac}\mathcal{E}^{ab_2\cdots c\cdots b_n} = \frac{2}{(n+1)!}\mathcal{D}^{ab_2\cdots b_nh_1\cdots h_n}F_{ah_1\cdots h_n}, \tag{7.13}$$

$$\begin{aligned}\partial_t\mathcal{E}^{a_1\cdots a_n} &= \cdots - \frac{2}{\sqrt{g}\,n!}\mathcal{D}^{a_1\cdots a_nb_1\cdots b_n}g_{b_1c_1}\cdots g_{b_nc_n}\mathcal{E}^{c_1\cdots c_n} \\ &\quad + \frac{2}{(n+1)!}\sqrt{g}g^{bh_0}g^{a_1h_1}\cdots g^{a_nh_n}g^{c_1h_{n+1}}\cdots g^{c_nh_{2n}}H_{h_0\cdots h_{2n}}F_{bc_1\cdots c_n}e^{\lambda_B\phi}.\end{aligned} \tag{7.14}$$

The equation for H which is affected (if n is odd) and its $d+1$ decomposition are

$$^{(D)}\nabla_{[\mu_0} H_{\mu_1\cdots\mu_{2n+1}]} = \frac{(2n+1)!}{(n+1)!(n+1)!} F_{[\mu_0\cdots\mu_n} F_{\mu_{n+1}\cdots\mu_{2n+1}]}, \tag{7.15}$$

$$e_{[a}(H_{b_0\cdots b_{2n}]}) - \frac{(2n+1)}{2} f^c_{[ab_0} H_{|c|b_1\cdots b_{2n}]} = \frac{(2n+1)!}{(n+1)!(n+1)!} F_{[ab_0\cdots b_{n-1}} F_{b_n\cdots b_{2n}]}, \tag{7.16}$$

$$\partial_t H_{a_0\cdots a_{2n}} = \cdots + \frac{(2n+2)!}{\sqrt{g}\,(n+1)!(n+1)!} g_{[a_0|b_1|}\cdots g_{a_{n-1}|b_n|}\mathcal{E}^{b_1\cdots b_n} e^{-\lambda_A\phi} F_{a_n\cdots a_{2n}]}. \tag{7.17}$$

The velocity-dominated matter constraint equations which are affected can be obtained from Equation (7.13) and (7.16) by substituting the corresponding velocity-dominated quantities for all variables. The only additional terms occurring in f are due to Equations (7.14) for $\mathcal{E}$ and (7.17) for H. Here again, the estimate that the additional terms in f are $O(t^\delta)$, is just as the estimate of terms appearing already in Section 5, either in Equation (5.7) for $\mathcal{D}$ or in Equation (5.8) for F.

8 Yang-Mills

We complete our analysis by proving that Yang-Mills couplings also enjoy the property of not modifying the conclusions. The action is (1.7), with a Yang-Mills field as source in addition to the scalar field considered in Section 4 and with $|\lambda| < \lambda_c$. The argument is again based on that of Sections 2–5 and it is enough here to note differences. The main one is that one must work with the vector potential instead of the fields themselves, because bare A's appear in the equations. We could, in fact, have developed the entire previous analysis in terms of the vector potentials, thereby reducing the number of matter constraint equations. We followed a manifestly gauge-invariant approach for easing the physical understanding, but this was not mandatory. The stress-energy tensor is

$$T_{\mu\nu} = {}^{(D)}\nabla_\mu\phi\,{}^{(D)}\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\,{}^{(D)}\nabla_\alpha\phi\,{}^{(D)}\nabla^\alpha\phi + [F_{\mu\alpha}\cdot F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}\cdot F^{\alpha\beta}]e^{\lambda\phi}. \tag{8.1}$$

We work in the temporal gauge, $A_0 = 0$. The matter fields satisfy the following equations.

$$^{(D)}\nabla_\alpha\,{}^{(D)}\nabla^\alpha\phi - \frac{\lambda}{4}F_{\alpha\beta}\cdot F^{\alpha\beta}e^{\lambda\phi} = 0 \tag{8.2}$$

$$^{(D)}\nabla_\mu(F^{\mu\nu}e^{\lambda\phi}) + [A_\mu, F^{\mu\nu}]e^{\lambda\phi} = 0, \tag{8.3}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{8.4}$$

The Lie Bracket has no intrinsic time dependence. The $d+1$ decomposition of the stress-energy tensor is expressed in terms of the spatial tensor density

$\mathcal{E}^a = \sqrt{g}\, F^{0a} e^{\lambda\phi}$ and the antisymmetric spatial tensor F_{ab}.

$$\rho = \frac{1}{2}\{(\partial_t\phi)^2 + g^{ab}e_a(\phi)e_b(\phi) + \frac{1}{g}g_{ab}\mathcal{E}^a\cdot\mathcal{E}^b e^{-\lambda\phi} + \frac{1}{2}g^{ab}g^{ch}F_{ac}\cdot F_{bh}e^{\lambda\phi}\}, \tag{8.5}$$

$$j_a = -\partial_t\phi\, e_a(\phi) + \frac{1}{\sqrt{g}}\mathcal{E}^b\cdot F_{ab}, \tag{8.6}$$

$$M^a{}_b = g^{ac}e_b(\phi)\,e_c(\phi) - \frac{1}{g}\{g_{bc}\mathcal{E}^a\cdot\mathcal{E}^c - \frac{1}{2}\delta^a{}_b g_{ch}\mathcal{E}^c\cdot\mathcal{E}^h\}e^{-\lambda\phi} + \{g^{ac}g^{hi}F_{ch}\cdot F_{bi} - \frac{1}{4}\delta^a{}_b g^{ch}g^{ij}F_{ci}\cdot F_{hj}\}e^{\lambda\phi}. \tag{8.7}$$

The matter constraint equation is

$$e_a(\mathcal{E}^a) + f^b_{ba}\,\mathcal{E}^a + [A_a, \mathcal{E}^a] = 0. \tag{8.8}$$

The matter evolution equations are

$$\partial_t^2\phi - (\mathrm{tr}k)\partial_t\phi = g^{ab}\nabla_a\nabla_b\phi + \frac{\lambda}{2g}g_{ab}\mathcal{E}^a\cdot\mathcal{E}^b e^{-\lambda\phi} - \frac{\lambda}{4}g^{ab}g^{ch}F_{ac}\cdot F_{bh}e^{\lambda\phi}, \tag{8.9}$$

$$\partial_t\mathcal{E}^a = e_b(\sqrt{g}g^{ac}g^{bh}F_{ch}e^{\lambda\phi}) + (f^i_{ib}g^{ac} + \frac{1}{2}f^a_{bi}g^{ic})\sqrt{g}g^{bh}F_{ch}e^{\lambda\phi} \tag{8.10}$$

$$\partial_t A_a = -\frac{1}{\sqrt{g}}g_{ab}\mathcal{E}^b e^{-\lambda\phi}. \tag{8.11}$$

Note that we use as basic matter variables A_a and $\mathcal{E}^b$ (the quantity F_{ab} being then defined in terms of A_a as $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$).

The Kasner-like evolution equations are Equations (2.7)–(2.10) and $\partial_t\, {}^0A_a = 0$. We consider analytic solutions of the Kasner-like evolution equations of the form (2.12)–(2.15) along with the quantity ${}^0A_a$ which is constant in time. Given a point $x_0 \in \Sigma$, we use an adapted spatial frame on a neighborhood of x_0, U_0, as in Section 3. Thus, ${}^0g_{ab}(t_0)$ and $K^a{}_b$ are specialized as in that section. There is one velocity-dominated matter constraint equation, obtained from Equations elconstraintym) by replacing $\mathcal{E}^a$ and A_a with ${}^0\mathcal{E}^a$ and ${}^0A_a$. If the velocity-dominated matter constraint is satisfied at some time $t_0 > 0$, then it is satisfied for all $t > 0$. Define

$${}^0\rho = \frac{1}{2}(\partial_t\, {}^0\phi)^2, \tag{8.12}$$

$${}^0j_a = -\partial_t\, {}^0\phi\, e_a({}^0\phi) + \frac{1}{\sqrt{{}^0g}}\, {}^0\mathcal{E}^b \cdot {}^0F_{ab}. \tag{8.13}$$

The velocity-dominated Einstein constraints are defined as in the other cases. Equations (2.21) and (2.22) are again satisfied, so if the velocity-dominated constraints are satisfied at some t_0, then they are satisfied for all $t > 0$. The restrictions defining the set V are as in Section 5, with $n_j = 1$. The relation of the unknown, u, in Equation (1.5) to the Einstein-matter variables is given by Equations (2.31)–(2.37) and

$$A_a = {}^0A_a + t^\beta \varphi_a. \tag{8.14}$$

The quantities $\mathcal{A}$ and f in Equation (1.5) are given by Equations (2.39)–(2.43) and

$$\begin{aligned} t\,\partial_t\chi + \beta\chi &= t^{\alpha_0-\beta}(\mathrm{tr}\,\kappa)(A + t^\beta\chi) + t^{2-\beta}\,{}^S g^{ab}\,{}^S\nabla_a\,{}^S\nabla_b\,{}^0\phi + t^{2-\zeta}\,{}^S\nabla^a\omega_a \\ &\quad + t^{2-\beta}\{\frac{\lambda}{2g}g_{ab}\mathcal{E}^a\cdot\mathcal{E}^b e^{-\lambda\phi} - \frac{\lambda}{4}g^{ab}g^{ch}F_{ac}\cdot F_{bh}e^{\lambda\phi}\}, \qquad (8.15) \\ t\,\partial_t\xi^a + \beta\xi^a &= t^{1-\beta}\{e_b(\sqrt{g}g^{ac}g^{bh}F_{ch}e^{\lambda\phi}) \\ &\quad + (f^i_{ib}g^{ac} + \frac{1}{2}f^a_{bi}g^{ic})\sqrt{g}g^{bh}F_{ch}e^{\lambda\phi}\}, \qquad (8.16) \\ t\,\partial_t\varphi_a + \beta\varphi_a &= -t^{1-\beta}\frac{1}{\sqrt{g}}g_{ab}\mathcal{E}^b e^{-\lambda\phi}. \qquad (8.17) \end{aligned}$$

The estimate that $f = O(t^\delta)$ is obtained as before, using $\mathcal{E}^a = O(1)$ and $F_{ab} = O(1)$. The matter constraint quantity, the left-hand side of Equation (8.8), is $o(1)$ and its time derivative vanishes, so the matter constraint is satisfied. The estimate of the matter terms in the Einstein constraints is obtained as in Section 5 for $n_j = 1$.

To conclude: the whole analysis goes through even in the presence of the Yang-Mills coupling terms and the system is asymptotically Kasner-like provided $|\lambda| < \lambda_c$, where λ_c is the same as in the abelian case and explicitly given by (6.16) and (6.19).

9 Self-interacting scalar field

Consider Einstein's equations, $D \geq 3$, with sources as in Sections 4, 5, 7 or 8, except that the massless scalar field, ϕ, is replaced by a self-interacting scalar field. That is, the Expression (1.8) is added to the action. Solutions with a monotone singularity can be constructed as in Sections 4–8, with assumptions regarding the function $V(\phi)$ which appears in (1.8) given below. There is no change in the velocity-dominated evolution equations and solutions, nor in the velocity-dominated constraints. The only change to Equation (1.5) is that two new terms appear in f. There is a new term, $t^{2-\alpha_0}\delta^a{}_b\, 2\, V(\phi)/(D-2)$, on the right-hand side of the evolution equation for $\kappa^a{}_b$ (through $M^a{}_b$). There is also a new term, $-t^{2-\beta}V'(\phi)$, on the right-hand side of the evolution equation for χ. For Equation (1.5) to be Fuchsian, it must be the case that $f = O(t^\delta)$ and, in addition, that f satisfy other regularity conditions [15, 28].

Some examples were considered in [31]. A trivial example is obtained by taking V to be a constant. Then the equation for the scalar field is not changed by the potential while its effect on the Einstein equations is equivalent to the addition of a cosmological constant. Thus we see that the analysis of [15] generalizes directly to the case of the Einstein-scalar field system with non-zero cosmological constant. Of course the analogous statement applies to the other dimensions and matter fields considered in previous sections. To get another simple example take $V(\phi) = \lambda\phi^p$ for a constant λ and an integer $p \geq 2$. Showing that the equation is Fuchsian involves examining the expression

$$V(A\ln t + B + t^\beta\psi) = \lambda(A\ln t + B + t^\beta\psi)^p \tag{9.1}$$

and corresponding expressions for the first and second derivatives of V. Of course in this particular case these are given by multiples of smaller powers of t. The aim is to estimate these quantities by suitable powers of t. In this case a Fuchsian system is always obtained. A linear massive scalar field is obtained by choosing $p = 2$. Another interesting possibility is to choose $V(\phi) = e^{\lambda\phi}$ for a constant λ, in which case the derivatives of V are also exponentials. Then

$$V(A\ln t + B + t^\beta\psi) = e^{\lambda B}t^{\lambda A}\exp(\lambda t^\beta\psi) \tag{9.2}$$

Note that such an exponential potential can be (formally) generated by adding, as matter field, a d-form $A_{\mu_1\cdots\mu_d}$ with dilaton coupling $\lambda_d = -\lambda$. Indeed, eliminating the field-strength $F = dA$ (which satisfies $e^{\lambda_d\phi}F = C\eta$, where C is a constant and η the volume form), leads to a term in the action proportional to $e^{-\lambda_d\phi}C^2$. A Fuchsian system is obtained provided the general "electric" p-form condition (5.10) (with $n_j = d$), $2p_1+\cdots+2p_d-\lambda_d A > 0$ is satisfied, i.e., (after using $p_1+\cdots+p_d = 1$ and $\lambda_d = -\lambda$) provided $\lambda A > -2$. This therefore yields a restriction on the data.

More generally, it is enough to have a function V on the real line which has an analytic continuation to the whole complex plane and which satisfies estimates of the form

$$\begin{aligned} t^{2-c_1}\tilde{V}(\tilde{A}\ln t + \tilde{B} + t^\beta\tilde{\psi}) &= O(1),\\ t^{2-c_2}\tilde{V}'(\tilde{A}\ln t + \tilde{B} + t^\beta\tilde{\psi}) &= O(1),\\ t^{2-c_3}\tilde{V}''(\tilde{A}\ln t + \tilde{B} + t^\beta\tilde{\psi}) &= O(1), \end{aligned} \tag{9.3}$$

for some positive numbers c_1, c_2 and c_3. Here $\tilde{A}$ and $\tilde{B}$ are the analytic continuations of $A(x)$ and $B(x)$, to some (small, simply connected) complex neighborhood of the range of a coordinate chart. And $\tilde{\psi}$ lies in some region of the complex plane containing the origin. For f to be regular, it must be the case that $c_1 \geq \alpha_0$ and $c_2 \geq \beta$, which can be achieved by reducing ϵ, if necessary, and also possibly U_0, so that previous assumptions are satisfied. By taking suitable account of the domains of the functions involved it is also possible to obtain an analogue of this result when the functions V and $\tilde{V}$ are only defined on some open subsets of $\mathbf{R}$ and $\mathbf{C}$.

The only other change to the construction given in Sections 4–8 is that $\rho \to \rho + V(\phi)$. It is still the case that ${}^{(D)}\nabla_\mu T^{\mu\nu} = 0$, so Equations (2.59) and (2.60) are satisfied. Equation (2.63) is satisfied due to the assumptions concerning $V(\phi)$, so the Einstein constraints are satisfied.

10 Conclusions

Our paper establishes the Kasner-like behaviour for vacuum gravity in spacetime dimensions greater than or equal to 11, as well as the Kasner-like behaviour for the Einstein-dilaton-matter systems with subcritical dilaton couplings. Our results can be summarized as follows

Theorem 10.1 *Let Σ be a d-dimensional analytic manifold, $d \geq 10$ and let $({}^0g_{ab}, {}^0k_{ab})$ be a C^ω solution of the Kasner-like vacuum Einstein equations on $(0,\infty)\times\Sigma$ such that $t\, tr^0 k = -1$ and such that the ordered eigenvalues of $-t\,{}^0k_{ab}$ satisfy $1 + p_1 - p_d - p_{d-1} > 0$.*

Then there exists an open neighborhood U of $\{0\}\times\Sigma$ in $[0,\infty)\times\Sigma$ and a C^ω solution (g_{ab}, k_{ab}) of the Einstein vacuum field equations on $U\cap((0,\infty)\times\Sigma)$ such that for each compact subset $K \subset \Sigma$ there are positive real numbers $\alpha^a{}_b$ for which the following estimates hold uniformly on K:

1. ${}^0g^{ac}g_{cb} = \delta^a{}_b + o(t^{\alpha^a{}_b})$
2. $k^a{}_b = {}^0k^a{}_b + o(t^{-1+\alpha^a{}_b})$

Theorem 10.2 *Let Σ be a d-dimensional analytic manifold, $d \geq 2$ and let*

$$X = ({}^0g_{ab},\, {}^0k_{ab},\, {}^0\phi,\, {}^0\mathcal{E}^{(j)a_1\cdots a_{n_j}},\, {}^0F^{(j)}_{a_0\cdots a_{n_j}}),$$

with j taking on values 1 through k for some non-negative integer k (possibly 0, in which case j takes on no values), $0 \leq n_j \leq d-1$. Let λ_j be constants in the subcritical range. Let X be a C^ω solution of the Kasner-like Einstein-matter equations on $(0,\infty)\times\Sigma$ such that $t\, tr^0 k = -1$, and such that the ordered eigenvalues of $-t\,{}^0k_{ab}$ satisfy $1 + p_1 - p_d - p_{d-1} > 0$ and, for each j, $2p_1 + \cdots + 2p_{n_j} - \lambda_j\, t\,\partial_t\, {}^0\phi > 0$ and $2p_1 + \cdots + 2p_{d-n_j-1} + \lambda_j\, t\,\partial_t\, {}^0\phi > 0$.

Then there exists an open neighborhood U of $\{0\}\times\Sigma$ in $[0,\infty)\times\Sigma$ and a C^ω solution $(g_{ab}, k_{ab}, \phi, \mathcal{E}^{(j)a_1\cdots a_{n_j}}, F^{(j)}_{a_0\cdots a_{n_j}})$ of the Einstein-matter field equations on $U\cap((0,\infty)\times\Sigma)$ such that for each compact subset $K\subset\Sigma$ there are positive real numbers $\beta, \alpha^a{}_b$, with $\beta < \alpha^a{}_b$, for which the following estimates hold uniformly on K:

1. ${}^0g^{ac}g_{cb} = \delta^a{}_b + o(t^{\alpha^a{}_b})$
2. $k^a{}_b = {}^0k^a{}_b + o(t^{-1+\alpha^a{}_b})$
3. $\phi = {}^0\phi + o(t^\beta)$
4. $\mathcal{E}^{(j)a_1\cdots a_{n_j}} = {}^0\mathcal{E}^{(j)a_1\cdots a_{n_j}} + o(t^\beta)$
5. $F^{(j)}_{a_0\cdots a_{n_j}} = {}^0F^{(j)}_{a_0\cdots a_{n_j}} + o(t^\beta)$

Remarks

1. Corresponding estimates hold for certain first order derivatives of the basic unknowns in Theorems 10.1 and 10.2 (*cf.* Theorem 2.1 in [15]). These are the derivatives which arise in the definition of new unknowns when second order equations are reduced to first order so as to produce a first order Fuchsian system.

2. Our analysis shows that a solution of the full subcritical Einstein-matter equations satisfying the estimates given in the theorems and the corresponding estimates for first order derivatives just mentioned is uniquely determined by the solution of the velocity-dominated equations (the integration functions are included in the zeroth order, Kasner-like solutions; the deviation from them is uniquely determined).

3. The Einstein-matter field equations may include interaction terms of Chern-Simons, Chapline-Manton and Yang-Mills type, and the scalar field may be self-interacting, with assumptions on $V(\phi)$ as stated in Section 9. If the jth field is a Yang-Mills field, then $F^{(j)}_{ab}$ is obtained from $A^{(j)}_a$ and ${}^0F^{(j)}_{ab}$ is obtained from ${}^0A^{(j)}_a$ through Equation (8.4). Note that the condition on $\mathrm{tr}\,{}^0k$ which is assumed in both theorems can always be arranged by means of a time translation.

4. The spacetimes of the class whose existence is established by these theorems have the desirable property that it is possible to determine the detailed nature of their singularities by algebraic calculations. This allows them to be checked for consistency with the cosmic censorship hypothesis. What should be done from this point of view is to check that some invariantly defined physical quantity is unbounded as the singularity at $t = 0$ is approached. This shows that $t = 0$ is a genuine spacetime singularity beyond which no regular extension of the spacetime is possible. For this purpose it is common to examine curvature invariants but in fact it is just as good if an invariant of the matter fields can be found which is unbounded in the approach to $t = 0$. This is particularly convenient in the cases where a dilaton is present. Then $\nabla^\alpha\phi\nabla_\alpha\phi$ is equal in leading order to the corresponding velocity-dominated quantity and the latter is easily seen to diverge like t^{-4} for $t \to 0$. The vacuum case is more difficult. It will be shown below that the approximation of the full solution by the velocity-dominated solution is sufficiently good that it is enough to do the calculation for the velocity-dominated metric. This means that it is enough to do the calculation for the Kasner metric in D dimensions. Note that the Kasner metric is invariant under reflection in each of the spatial coordinates. Hence curvature components of the form R_{0abc} vanish, as do components of the form R_{0a0b} with $a \neq b$. Hence the Kretschmann scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ is a sum of non-negative terms of the form $R^{abcd}R_{abcd}$ and $(R^a{}_{0a0})^2$. In order to show that the Kretschmann scalar

is unbounded it is enough to show that one of these terms is unbounded. A simple calculation shows that $(R^a{}_{0a0})^2 = p_a^2(1-p_a)^2 t^{-4}$ in a Kasner spacetime. Thus the curvature invariant under consideration can only be bounded as $t \to 0$ if all Kasner exponents are zero or one, which does not occur for the solutions we construct. To see that the approximation of the full solution by the velocity-dominated solution is valid for determining the asymptotics of the Kretschmann scalar it is enough to note that all terms appearing in the Kretschmann scalar which were not just considered are $o(t^{-4})$. Only two estimates additional to those already obtained are needed – for these, the estimates $\tilde{R}_{abc}{}^h = O(t^{-2+\epsilon})$ and $\tilde{\nabla}_a \tilde{k}^b_c = O(t^{-2+\epsilon})$ are sufficient. Both of these estimates are straightforward to obtain. The main input is $\tilde{\Gamma}^c_{ab} = O(t^{-1+4\sigma-2\epsilon-\delta})$ (i.e., the connection coefficients *do not* need to be expanded). The expression for the Kretschmann scalar is

$$\begin{aligned}
&4((\operatorname{tr} k)k^a{}_b - k^a{}_c\kappa^c{}_b)((\operatorname{tr} k)k^b{}_a - k^b{}_h k^h{}_a)\\
&+(k^a{}_b k^c{}_h - k^a{}_h k^c{}_b)(k^b{}_a k^h{}_c - k^h{}_a k^b{}_c)\\
&+4\{(R^a{}_b - M^a{}_b)(R^b{}_a - M^b{}_a) + 2(R^a{}_b - M^a{}_b)((\operatorname{tr} k)k^b{}_a - k^b{}_h k^h{}_a)\\
&-2(\tilde{\nabla}_a \tilde{k}^b{}_c)(\tilde{\nabla}_h \tilde{k}^c{}_b)\tilde{g}^{ah} - 2(\tilde{\nabla}_a \tilde{k}^b{}_c)(\tilde{\nabla}_b \tilde{k}^a{}_h)\tilde{g}^{ch}\} - \tilde{g}^{ab}\tilde{g}^{ch}\tilde{R}_{aci}{}^j \tilde{R}_{bhj}{}^i\\
&+2\tilde{R}_{abc}{}^h(\tilde{k}^a{}_i \tilde{k}^b{}_h - \tilde{k}^a{}_h \tilde{k}^b{}_i)\tilde{g}^{ci}.
\end{aligned}$$

Apart from the Kasner terms (which can each be written as two factors, with each factor $O(t^{-2})$), the remaining terms can each be written as two factors, with each factor $O(t^{-2})$ and at least one of the two factors $o(t^{-2})$.

5. We have constructed large classes of solutions of the Einstein-matter equations with velocity-dominated singularities for matter models defined by those field theories where the BKL picture predicts that solutions of this kind should exist. No symmetry assumptions were made. When symmetry assumptions are made there are more possibilities of finding specialized classes of spacetimes with velocity-dominated singularities. See for instance [45], where there are results for the Einstein-Maxwell-dilaton and other systems under symmetry assumptions. There are also results for the case where the Einstein equations are coupled to phenomenological matter models such as a perfect fluid and certain symmetry assumptions are made. For one of the most general results of this kind so far see [46].

6. When solutions are constructed by Fuchsian methods as is done is this paper there is the possibility of algorithmically constructing an expansion of the solution about the singularity to all orders which is convergent when the input data are analytic, as in this paper. (If the input data are only C^∞ the expansion is asymptotic in a rigorous sense when Fuchsian techniques can be applied.) At the same time, there is the possibility of providing a rigorous confirmation of the reliability of existing expansions such as those of [24] and [25]. This is worked out for the case of [24] in [28].

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References

[1] T. Damour and M. Henneaux, Chaos in superstring cosmology, *Phys. Rev. Lett.* **85**, 920 (2000) [arXiv:hep-th/0003139].

[2] T. Damour and M. Henneaux, Oscillatory behaviour in homogeneous string cosmology models, *Phys. Lett. B* **488**, 108 (2000) [arXiv:hep-th/0006171].

[3] T. Damour and M. Henneaux, E(10), BE(10) and arithmetical chaos in superstring cosmology, *Phys. Rev. Lett.* **86**, 4749 (2001) [arXiv:hep-th/0012172].

[4] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, Oscillatory approach to a singular point in the relativistic cosmology, *Adv. Phys.* **19**, 525 (1970); A general solution of the Einstein equations with a time singularity, *Adv. Phys.* **31**, 639 (1982).

[5] S. Deser, R. Jackiw and S. Templeton, Topologically Massive Gauge Theories, *Annals Phys.* **140**, 372 (1982) [Erratum-*ibid.* **185**, 406.1988 APNYA,281,409 (1982)].

[6] G.F. Chapline and N.S. Manton, Unification Of Yang-Mills Theory And Supergravity In Ten-Dimensions, *Phys. Lett. B* **120**, 105 (1983).

[7] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, Ten-Dimensional Maxwell-Einstein Supergravity, Its Currents, And The Issue Of Its Auxiliary Fields, *Nucl. Phys. B* **195**, 97 (1982).

[8] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, Inconsistency of interacting, multigraviton theories, *Nucl. Phys. B* **597**, 127 (2001) [arXiv:hep-th/0007220].

[9] T. Damour and A.M. Polyakov, The String dilaton and a least coupling principle, *Nucl. Phys. B* **423**, 532 (1994) [arXiv:hep-th/9401069]; T. Damour and A. Vilenkin, String theory and inflation, *Phys. Rev. D* **53**, 2981

(1996) [arXiv:hep-th/9503149]; M. Gasperini, F. Piazza and G. Veneziano, Quintessence as a run-away dilaton, arXiv:gr-qc/0108016.

[10] J.D. Barrow and S. Hervik, Magnetic brane-worlds, arXiv:gr-qc/0109084.

[11] D. Eardley, E. Liang and R. Sachs, Velocity-Dominated Singularities In Irrotational Dust Cosmologies, *J. Math. Phys.* **13**, 99 (1972).

[12] C.W. Misner, Mixmaster universe, *Phys. Rev. Lett.* **22**, 1071–1074 (1969).

[13] J. Demaret, M. Henneaux and P. Spindel, Nonoscillatory Behavior In Vacuum Kaluza-Klein Cosmologies, *Phys. Lett.* **164B**, 27 (1985);
J. Demaret, Y. De Rop and M. Henneaux, Are Kaluza-Klein Models Of The Universe Chaotic?, *Int. J. Theor. Phys.* **28**, 1067 (1989).

[14] V.A. Belinskii and I.M. Khalatnikov, Effect of scalar and vector fields on the nature of the cosmological singularity, *Sov. Phys. JETP* **36**, 591–597 (1973).

[15] L. Andersson and A.D. Rendall, Quiescent cosmological singularities, *Commun. Math. Phys.* **218**, 479–511 (2001) [arXiv:gr-qc/0001047].

[16] R.T. Jantzen, Finite-dimensional Einstein-Maxwell-scalar-field system, *Phys. Rev. D* **33**, 2121–2135 (1986).

[17] V.G. LeBlanc, Asymptotic states of magnetic Bianchi I cosmologies, *Class. Quantum Grav.* **14**, 2281–2301 (1997); Bianchi II magnetic cosmologies, *Class. Quantum Grav.* **15**, 1607–1626 (1998).

[18] M. Weaver, Dynamics of magnetic Bianchi VI_0 cosmologies, *Class. Quantum Grav.* **17**, 421–434 (2000) [arXiv:gr-qc/9909043].

[19] B.K. Berger, D. Garfinkle, J. Isenberg, V. Moncrief and M. Weaver, The singularity in generic gravitational collapse is spacelike, local, and oscillatory, *Mod. Phys. Lett. A* **13**, 1565–1574 (1998) [arXiv:gr-qc/9805063].

[20] B.K. Berger, J. Isenberg and M. Weaver, Oscillatory approach to the singularity in vacuum spacetimes with T^2 isometry, *Phys. Rev. D* **64**, 084006–1-20 (2001) [arXiv:gr-qc/0104048].

[21] B.K. Berger and V. Moncrief, Evidence for an oscillatory singularity in generic U(1) symmetric cosmologies on $T^3 \times R$, *Phys. Rev. D* **58**, 064023–1-8 (1998) [arXiv:gr-qc/9804085].

[22] M. Weaver, J. Isenberg and B.K. Berger, Mixmaster behavior in inhomogeneous cosmological spacetimes, *Phys. Rev. Lett.* **80** 2984–2987 (1998) [arXiv:gr-qc/9712055].

[23] D. Garfinkle, Harmonic coordinate method for simulating generic singularities, arXiv:gr-qc/0110013.

[24] B. Grubišić and V. Moncrief, Asymptotic behaviour of the $T^3 \times R$ Gowdy spacetimes, *Phys. Rev. D* **47** 2371–2382 (1993).

[25] A. Buonanno, T. Damour and G. Veneziano, Pre-big bang bubbles from the gravitational instability of generic string vacua, *Nucl. Phys. B* **543**, 275 (1999) [arXiv:hep-th/9806230].

[26] B.K. Berger and V. Moncrief, Exact U(1) symmetric cosmologies with local Mixmaster dynamics, *Phys. Rev. D* **62** 02359 (2000) [arXiv:gr-qc/0001083].

[27] H. Ringström, The Bianchi IX attractor, *Ann. H. Poincaré* **2**, 405–500 (2001) [arXiv:gr-qc/0006035].

[28] S. Kichenassamy, and A.D. Rendall, Analytic description of singularities in Gowdy spacetimes, *Class. Quantum Grav.* **15** 1339–1355 (1998).

[29] A.D. Rendall, Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity, *Class. Quantum Grav.* **17** 3305-3316 (2000) [arXiv:gr-qc/0004044].

[30] A.D. Rendall, Applications of the theory of evolution equations to general relativity, arXiv:gr-qc/0109028.

[31] A.D. Rendall, Blow-up for solutions of hyperbolic PDE and spacetime singularities, in *Proceedings of Journees EDP Atlantique*, 2000 [arXiv:gr-qc/0006060].

[32] J. Demaret, J.L. Hanquin, M. Henneaux, P. Spindel and A. Taormina, The Fate Of The Mixmaster Behavior In Vacuum Inhomogeneous Kaluza-Klein Cosmological Models, *Phys. Lett. B* **175**, 129 (1986).

[33] B.K. Berger, Influence of scalar fields on the approach to a cosmological singularity, *Phys. Rev. D* **61**, 023508–1-6 (1999) [arXiv:gr-qc/9907083].

[34] B.K. Berger and D. Garfinkle, Phenomenology of the Gowdy universe on $T^3 \times R$, *Phys. Rev. D* **57**, 4767–4777 (1998) [arXiv:gr-qc/9710102].

[35] A.D. Rendall and M. Weaver, Manufacture of Gowdy spacetimes with spikes, *Class. Quantum Grav.* **18**, 2959–2975 (2001) [arXiv:gr-qc/0103102].

[36] D.M. Chitre, Ph. D. thesis, University of Maryland, 1972.

[37] C.W. Misner, in: D. Hobill et al. (Eds), Deterministic chaos in general relativity, Plenum, 1994, pp. 317–328 [gr-qc/9405068].

[38] A.A. Kirillov and V.N. Melnikov, Dynamics of Inhomogeneities of Metric in The Vicinity of a Singularity in Multidimensional Cosmology *Phys. Rev. D* **52**, 723 (1995) [gr-qc/9408004]; V.D. Ivashchuk and V.N. Melnikov, Billiard Representation for Multidimensional Cosmology with Multicomponent Perfect Fluid Near the Singularity, *Class. Quantum Grav.* **12**, 809 (1995).

[39] T. Damour, M. Henneaux, B. Julia and H. Nicolai, Hyperbolic Kac-Moody algebras and chaos in Kaluza-Klein models, *Phys. Lett. B* **509**, 323 (2001) [arXiv:hepn-th/0103094].

[40] H. Nicolai, Two-dimensional gravities and supergravities as integrable system, DESY-91-038 *Lectures presented at 30th Schladming Winter School, Schladming, Austria, Feb 27–Mar 5, 1991.*

[41] L. Andersson, The global existence problem in general relativity, arXiv:gr-qc/9911032.

[42] Y. Choquet-Bruhat and J.W. York, The Cauchy problem, in A. Held (ed.), *General Relativity* Plenum, New York, 1980.

[43] Y. Choquet-Bruhat, J. Isenberg and J.W. York, Einstein constraints on asymptotically Euclidean manifolds, *Phys. Rev. D* **61**, 084034 (2000).

[44] N.Ó. Murchadha and J.W. York, Existence and uniqueness of solutions of the Hamiltonian constraint of general relativity on compact manifolds, *J. Math. Phys.* **14**, 1551–1557, 1973.

[45] M. Narita, T. Torii and K. Maeda, Asymptotic singular behaviour of Gowdy spacetimes in string theory, *Class. Quantum Grav* **17**, 4597–4613 (2000) [arXiv:gr-qc/0003013].

[46] K. Anguige, A class of perfect-fluid cosmologies with polarized Gowdy symmetry and a Kasner-like singularity, arXiv:gr-qc/0005086.

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