Ann. Henri Poincaré 1 (2000) 499 – 541
© Birkhäuser Verlag, Basel, 2000
1424-0637/00/030499-43 $ 1.50+0.20/0

Annales Henri Poincaré

Sine-Gordon Revisited

J. Dimock and T. R. Hurd

Abstract. We study the sine-Gordon model in two dimensional space time in two different domains. For $\beta > 8\pi$ and weak coupling, we introduce an ultraviolet cutoff and study the infrared behavior. A renormalization group analysis shows that the model is asymptotically free in the infrared. For $\beta < 8\pi$ and weak coupling, we introduce an infrared cutoff and study the ultraviolet behavior. A renormalization group analysis shows that the model is asymptotically free in the ultraviolet.

I Introduction

We are concerned with the two dimensional sine-Gordon model. The model is characterized by its partition function which is formally

$$Z = \int \exp\left(\zeta \int \cos(\phi(x))dx \;-\; \frac{1}{2\beta}\int |\partial\phi(x)|^2 dx\right) \prod_{x\in\mathbf{R}^2} d\phi(x)\ . \tag{1}$$

It is of interest both as a Euclidean quantum field theory and because it describes the classical statistical mechanics of a Coulomb gas with inverse temperature β and activity $\zeta/2$.

The expression for Z is ill-defined. To make sense of it we first replace the plane $\mathbf{R}^2$ by the torus $\Lambda_M = \mathbf{R}^2/L^M\mathbf{Z}^2$, where M is a non-negative integer and L is a fixed large positive constant. Then the quadratic term is combined with the non-existent Lebesgue measure to give a Gaussian measure. Introduction of a short distance cutoff at scale L^{-N} gives the Gaussian measure $\mu_{\beta v^M_{-N}}$ with covariance

$$\beta v^M_{-N}(x-y) = \beta|\Lambda_M|^{-1} \sum_{p\in\Lambda^*_M, p\neq 0} \frac{e^{ip(x-y)}}{p^2} e^{-p^4 L^{-4N}} \tag{2}$$

where $\Lambda^*_M = (2\pi L)^{-M}\mathbf{Z}^2$. Since $p = 0$ is excluded the measure is supported on fields ϕ with $\int \phi = 0$. Furthermore the covariance is smooth and so the measure is supported on smooth functions. Thus the cutoff expression

$$Z = \int \exp\left(\zeta \int_{\Lambda_M} \cos(\phi(x))dx\right)\ d\mu_{\beta v^M_{-N}}(\phi). \tag{3}$$

is well-defined. We are interested in studying the limits $N \to \infty$ (the UV problem) and $M \to \infty$ (the IR problem).

There are two distinct domains in which these problems are tractable. For $\beta > 8\pi$ and ζ small it turns out that the long distance behavior differs only slightly from that of the free model ($\zeta = 0$) and thus the IR problem can be controlled. For $\beta < 8\pi$ and ζ small it turns out that the short distance behavior differs only slightly from free and thus the UV problem can be controlled. The purpose of this paper is to carry out the analysis in each case using a renormalization group (RG) method.

Each of these problems have been previously studied by the authors in [13] and [15]. Unfortunately there is an error which occurs in both papers and spoils the proofs of the main results.[1] In our present paper we are at last able to fix this error, and reinstate our earlier results. The fix requires some substantial modifications to the method, and so we give here reasonably self-contained proofs of the main technical lemmas.

We first discuss the IR problem for $\beta > 8\pi$. We study the expression (3) with the ultraviolet cutoff N fixed: for simplicity we take $N = 0$. The RG method involves the introduction of a sum over scales. For any $0 \leq j \leq M$ we have

$$v_0^M(x-y) = \sum_{k=0}^{j-1} C^{M-k}(L^{-k}(x-y)) + v_0^{M-j}(L^{-j}(x-y)) \; . \tag{4}$$

The slice covariances are defined by[2]

$$C^M(x-y) = |\Lambda_M|^{-1} \sum_{p\in\Lambda_M^*, p\neq 0} \frac{e^{ip(x-y)}}{p^2}(e^{-p^4} - e^{-L^4p^4}) \; . \tag{5}$$

The integral over $\mu_{\beta v_0^M}$ in the partition function can then be evaluated by successively taking convolutions with $\mu_{\beta C}$ and then scaling down by L. After j steps we have the expression

$$Z = \int \mathcal{Z}_j(\phi)\; d\mu_{\beta v_0^{M-j}}(\phi) \tag{6}$$

with successive densities $\mathcal{Z}_j$ defined on Λ_{M-j} and related by

$$\mathcal{Z}_{j+1}(\phi) = (\mu_{\beta C} * \mathcal{Z}_j)(\phi_L) = \int \mathcal{Z}_j(\phi_L + \zeta) d\mu_{\beta C}(\zeta) \tag{7}$$

where $\phi_L(x) = \phi(x/L)$ is the canonical rescaling of the field for $d = 2$. Equation (7) is the RG map. We want to study the flow starting with $\mathcal{Z}_0(\phi) = \exp(\zeta \int \cos\phi)$.

[1]The problem is that for the homotopy property one needs κ small, but the limitation on κ cannot be made independently of L as was implicitly assumed. In fact one needs $\kappa \leq \mathcal{O}(L^{-2})$ or smaller. Then the use of Sobolev inequalities require $\kappa(h_1^*)^2 \geq \mathcal{O}(1)$ and hence $h_1^* \geq \mathcal{O}(L)$. This spoils the estimate above line (49) in [15]. There is a similar problem in [13].

[2]We have chosen to take e^{-p^4} rather than say e^{-p^2} in (2),(5) in order to have a smoother approach to infinite volume at $p = 0$.

To track the flow we must isolate the fastest growing parts of $\mathcal{Z}_j$ during each RG step. We extract a constant part and a gradient part and instead of (7) now define $\mathcal{Z}_{j+1}$ by

$$\mathcal{Z}_{j+1}(\phi)\ \exp\left(\delta E_j|\Lambda_{M-j}| - \frac{\delta\sigma_j}{2\beta}\int_{\Lambda_{M-j-1}}(\partial\phi)^2\right) = \left(\mu_{\beta C} * \mathcal{Z}_j\right)(\phi_L) \tag{8}$$

with special choices of $\delta E_j, \delta\sigma_j$. The quadratic factor is absorbed into the measure at each step and so instead of (6) we have for some constants $\mathcal{E}_j, \sigma_j$

$$Z = e^{\mathcal{E}_j}\int \mathcal{Z}_j(\phi)\ d\mu_{\beta v_0^{M-j}(\sigma_j)}(\phi)\ , \tag{9}$$

where

$$v_0^M(\sigma; x-y) = |\Lambda_M|^{-1}\sum_{p\in\Lambda_M^*, p\neq 0}\frac{e^{ip(x-y)}}{p^2}(e^{p^4}+\sigma)^{-1}\ . \tag{10}$$

The successive values of σ_j are given by $\sigma_{j+1} = \sigma_j + \delta\sigma_j$ and there is a similar formula for $\mathcal{E}_{j+1}$ in terms of $\mathcal{E}_j, \delta E_j$ and $\delta\sigma_j$.

To state the main result we need one more ingredient. This is a local structure for the densities $\mathcal{Z}_j$. Following Brydges and Yau [11] densities are represented by polymer expansions $\mathcal{Z}_j(\phi) = \mathcal{E}xp(\Box + K_j)(\Lambda_j, \phi)$ as we now explain. A *closed polymer* X is a union of closed unit squares centered on lattice points. A *polymer activity* is a function $K(X, \phi)$ depending on polymers X and fields ϕ with the property that the dependence on ϕ is localized in X. One can define a product on polymer activities and an associated exponential function $(\mathcal{E}xp(K))(X, \phi)$. If $\Box$ is the characteristic function of open unit cells then

$$\mathcal{E}xp(\Box + K)(X,\phi) = \sum_{\{X_i\}}\prod_i K(X_i,\phi) \tag{11}$$

where the sum is over collections of disjoint polymers $\{X_i\}$ in X. For an exposition of polymers see [3].

Now we can state the IR result:

Theorem 1 *Let $\beta > 8\pi$, let $\epsilon > 0$, let L be sufficiently large, and let $|\zeta|$ be sufficiently small. Then for $j = 0, 1, 2, ..$ the partition function Z defined by (3) with $N = 0$ can be written*

$$Z = e^{\mathcal{E}_j}\int \mathcal{E}xp(\Box + K_j)(\Lambda_{M-j}, \phi) d\mu_{\beta v_0^{M-j}(\sigma_j)}(\phi) \tag{12}$$

where $\mathcal{E}_j/|\Lambda_M|$ and σ_j are bounded and $\mathcal{O}(|\zeta|)$ uniformly in M. The polymer activities K_j are even and 2π–periodic in ϕ. There is a norm $\|\cdot\|_\infty$ such that

$$\|K_j\|_\infty \le \delta^j|\zeta|^{1-\epsilon} \tag{13}$$

where $\delta = \mathcal{O}(1)\max\{L^{-2}, L^{2-\beta/4\pi}\} < 1/4$

Here and throughout the paper $\mathcal{O}(1)$ means a constant which is independent of L, ζ, M, j. The norm $\|K_j\|_\infty$ of $K_j(X, \phi)$ enforces conditions of growth and analyticity in ϕ and requires tree decay in X. A more precise version of the theorem will be stated later when we come to the proof.

The point is that K_j shrinks uniformly in M so that the dominant contribution as $j \to M$ is from the Gaussian measure. The result gives a uniform bound on the energy density $\log Z/|\Lambda_M|$ and it should be possible to also take the limit $M \to \infty$. Everything should also be analytic in ζ in a complex neighborhood of the origin. The only difficult part here is working with complex measures; see [11] for a treatment of this problem for the closely related dipole gas.

A modification of this theorem to include local perturbations should make it possible to study correlation functions for the model, proving the existence of the $M \to \infty$ limit and showing that the long distance behavior of correlations is essentially the same as free. See [19] for results of this nature, and [14], [8] for the closely related dipole gas.

Let us mention some earlier work on this model. It was first treated heuristically by Kosterlitz and Thouless [20]. Fröhlich and Spencer later gave a rigorous treatment for β large [17] by a special method (not the RG). The range of validity was extended to $\beta > 8\pi$ by Marchetti and Klein [21].

Now we discuss the UV problem for $\beta < 8\pi$. We start with a fixed torus Λ_M. For simplicity take the unit torus Λ_0 so the starting covariance is v^0_{-N}. We also make a renormalization replacing $\cos(\phi(x))$ by the Wick ordered version

$$: \cos(\phi(x)) :_{\beta v^0_{-N}} = \exp(\beta v^0_{-N}(0)/2) \cos(\phi(x)) \ . \tag{14}$$

Thus we study the partition function

$$Z = \int \exp\left(\zeta \int_{\Lambda_0} : \cos(\phi(x)) :_{\beta v^0_{-N}} dx\right) d\mu_{\beta v^0_{-N}}(\phi) \ . \tag{15}$$

We scale up to get an expression for Z on Λ_N. Absorbing the Wick ordering constant into the coupling constant one finds that

$$Z = \int \exp\left(\zeta_{-N} \int_{\Lambda_N} \cos(\phi(x)) dx\right) d\mu_{\beta v^N_0}(\phi) \tag{16}$$

where for any $j \leq 0$

$$\zeta_j = L^{-2|j|} \exp(\beta v^0_{-|j|}(0)/2) \zeta \approx L^{-(2-\beta/4\pi)|j|} \zeta \ . \tag{17}$$

The UV problem of controlling the limit $N \to \infty$ by an RG analysis looks very much like the IR problem. The main difference is that the coupling constants ζ_j start out ultra small at $j = -N$ and grow to a small value $\zeta_0 = \zeta$, instead of starting out small and then shrinking. A technical simplification is that the field strength extraction is no longer needed and we can take $\sigma_j = 0$.

We define

$$V(X,\phi) = \begin{cases} \int_\Delta \cos(\phi(x))dx & X = \Delta = \text{unit square} \\ 0 & |X| \geq 2 \end{cases} \tag{18}$$

and then the result is :

Theorem 2 *Let $\beta < 8\pi$, let $\epsilon > 0$, let L be sufficiently large, and let $|\zeta|$ be sufficiently small. Then for $j = -N, -N+1, \ldots, 0$ the partition function given by (15) or (16) can be written*

$$Z = e^{\mathcal{E}_j} \int \mathcal{E}xp(\Box + K_j)(\Lambda_{|j|}, \phi) d\mu_{\beta v_0^{|j|}}(\phi) \tag{19}$$

where

$$\begin{aligned} \mathcal{E}_j &= \sum_{k=-N}^{j-1} \delta E_k |\Lambda_{|k|}| \ , \\ K_j &= \zeta_j V + \tilde{K}_j \ . \end{aligned} \tag{20}$$

The $\tilde{K}_j$ are even and 2π–periodic in ϕ, and $\delta E_j, \tilde{K}_j$ are analytic in ζ and satisfy

$$\begin{aligned} |\delta E_j| &\leq |\zeta_j|^{2-\epsilon} \\ \|\tilde{K}_j\|_\infty &\leq |\zeta_j|^{2-\epsilon} \end{aligned} \tag{21}$$

for some norm $\|\cdot\|_\infty$.

For $\beta < 4\pi$ the theorem implies that Z is uniformly bounded and analytic in ζ. For $4\pi \leq \beta < 8\pi$ it isolates the divergence in Z. One can also show that $\delta E_j, K_j$ have limits as $N \to \infty$, and hence so does Z (for $\beta < 4\pi$). [15]

A modification of this theorem to include local perturbations should make it possible to study correlation functions for $\beta < 8\pi$. (The potentially divergent factor $\mathcal{E}_j$ does not contribute to correlation functions). One should be able to take the $N \to \infty$ limit and study the short distance behavior of correlations. See [15],[19] for results of this nature. Also see [12] for a proof that at $\beta = 4\pi$ the theory is equivalent to a theory of massive free fermions.

Earlier work on this problem can be found in [16], [2], [23] [22].

II Estimates on the RG map

Our treatment of the RG map on polymer activities is similar to that used in previous papers [13],[15], [5], [6]. However there are essential modifications: references [5], [6], which we follow as much as possible, use open polymers while we have to use closed polymers as in [13],[15] (see the discussion in the next section). Our norms are now simpler as well, a simplification we pay for with some harder proofs.

In this chapter, we analyze a single RG map on a torus $\Lambda = \Lambda_M = \mathbf{R}^d / L^M \mathbf{Z}^d$ of arbitrary dimension $d \geq 2$. We work with the fixed covariance

$$C^M(\sigma, x-y) = |\Lambda_M|^{-1} \sum_{p \in \Lambda_M^*, p \neq 0} \frac{e^{ipx}}{p^2} \left[(e^{p^4} + \sigma)^{-1} - (e^{L^4 p^4} + \sigma)^{-1} \right] \tag{22}$$

and $|\sigma|$ assumed small, although the results holds for a much larger class. We start by defining our norms. Then we consider separately the three pieces of the RG: fluctuation, extraction, and scaling. Finally we put them together in Theorem 10 to give an overall estimate on the RG map.

II.1 Norms

Let the Banach spaces $\mathcal{C}^r(X), \tilde{\mathcal{C}}^s(X)$ of smooth fields $\phi(x)$ on a closed polymer X be defined respectively for fixed $r, s \geq 0$ by the following norms:

$$\begin{aligned} \|\phi\|_X \equiv \|\phi\|_{\infty,r,X} &= \sup_{|\alpha| \leq r,\ x \in X} |\partial^\alpha \phi(x)| \\ \|\phi\|_{s,X} \equiv \|\phi\|_{2,s,X} &= \left[\sum_{|\alpha| \leq s} \int_X |\partial^\alpha \phi(x)|^2 \, dx \right]^{1/2} \end{aligned} \tag{23}$$

We assume $s > d/2 + r$ to ensure a Sobolev inequality $\|\phi\|_{\infty,r,X} \leq \mathcal{O}(1) \|\phi\|_{2,s,X}$ and the corresponding dense embedding $\tilde{\mathcal{C}}^s(X) \subset \mathcal{C}^r(X)$.

Let $K(X, \phi)$ be a smooth function on $\tilde{\mathcal{C}}^s(X)$. Thus we assume the existence of all derivatives $K_n(X, \phi)$. These are continuous symmetric multilinear functionals on $\tilde{\mathcal{C}}^s(X)$. In fact we make a stronger assumption that these derivatives have continuous extensions to $\mathcal{C}^r$ by demanding the finiteness of the following norm

$$\|K_n(X, \phi)\| = \sup_{\substack{f_i \in \tilde{\mathcal{C}}^s(X) \\ \|f_i\|_{\infty,r,X} \leq 1}} |K_n(X, \phi; f_1, ..., f_n)|. \tag{24}$$

A *large field regulator* is a functional of the form

$$G(\kappa, X, \phi) = G'(\kappa, X, \phi) \delta G(\kappa, \partial X, \phi) \tag{25}$$

where

$$\begin{aligned} G'(\kappa, X, \phi) &= \exp(\kappa \sum_{1 \leq |\alpha| \leq s} \int_X |\partial^\alpha \phi|^2) \\ \delta G(\kappa, \partial X, \phi) &= \exp(\kappa c \sum_{|\alpha|=1} \int_{\partial X} |\partial^\alpha \phi|^2) \end{aligned} \tag{26}$$

with constants $\kappa, c \leq 1$ to be specified.

A *large set regulator* $\Gamma(X)$ has the form

$$\Gamma(X) = A^{|X|}\Theta(X) \tag{27}$$

for a parameter $A \geq 1$ and factor $\Theta(X)$ such that $\Theta(X)^{-1}$ has polynomial tree decay (see [11], [6] for the exact definition). For our present paper we fix $A = L^{d+3}$, and also define $\Gamma_p(X) = 2^{p|X|}\Gamma(X)$ for any $p = \pm 1, \pm 2, \ldots$.

In terms of regulators G, Γ and a further parameter $h \geq 0$ we define the norms :

$$\begin{aligned} \|K\|_{G,h,\Gamma} &= \sum_{X \supset \Delta} \Gamma(X) \|K(X)\|_{G,h} \ , \\ \|K(X)\|_{G,h} &= \sum_{n=0}^{\infty} \frac{h^n}{n!} \|K_n(X)\|_G \ , \\ \|K_n(X)\|_G &= \sup_{\phi \in \tilde{\mathcal{C}}^s(X)} \|K_n(X,\phi)\| G(X,\phi)^{-1}. \end{aligned} \tag{28}$$

The sum over X is independent of the unit block Δ for translation invariant K.

These norms are simpler than the norms in earlier versions of this formalism in which one first localizes the derivatives in unit blocks, then takes the supremum over the fields, and finally sums over blocks. The previous version (designed for models in $d > 2$) controls the fluctuation step in an elegant manner, but in $d = 2$ leads to unbounded growth in the parameter h in the scaling step. The present norms require a different treatment of fluctuation, but avoid growth in h.

Another point concerns the boundary term $\delta G(\kappa, \partial X, \phi)$ in the large field regulator $G(\kappa, X, \phi)$. It is present to absorb the growth of $G'(\kappa, X, \phi)$ a feature upon which we elaborate in the next section. However we also need $G(X)G(Y) \leq G(X \cup Y)$ for disjoint polymers. The boundary term spoils this if the polymers are open since disjoint polymers may have pieces of their boundaries in common. This is the reason we have taken closed polymers.

II.2 Fluctuation

Given a localized density $\mathcal{E}xp(\Box + K)$ and the Gaussian measure μ_C we want to find new polymer activities $\mathcal{F}K$ such that $\mu_C * \mathcal{E}xp(\Box + K) = \mathcal{E}xp(\Box + \mathcal{F}K)$ and such that we more or less preserve control over size and localization. We accomplish this using the framework of Brydges and Yau [11] (see also Brydges and Kennedy [9]). Those authors actually give two constructions. The first is by solving a functional Hamilton-Jacobi equation for $\mu_{tC} * \mathcal{E}xp(\Box + K)$. This is elegant and efficient, but with our new norms we cannot take advantage of it (since we can no longer keep the large set regulator Γ constant). Instead we use the second construction of Brydges and Yau, an explicit cluster expansion. Cluster expansions have a long history starting with [18].

We begin with the purely combinatoric part. Let $F(s)$ be a continuously differentiable function of $s = \{s_{ij}\}$ where $0 \leq s_{ij} \leq 1$ and where ij runs over the distinct unordered pairs (*bonds*) from some finite index set. A *graph* G on this set is a collection of bonds , and it is called a *forest* if it has no closed loops. The set of all forests is denoted $\mathcal{F}$. Finally we define

$$\sigma_{ij}(G,s) = \inf\{s_b : b \in \text{path joining } ij \text{ in } G\} \tag{29}$$

with the convention that $\sigma_{ij}(G,s) = 0$ if there is no path joining ij in G. Then for $1 = \{1, 1, ..., 1\}$

$$F(1) = \sum_{G\in\mathcal{F}} \int \prod_{b\in G} ds_b \, (\prod_{b\in G} \partial_{s_b} F)(\sigma(G,s)) \tag{30}$$

where the $G = \emptyset$ term is interpreted as $F(0)$. For the proof see Abdesselam and Rivasseau [1] or Brydges and Martin, Theorem VIII.2 [10].

Now for any X, Y define

$$C(X,Y)(x,y) = \frac{1}{2}[\chi_X(x)\chi_Y(y) + \chi_Y(x)\chi_X(y)]C(x-y) \tag{31}$$

and let $C_X = C(X,X)$ be the restriction of C to X. Suppose that $\{X_i\}$ is a collection of disjoint polymers whose union is X. Then the restriction C_X can be written

$$C_X = \sum_{i,j} C(X_i, X_j) \tag{32}$$

with the sum over ordered pairs. We weaken the coupling between X_i, X_j with parameters s_{ij} and define

$$C_X(s) = \sum_{i,j} C(X_i, X_j)s_{ij} \tag{33}$$

where $s_{ii} = 1$. Now while $C_X(s)$ is not necessarily positive definite, $C_X(\sigma(T,s))$ is positive definite for any s and any tree T [9], [10].

Let Δ_C be the functional Laplacian given formally by:

$$\Delta_C = \frac{1}{2}\int C(x,y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}dxdy \, . \tag{34}$$

Lemma 3 *[11], [1] For smooth polymer activities K*

$$\mu_C * \mathcal{E}xp(\Box + K) = \mathcal{E}xp(\Box + \mathcal{F}K) \tag{35}$$

with

$$\mathcal{F}K(X) = \sum_{\{X_i\},T\to X} \int ds^T \; \mu_{C_X(\sigma(T,s))} * \prod_{ij\in T}(-2\Delta_{C(X_i,X_j)}) \prod_i K(X_i) \tag{36}$$

where the sum is over collections of disjoint polymers $\{X_i\}$ whose union is X, and over tree graphs T on $\{X_i\}$. If $\{X_i\} = \{X\}$ the term is interpreted as $\mu_C * K(X)$.

Proof. We start with

$$\mu_C * \mathcal{E}xp(\Box + K)(X) = \sum_{\{X_i\}} \mu_C * \prod_i K(X_i) \ . \tag{37}$$

In the expression $\mu_C * \prod_i K(X_i, \phi)$ we regard the product as a function of fields ϕ on X only, and replace the covariance C by C_X.[3] If $\{X_i\} = \{X\}$ has only one element we leave the expression alone. Otherwise there are two or more subsets and for each $\{X_i\}$ we analyze $F(1) = \mu_{C_X} * \prod_i K(X_i)$ by introducing the interpolation $F(s) = \mu_{C_X(s)} * \prod_i K(X_i)$ with $C_X(s)$ given by (33), and then making the expansion (30). This gives the expression

$$\sum_{\{X_i\}} \sum_G \int ds^G \left(\partial^G \mu_{C_X(s)} * \prod_i K(X_i) \right)_{s=\sigma(G,s)} . \tag{38}$$

Now the graph G can be regarded as a union of trees $\{T_k\}$. Grouping together the polymers $\{X_i\}$ linked by the trees yields new disjoint polymers $\{Y_k\}$. The covariance $C_X(s)$ preserves the $\{Y_k\}$ since $\sigma_{ij}(G, s) = 0$ for blocks X_i, X_j in different trees. We can write $C_X(s) = \oplus_k C_{Y_k}(s)$. Then the integrand above factors and we have

$$\sum_{\{X_i\}} \sum_{\{T_k\}} \prod_k \left[\int ds^{T_k} \left(\partial^{T_k} \mu_{C_{Y_k}(s)} * \prod_{i:X_i \subset Y_k} K(X_i) \right)_{s=\sigma(T_k,s)} \right] . \tag{39}$$

Now we group together the terms in the sum by the $\{Y_k\}$ they determine and find

$$\sum_{\{Y_k\}} \prod_k \mathcal{F}K(Y_k) = \mathcal{E}xp(\Box + \mathcal{F}K) \tag{40}$$

where

$$\mathcal{F}K(Y) = \sum_{\{X_i\}, T \to Y} \int ds^T \left(\partial^T \mu_{C_Y(s)} * \prod_i K(X_i) \right)_{s=\sigma(T,s)} . \tag{41}$$

The result now follows since $\partial C_Y(s)/\partial s_{ij} = 2C(X_i, X_j)$ and hence $\partial/\partial s_{ij}(\mu_{C_Y(s)} * F) = \mu_{C_Y(s)} * (-2\Delta_{C(X_i,X_j)})F$ and hence

$$\partial^T \mu_{C_Y(s)} * \prod_i K(X_i) = \mu_{C_Y(s)} * \prod_{ij \in T} (-2\Delta_{C(X_i,X_j)}) \prod_i K(X_i) \ . \tag{42}$$

□

The behavior in ϕ of $\mathcal{F}K(X, \phi)$ will turn out to be slightly worse than that for $K(X, \phi)$ which means we have to take a larger large field regulator. It is convenient

[3] Our assumption that $K(X, \phi)$ has ϕ dependence localized in X means that the function is measurable with respect to Σ_X, the σ-algebra generated by $\{\phi_x\}_{x \in X}$.

to choose a regulator which is a scaling of the original. Let the field scaled up by $\ell > 1$ be defined by

$$\phi_\ell(x) = \ell^{-(d-2)/2}\phi(x/\ell) \tag{43}$$

(our convention here is different from earlier papers). Then define

$$\begin{aligned} G_\ell(\kappa, X, \phi) &= G(\kappa, \ell^{-1}X, \phi_{\ell^{-1}}) \\ &= \exp\left(\kappa \sum_{1\leq|\alpha|\leq s} \ell^{2|\alpha|-2} \int_X |\partial^\alpha\phi|^2 + \kappa c \sum_{|\alpha|=1} \ell \int_{\partial X} |\partial^\alpha\phi|^2\right). \end{aligned} \tag{44}$$

For the applications we have in mind we need $1 < \ell \leq L$: for definiteness we take $\ell = 2$ in the following.

For unit blocks Δ, Δ' define

$$C_*(\Delta, \Delta') = \|C(\Delta, \Delta')\| d(\Delta, \Delta')^{2d}\theta(\Delta, \Delta')\ . \tag{45}$$

Here the norm is the $\mathcal{C}^r$ norm in each variable, $d(\Delta, \Delta')$ is Euclidean distance between block centres, and θ is the distance function built into the tree decay factor Θ in (27). Now define

$$\|C\|_* = \sup_\Delta \sum_{\Delta'\neq\Delta} C_*(\Delta, \Delta')\ . \tag{46}$$

Theorem 4 *Let $\kappa c^{-1}L^2$ be sufficiently small. Then there is a constant γ depending only on the dimension such that if $0 < \delta h < h$ and for some p*

$$\delta h^2 \geq 8\gamma^2\ \|C\|_*\ \|K\|_{G(\kappa),h,\Gamma_{p+3}} \tag{47}$$

then

$$\|\mathcal{F}K\|_{G_\ell(\kappa),h-\delta h,\Gamma_p} \leq 2\|K\|_{G(\kappa),h,\Gamma_{p+3}}\ . \tag{48}$$

Remark. The linearization $\mathcal{F}_1 K = \mu_C * K$ satisfies the same bound (or even the better bound with $\delta h = 0$).

Proof. We adapt the analysis of [11] to our norms. In (36) change to a sum on disjoint ordered polymers $(X_1, ..., X_N)$ and regard T as a tree on $(1, ..., N)$. Then we have

$$\begin{aligned} \mathcal{F}K(X) &= \mu_C * K(X) \\ &+ \sum_{N=2}^\infty \frac{1}{N!} \sum_{(X_1,\dots,X_N)} \sum_T \int ds^T \mu_{C_X(\sigma(T,s))} * \prod_{ij\in T} \Delta_{C(X_i,X_j)} \prod_{i=1}^N K(X_i)\ . \end{aligned} \tag{49}$$

We next introduce a sum over unit blocks: for $b = \{ij\}$

$$\Delta_{C(X_i,X_j)} = \sum_{\Delta_{bi}\in X_i, \Delta_{bj}\in X_j} \Delta_{C(\Delta_{bi},\Delta_{bj})} \tag{50}$$

Taking derivatives and norms yields

$$\|(\mathcal{F}K)_n(X)\| \leq \|\mu_C * K_n(X)\| \ + \ \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{(X_1,\dots,X_N)} \sum_{T} \sum_{\{\Delta_{bi},\Delta_{bj}\}}$$
$$\sum_{n_1,\dots,n_N} \frac{n!}{n_1!\dots n_N!} \int ds^T \|\mu_{C_X(\sigma(T,s))} * \left[\prod_{b\in T} \Delta_{C(\Delta_{bi},\Delta_{bj})} \prod_{i=1}^{N} K_{n_i}(X_i) \right] \| \ . \quad (51)$$

In Lemma 5 to follow we show that for $\kappa c^{-1} L^2$ sufficiently small

$$\mu_{C_X(\sigma(T,s))} * G(\kappa, X) \leq G_\ell(\kappa, X) 2^{|X|} \quad (52)$$

which makes it possible to estimate the above convolutions. Using this and $G(\kappa, X) = \prod_i G(\kappa, X_i)$ (since the X_i are disjoint) we find (see [6] for more details)

$$\|(\mathcal{F}K)_n(X)\|_{G_\ell(\kappa)} \leq \|K_n\|_{G(\kappa)} \ + \ \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{(X_1,\dots,X_N)} \sum_{T} \sum_{\{\Delta_{bi},\Delta_{bj}\}}$$
$$\sum_{n_1,\dots,n_N} \frac{n!}{n_1!\dots n_N!} \prod_{b\in T} \|C(\Delta_{bi},\Delta_{bj})\| \prod_{i=1}^{N} \|K_{n_i+d_i}(X_i)\|_{G(\kappa)} \ 2^{|X_i|} \ . \quad (53)$$

Here d_i is the incidence number for the i^{th} vertex in the graph T.

Now multiply by $(h-\delta h)^n/n!$ and sum over n to obtain

$$\|\mathcal{F}K(X)\|_{G_\ell(\kappa),h-\delta h} \quad \leq \quad \|K_n(X)\|_{G(\kappa),h} \ + \ \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{(X_1,\dots,X_N)} \sum_{T} \sum_{\{\Delta_{bi},\Delta_{bj}\}}$$
$$\prod_{b\in T} \|C(\Delta_{bi},\Delta_{bj})\| \prod_{i=1}^{N} \Big(\frac{d}{dh}\Big)^{d_i} \|K(X_i)\|_{G(\kappa),h-\delta h} \ 2^{|X_i|} \ . \quad (54)$$

A Cauchy bound yields

$$(\frac{d}{dh})^{d_i} \|K(X_i)\|_{G(\kappa),h-\delta h} \leq \Big(\delta h\Big)^{-d_i} d_i! \|K(X_i)\|_{G(\kappa),h} \ . \quad (55)$$

It is proved in Lemma 6 to follow that for any i we have

$$d_i! \leq \gamma^{d_i} \prod_{b\ni i} d(\Delta_{bi},\Delta_{bj})^d \quad (56)$$

for some constant γ. Taking into account that $\sum_i d_i = 2N-2$ this gives

$$\prod_i d_i! \leq \gamma^{2N-2} \prod_b d(\Delta_{bi},\Delta_{bj})^{2d} \quad (57)$$

and so

$$\|\mathcal{F}K(X)\|_{G_\ell(\kappa),h-\delta h} \leq \|K(X)\|_{G(\kappa),h} + \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{(X_1,\dots,X_N)} \sum_{T} \sum_{\{\Delta_{bi},\Delta_{bj}\}}$$
$$(\gamma\ \delta h^{-1})^{2N-2} \prod_{b\in T} \|C(\Delta_{bi},\Delta_{bj})\|_\infty d(\Delta_{bi},\Delta_{bj})^{2d} \prod_{i=1}^{N} \|K(X_i)\|_{G(\kappa),h}\ 2^{|X_i|}\ . \quad (58)$$

Now multiply by

$$\Gamma_p(X) \leq \prod_i \Gamma_p(X_i) \prod_b \theta(\Delta_{bi},\Delta_{bj}) \quad (59)$$

and identify $\prod_b C_*(\Delta_{bi},\Delta_{bj})$. Next sum over $X \supset \Delta$ and dominate the expression by a sum over i_0 and a sum over unrestricted disjoint $(X_1,...,X_N)$ such that $X_{i_0} \supset \Delta$. To estimate this sum and the sum over $\{\Delta_{bi},\Delta_{bj}\}$, we start at the twigs of the tree and work inward leaving to the last the set X_{i_0} which is pinned. Suppose that when we come to a vertex i we have gained a factor $|X_i|^{d_i-1}$ from the previous estimates. If $b=\{ij\}$ is the remaining inward bond at this vertex and $\Delta=\Delta_{bi},\Delta'=\Delta_{bj}$, then we have

$$\begin{aligned}
& \sum_{X_i} \sum_{\Delta\in X_i,\Delta'\in X_j} C_*(\Delta,\Delta')\|K(X_i)\|_{G(\kappa),h}\Gamma_{p+1}(X_i)|X_i|^{d_i-1} \\
\leq\ & \sum_{X_i} \sum_{\Delta\in X_i,\Delta'\in X_j} C_*(\Delta,\Delta')\|K(X_i)\|_{G(\kappa),h}\Gamma_{p+3}(X_i)(d_i-1)! \\
\leq\ & \sum_{\Delta'\in X_j,\Delta} C_*(\Delta,\Delta')\|K\|_{G(\kappa),h,\Gamma_{p+3}}(d_i-1)! \\
\leq\ & \|C\|_*\|K\|_{G(\kappa),h,\Gamma_{p+3}}|X_j|(d_i-1)!\ . \qquad (60)
\end{aligned}$$

This gives a factor $|X_j|$ for the j vertex. The case for $i=i_0$ is special and we have $d_{i_0}! \leq (N-1)(d_{i_0}-1)!)$. There is also a factor N for the sum over i_0 and combining all the above yields

$$\|\mathcal{F}K\|_{G_\ell(\kappa),h-\delta h,\Gamma_p} \leq \|K\|_{G(\kappa),h,\Gamma_{p+3}}(1+\sum_{N=2}^{\infty}\frac{\alpha^{N-1}}{(N-2)!}\sum_T\prod_{i=1}^{N}(d_i-1)!) \quad (61)$$

where $\alpha = \gamma^2\ \delta h^{-2}\ \|C\|_*\|K\|_{G(\kappa),h,\Gamma_{p+3}}$. But the number of trees with given incidence numbers d_i is $(N-2)!/\prod_i(d_i-1)!$ by Cayley's theorem, and the number of choices of d_i is bounded by $2^{2N-2}=4^{N-1}$. Thus the sum over T is bounded by $(N-2)!4^{N-1}$. Then the sum over N is bounded by $\sum_{N=2}^{\infty}(4\alpha)^{N-1}$ and this is less that 1 since our basic assumption is $4\alpha \leq 1/2$

□

This completes the proof of the theorem, except for the following two results which we skipped.

Lemma 5 *Let $\kappa c^{-1}L^2$ be sufficiently small. Then*

$$\mu_{C_X(\sigma(T,s))} * G(\kappa, X) \leq G_\ell(\kappa, X) 2^{|X|} \ . \tag{62}$$

Proof. (see [3] for more details) Consider for $0 \leq t \leq 1$ the family of large field regulators

$$G_t(\kappa, X) = 2^{t|X|} \left[G_\ell(\kappa, X)\right]^t \left[G(\kappa, X)\right]^{1-t} \ . \tag{63}$$

We prove for $0 \leq t \leq 1$ that

$$\mu_{tC_X(\sigma(T,s))} * G_0(\kappa, X) \leq G_t(\kappa, X) \ . \tag{64}$$

The result we want comes at $t = 1$.

We have $G_t(X) = \exp(U(t, X))$ where (with $\ell = 2$)

$$U(t, X) = t\log(2)|X| + \kappa \sum_{1\leq|\alpha|\leq s} \int_X |\partial^\alpha \phi|^2 \cdot (2^{2|\alpha|-2} t + (1-t)) + \kappa c \int_{\partial X} |\partial\phi|^2 (1+t) \tag{65}$$

The bound (64) is implied by

$$\Delta_{C_X(\sigma(T,s))} U + \frac{1}{2} C_X(\sigma(T,s)) \left(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi}\right) \leq \frac{\partial U}{\partial t} \ . \tag{66}$$

Showing (66) is a somewhat lengthy computation in which every term on the left is bounded by corresponding terms on the right for κ sufficiently small. The terms with $|\alpha| = 1$ are special since there is no corresponding term on the right. Instead one integrates by parts. This adds derivatives and boundary terms both of which can be bounded.

The condition on κ turns out to be that the following quantities be sufficiently small :

$$\begin{aligned} &\kappa \sup_{1\leq|\alpha|,|\beta|\leq s} \sup_{x\in X} |(\partial_x^\alpha \partial_y^\beta C_X(\sigma(T,s)))(x,x)| \\ &\kappa c^{-1} \sup_{0\leq|\alpha|,|\beta|\leq s} \sup_{x\in X} \int_X |(\partial_x^\alpha \partial_y^\beta C_X(\sigma(T,s)))(x,y)| dy \\ &\kappa c^{-1} \sup_{0\leq|\alpha|,|\beta|\leq s} \sup_{x\in X} \int_{\partial X} |(\partial_x^\alpha \partial_y^\beta C_X(\sigma(T,s)))(x,y)| dy \ . \end{aligned} \tag{67}$$

These quantities are bounded by the corresponding quantities with $\sigma = 1$. Note from Lemma 22 in the appendix, $(\partial_x^\alpha \partial_y^\beta C)(x,x)$ is bounded by $\mathcal{O}(1)$. The second and third quantities are bounded by same expressions with $X = \Lambda$ and $X =$ the $d-1$ dimensional "checkerboard" in Λ. For both these integrals, we use Lemma 22 again and find the worst bound is $\kappa c^{-1}L^2$. Hence the result follows.

□

Lemma 6 *Let Δ and $\Delta_1, ...\Delta_n$ be distinct unit blocks. Then there is a constant γ depending only on the dimension d such that*

$$n! \leq \gamma^n \prod_{j=1}^{n} d(\Delta, \Delta_j)^d \ . \tag{68}$$

Remark. Bounds of this type were introduced in [18].

Proof. Let m_r be the number of unit blocks intersecting a ball of radius r centered on a lattice point, and select γ so $m_r \leq \gamma r^d$ for all $r > 1$. Order the blocks so that

$$d(\Delta, \Delta_1) \leq ... \leq d(\Delta, \Delta_n) \ . \tag{69}$$

Then the ball of radius $r_k = d(\Delta, \Delta_k)$ around the center of Δ intersects m_{r_k} unit blocks and $m_{r_k} \geq k$. Then $k \leq m_{r_k} \leq \gamma r_k^d$ and we have

$$n! = \prod_{k=1}^{n} k \leq \prod_{k=1}^{n} \gamma r_k^d = \gamma^n \prod_{k=1}^{n} d(\Delta, \Delta_k)^d \ . \tag{70}$$

□

II.3 Extraction

In the extraction step we remove a polymer activity F from the general activity K. Usually F is some low order terms in K but we do not assume this at first. The extraction is defined so that

$$\mathcal{E}xp(\Box + K)(\Lambda, \phi) = \exp\left(\sum_{X \subset \Lambda} F(X, \phi)\right) \ \mathcal{E}xp(\Box + \mathcal{E}(K, F))(\Lambda, \phi) \tag{71}$$

with new polymer activities $\mathcal{E}(K, F)$. To specify $\mathcal{E}(K, F)$ we define

$$\begin{aligned} \tilde{K}(X) &= K(X) - (e^F - 1)^+(X) \ , \\ (e^F - 1)^+(Y) &= \sum_{\{Y_j\} \to Y} \prod_j (e^{F(Y_j)} - 1) \end{aligned} \tag{72}$$

where the sum is over collections $\{Y_j\}$ of distinct polymers which are overlap connected and whose union is Y. Then formula (71) holds with $\mathcal{E}(K, F)$ given by

$$\mathcal{E}(K, F)(Z) = \sum_{\{X_i\}, \{Y_j\} \to Z} \prod_i \tilde{K}(X_i) \prod_j (e^{-F(Y_j)} - 1) \tag{73}$$

where the sum is over collections of disjoint subsets $\{X_i\}$ and collections of distinct subsets $\{Y_j\}$ each intersecting some X_i, so that the $\{X_i\}, \{Y_j\}$ are overlap connected and their union is Z. This version of extraction is taken from [14], to which we refer for a proof. The linearization of $\mathcal{E}(K,F)$ in K and F is $\mathcal{E}_1(K,F) = K - F$: this is the sense in which F has been removed from K.

To obtain estimates on $\mathcal{E}(K,F)$ we will need estimates like $G(X) \leq G(Z)$ when $X \subset Z$. For this to be true we have to be able to dominate δG by G' so we can "dissolve" the pieces of ∂X which do not contribute to ∂Z. This means that the constant c in δG has to be sufficiently small. Let c_s be the Sobolev constant defined so that for $x \in \Delta$, the closed unit block, we have $|\partial\phi(x)|^2 \leq c_s \sum_{1\leq|\alpha|\leq s} \int_\Delta |\partial^\alpha \phi|^2$.

Lemma 7 *For $X \subset Z$, $\kappa > 0$ and $c < (2d\ c_s)^{-1}$ we have*

$$G(\kappa, X) \leq G(\kappa, Z)\ . \tag{74}$$

If $c < (4d\ c_s)^{-1}$ the same bound holds with G replaced by G_ℓ, $\ell = 2$.

Proof. Let f be a face ($d-1$ cell) in ∂X which does not contribute to ∂Z. Any such face f must be also be a face for some Δ in $Z - X$. Then we can "dissolve" the boundary by using the Sobolev inequality and the bound on c to obtain

$$\delta G(\kappa, f) \leq G'(\kappa/2d, \Delta)\ . \tag{75}$$

Each Δ arises from at most $2d$ faces and so

$$\delta G(\kappa, \partial X - \partial Z) \leq G'(\kappa, Z - X)\ . \tag{76}$$

Thus we have

$$\begin{aligned} G(\kappa, X) &= G'(\kappa, X)\delta G(\kappa, \partial X - \partial Z)\delta G(\kappa, \partial Z \cap \partial X) \\ &\leq G'(\kappa, Z) G(\kappa, \partial Z \cap \partial X)\ . \end{aligned} \tag{77}$$

Since $\delta G(\kappa, \partial Z \cap \partial X) \leq \delta G(\kappa, \partial Z)$ the result follows.

□

We now assume F satisfies the following *localization* property: $F(X,\phi)$ has the decomposition

$$F(X,\phi) = \sum_{\Delta \subset X} F(X, \Delta, \phi) \tag{78}$$

where Δ is summed over unit blocks, and $F(X,\Delta,\phi)$ has the ϕ dependence localized in Δ.

We also need stability conditions on the perturbation F. Let $f(X)$ be a collection of constants. We say that F is *stable* for $(G, h, f(X))$ if for complex $z(X)$

$$\sup_{|z(X)| f(X) \leq 1} \| \exp\left\{ \sum_{X \supset \Delta} z(X) F(X, \Delta) \right\} \|_{G,h} \leq 2\ . \tag{79}$$

For a method to verify the stability hypothesis see the appendix.

Theorem 8 *Let* $c < (2d\ c_s)^{-1}$*. Suppose that* F *is stable for* $(G(\kappa), h, f(X))$ *and for* $(G'(\delta\kappa), h, \delta f(X))$ *and that* $\|f\|_{\Gamma_{p+4}}, \|\delta f\|_{\Gamma_{p+2}}$ *and* $\|K\|_{G(\kappa),h,\Gamma_{p+2}}$ *are sufficiently small. Then there is a constant* $\mathcal{O}(1)$ *such that*

$$\|\mathcal{E}(K,F)\|_{G(\kappa+\delta\kappa),h,\Gamma_p} \leq \mathcal{O}(1)(\|K\|_{G(\kappa),h,\Gamma_{p+2}} + \|f\|_{\Gamma_{p+4}}) \ . \tag{80}$$

For $c < (4d\ c_s)^{-1}$ *the same bound holds with each* G *replaced by* G_ℓ*,* $\ell = 2$*.*

Proof. The proof is similar to [6] where however the extraction is not global. We start with (73) which can be written

$$\mathcal{E}(K,F)(Z) = \sum_{\{X_i\},\{Y_j\}\to Z} \prod_i \tilde{K}(X_i) \prod_j \frac{1}{2\pi i} \int \frac{dz_j}{z_j(z_j-1)} \exp\{-z_j F(Y_j)\} \ . \tag{81}$$

The integral is over the circles $|z_j|\delta f(Y_j) = 1$. Inserting $F(Y) = \sum_{\Delta\subset Y} F(Y,\Delta)$ we can rewrite this as

$$\begin{aligned}&\mathcal{E}(K,F)(Z)\\ =&\sum_{\{X_i\},\{Y_j\}\to Z} \prod_i \tilde{K}(X_i) \prod_j \frac{1}{2\pi i} \int \frac{dz_j}{z_j(z_j-1)} \prod_{\Delta\subset Z} \exp\left\{-\sum_j z_j F(Y_j,\Delta)\right\}\end{aligned} \tag{82}$$

Now we note

$$\prod_i G(\kappa, X_i) \prod_{\Delta\subset Z} G'(\delta\kappa, \Delta) \leq G(\kappa+\delta\kappa, Z) \ . \tag{83}$$

This follows from $\prod_i G(\kappa, X_i) = G(\kappa, \cup_i X_i) \leq G(\kappa, Z)$ (by the lemma) and from $\prod_{\Delta\subset Z} G'(\delta\kappa,\Delta) = G'(\delta\kappa, Z) \leq G(\delta\kappa, Z)$. Using this estimate and the multiplicative property of the norm we obtain

$$\begin{aligned}\|\mathcal{E}(K,F)(Z)\|_{G(\kappa+\delta\kappa),h} \quad &\leq \quad \sum_{\{X_i\},\{Y_j\}\to Z} \prod_i \|\tilde{K}(X_i)\|_{G(\kappa),h} \prod_j \mathcal{O}(1)\delta f(Y_j)\\ &\sup_{|z_j|\delta f(Y_j)\leq 1} \prod_{\Delta\subset Z} \|\exp\left\{-\sum_j z_j F(Y_j,\Delta)\right\}\|_{G'(\delta\kappa),h} \ .\end{aligned} \tag{84}$$

By our second stability assumption the last factor is bounded by $2^{|Z|}$. Now we write

$$\sum_{\{X_i\},\{Y_j\}} = \sum_{N,M} \frac{1}{N!M!} \sum_{(X_1,\ldots,X_N),(Y_1,\ldots,Y_M)}$$

where the sum is over ordered sets, but otherwise the restrictions apply. We multiply by $\Gamma_p(Z)$, identify $2^{|Z|}\Gamma_p(Z) = \Gamma_{p+1}(Z)$ and use $\Gamma_{p+1}(Z) \leq \prod_i \Gamma_{p+1}(X_i) \prod_j \Gamma_{p+1}(Y_j)$ which follows from the overlap connectedness. Then sum over Z

with a pin, and use a spanning tree argument and the small norm hypotheses to obtain

$$\begin{aligned}\|\mathcal{E}(K,F)\|_{G(\kappa+\delta\kappa),h,\Gamma_p} &\leq \sum_{N\geq 1, M\geq 0} \frac{(N+M)!}{N!M!}(\mathcal{O}(1))^{N+M}\|\tilde{K}\|^N_{G(\kappa),h,\Gamma_{p+2}}\|\delta f\|^M_{\Gamma_{p+2}} \\ &\leq \mathcal{O}(1)\|\tilde{K}\|_{G(\kappa+\delta\kappa),h,\Gamma_{p+2}} \ . \end{aligned} \tag{85}$$

(In the last step use $(N+M)!/N!M! \leq 2^{N+M}$.)

Recall that $\tilde{K} = K - (e^F - 1)^+$. We write

$$(e^F-1)^+(Y) = \sum_{\{Y_j\}}\prod_j \frac{1}{2\pi i}\int \frac{dz_j}{z_j(z_j-1)}\exp\{z_jF(Y_j)\} \tag{86}$$

now with the integral over $|z_j|f(Y_j) = 1$. Proceeding as above and using the first stability assumption we have

$$\|(e^F-1)^+(Y)\|_{G(\kappa),h} \leq 2^{|Y|}\sum_{\{Y_j\}}\prod_j \mathcal{O}(1)f(Y_j) \tag{87}$$

and hence

$$\|(e^F-1)^+\|_{G(\kappa),h,\Gamma_{p+2}} \leq \sum_{N=1}^{\infty}(\mathcal{O}(1))^N\|f\|^N_{\Gamma_{p+4}} \leq \mathcal{O}(1)\|f\|_{\Gamma_{p+4}} \ . \tag{88}$$

This gives the result.

□

II.4 Scaling

In the scaling step we define new polymer activities $\mathcal{S}(K)$ so that

$$\mathcal{E}xp(\Box + K)(\Lambda,\phi_L) = \mathcal{E}xp(\Box + \mathcal{S}(K))(L^{-1}\Lambda,\phi) \ . \tag{89}$$

Here the scaled field is $\phi_L(x) = L^{-\alpha}\phi(x/L)$ with $\alpha = \dim\phi = (d-2)/2$. After a rearrangement one finds

$$\mathcal{S}(K)(X,\phi) = \sum_{\{Y_i\}\to LX}\prod_i K(Y_i,\phi_L) \tag{90}$$

where the Y_i are disjoint but the L-closures $\bar{Y}_i^L$ overlap and fill LX.

Theorem 9 *Let $c < (2d\ L^{d/2}\ c_s)^{-1}$ and define $h_L = L^{-\alpha}h$. For any positive p,q there is a constant $\mathcal{O}(1)$ such that*

$$\|\mathcal{S}(K)\|_{G(\kappa),h,\Gamma_p} \leq \mathcal{O}(1)L^d\|K\|_{G_L(\kappa),h_L,\Gamma_{p-q}} \tag{91}$$

provided $\|K\|_{G_L(\kappa),h_L,\Gamma_{p-q}}$ is sufficiently small.

Proof. Let $Y = \cup_i Y_i$. Since $L^{-1}Y \subset X$ we have by a generalization of Lemma 7 and the bound $c < (2d\ L^{d/2}\ c_s)^{-1}$

$$\prod_i G(\kappa, L^{-1}Y_i) = G(\kappa, L^{-1}Y) \leq G(\kappa, X)\ . \tag{92}$$

The point here is that we need the Sobolev inequality on the L^{-1} scale which means that we must replace c_s by the larger $L^{d/2}c_s$.

In the definition of $\mathcal{S}(K)$ we write $K(Y_i, \phi_L) = K_{L^{-1}}(L^{-1}Y_i, \phi)$ and by (92) and the multiplicative property of the norm we have

$$\|\mathcal{S}(K)(X)\|_{G(\kappa),h} \leq \sum_{\{Y_i\}\to LX} \prod_i \|K_{L^{-1}}(L^{-1}Y_i)\|_{G(\kappa),h}\ . \tag{93}$$

However $\|K_{L^{-1}}(L^{-1}Y)\|_{G(\kappa),h} \leq \|K(Y)\|_{G_L(\kappa),h_L}$ and so

$$\|\mathcal{S}(K)(X)\|_{G(\kappa),h} \leq \sum_{\{Y_i\}\to LX} \prod_i \|K(Y_i)\|_{G_L(\kappa),h_L}\ . \tag{94}$$

Now multiply by $\Gamma_p(X)$. By the connectedness we have $\Gamma_p(X) \leq \prod_i \Gamma_p(L^{-1}\bar{Y}_i^L)$. Furthermore we have the bound [11] for some constant $\mathcal{O}(1)$:

$$\Gamma_p(L^{-1}\bar{Y}^L) \leq \mathcal{O}(1)\Gamma_{p-q}(Y) \tag{95}$$

Summing over X with a pin and using a spanning tree argument we obtain

$$\|\mathcal{S}(K)\|_{G(\kappa),h,\Gamma_p} \leq \sum_{N=1}^{\infty} (\mathcal{O}(1)L^d\|K\|_{G_L(\kappa),h_L,\Gamma_{p-q}})^N\ . \tag{96}$$

This gives the result.

□

Remark. The linearization given by

$$\mathcal{S}_1(K)(X,\phi) = \sum_{\bar{Y}^L = LX} K(Y, \phi_L) \tag{97}$$

also satisfies the same bound.

II.5 Summary

We combine the three steps into one theorem which tells how the polymer activity changes under a single RG step. Our assumptions on the polymer activity K, the extraction F, and parameters $\kappa, \delta\kappa, h, \delta h$ are as follows:

1. $\|K\|_{G(\kappa),h,\Gamma}$ is sufficiently small.

2. The constants κ, c in $G(\kappa)$ satisfy $c \leq (2d\ L^{d/2}\ c_s)^{-1}$ and $\kappa c^{-1} L^2$ is sufficiently small.

3. The inequality $(\delta h)^2 \geq 8\gamma^2\ \|C\|_*\ \|K\|_{G(\kappa),h,\Gamma}$ holds.

4. The extraction F is stable for $(G_\ell(\kappa), h - \delta h, f(X))$ and for $(G'_\ell(\delta\kappa), h - \delta h, \delta f(X))$ with constants $f(X), \delta f(X)$ such that $\|f\|_{\Gamma_{-1}}, \|\delta f\|_{\Gamma_{-3}}$ are sufficiently small and such that $\|f\|_{\Gamma_{-1}} \leq \mathcal{O}(1)\|K\|_{G(\kappa),h,\Gamma}$.

Theorem 10 *Under the above assumptions*

$$(\mu_C * \mathcal{E}xp(\Box + K)(\Lambda))\,(\phi_L) = \exp\left(\sum_{X \subset \Lambda} F(X, \phi_L)\right)\ \mathcal{E}xp(\Box + \mathcal{R}(K, F))(L^{-1}\Lambda, \phi) \tag{98}$$

where

$$\mathcal{R}(K, F) = \mathcal{S}(\mathcal{E}(\mathcal{F}(K), F))\ . \tag{99}$$

In addition

$$\|\mathcal{R}(K, F)\|_{G(\kappa+\delta\kappa),h-\delta h,\Gamma} \leq \mathcal{O}(1) L^d \|K\|_{G(\kappa),h,\Gamma}\ . \tag{100}$$

Proof. If $K^\# = \mathcal{F}(K)$ then by conditions 2,3, Theorem 4 is applicable and so

$$(\mu_C * \mathcal{E}xp(\Box + K)(\Lambda))\,(\phi) = \mathcal{E}xp(\Box + K^\#)(\Lambda, \phi) \tag{101}$$

and

$$\|K^\#\|_{G_\ell(\kappa),h-\delta h,\Gamma_{-3}} \leq 2\|K\|_{G(\kappa),h,\Gamma}\ . \tag{102}$$

Then we extract F and we find

$$\mathcal{E}xp(\Box + K^\#)(\Lambda, \phi) = \exp\left(\sum_{X \subset \Lambda} F(X, \phi)\right) \mathcal{E}xp(\Box + K^*)(\Lambda, \phi) \tag{103}$$

where $K^* = \mathcal{E}(K^\#, F)$. The hypotheses of Theorem 8 hold for $K^\#$ and $p = -5$: one has that $\|K^\#\|_{G_\ell(\kappa),h-\delta h,\Gamma_{-3}}$ is sufficiently small by assumption 1 and (102). Therefore

$$\|K^*\|_{G_\ell(\kappa+\delta\kappa),h-\delta h,\Gamma_{-5}} \leq \mathcal{O}(1)(\|K^\#\|_{G_\ell(\kappa),h-\delta h,\Gamma_{-3}} + \|f\|_{\Gamma_{-1}}) \leq \mathcal{O}(1)\|K\|_{G(\kappa),h,\Gamma} \tag{104}$$

Finally we scale and find by Theorem 9 that

$$\mathcal{E}xp(\Box + K^*)(\Lambda, \phi_L) = \mathcal{E}xp(\Box + K')(L^{-1}\Lambda, \phi) \tag{105}$$

where $K' = \mathcal{S}(K^*) = \mathcal{R}(K, F)$, and since $\|K^*\|_{G_\ell(\kappa+\delta k),h-\delta h,\Gamma_{-5}}$ is sufficiently small we have

$$\begin{aligned} \|K'\|_{G(\kappa+\delta k),h-\delta h,\Gamma} &\leq \mathcal{O}(1) L^d \|K^*\|_{G_L(\kappa+\delta k),(h-\delta h)_L,\Gamma_{-5}} \\ &\leq \mathcal{O}(1) L^d \|K^*\|_{G_\ell(\kappa+\delta k),h-\delta h,\Gamma_{-5}} \\ &\leq \mathcal{O}(1) L^d \|K\|_{G(\kappa),h,\Gamma}\ . \end{aligned} \tag{106}$$

This completes the proof.

□

Remark. The linearization $\mathcal{R}_1(K,F) = \mathcal{S}_1\mathcal{E}_1(\mathcal{F}_1 K, F)$ satisfies the same bound.

III More estimates

The last theorem exhibits the obstruction to iterating the RG, namely the L^d growth factor. The aim in what follows is to exhibit special cases where one can beat this growth factor. There are three mechanisms which are more or less model independent: higher order terms, large sets, and scaling for small sets with extractions. A fourth mechanism is estimates on the fluctuation integral for small sets and charged polymers and is special to the two dimensional sine-Gordon model. We discuss each of these in turn.

III.1 Higher order terms

We show that if K, F are small enough then the higher order terms in $\mathcal{R}(K,F)$ are even smaller. This fact, which follows from the next proposition with $D = \mathcal{O}(L^d)$, will allow us to restrict attention to the linearized RG.

Lemma 11 *Suppose that K, F are small enough so that sK, sF satisfy the hypotheses of Theorem 10 for all complex s in the disc $|s| \leq D$ for some $D \geq 2$. Then*

$$\mathcal{R}(K,F) = \mathcal{R}_1(K,F) + \mathcal{R}_{\geq 2}(K,F) \tag{107}$$

where $\mathcal{R}_1(K,F)$ is the linearization and

$$\|\mathcal{R}_{\geq 2}(K,F)\|_{G(\kappa+\delta\kappa),h-\delta h,\Gamma} \leq \mathcal{O}(1) D^{-1} L^d \|K\|_{G(\kappa),h,\Gamma}\ . \tag{108}$$

Proof. By Theorem 10 we have that $\mathcal{R}(sK, sF)$ is well-defined for $|s| \leq D$ and satisfies

$$\|\mathcal{R}(sK,sF)\|_{G(\kappa+\delta\kappa),h-\delta h,\Gamma} \leq \mathcal{O}(1) D L^d \|K\|_{G(\kappa),h,\Gamma}\ . \tag{109}$$

Furthermore it is not difficult to see that $\mathcal{R}(sK, sF)$ is analytic in s . Expand around $s = 0$ and evaluate at $s = 1$ and obtain (107) with the remainder given by

$$\mathcal{R}_{\geq 2}(K,F) = \frac{1}{2\pi i} \oint_{|s|=D} \frac{\mathcal{R}(sK,sF)\ ds}{s^2(s-1)}\ . \tag{110}$$

Using the bound (109) and picking up an extra factor $|s^{-2}| = D^{-2}$ we have the result.

□

III.2 Large sets

We next study the linearization $\mathcal{R}_1(K,F)$ on large sets, that is on large polymers. A polymer X is called *small* if it is connected and has $|X| \le 2^d$. Otherwise it is a *large* polymer.

The following gives favourable bounds for large sets :

Lemma 12 *Let K be supported on large sets. Then for any $p,q>0$*

$$\|\mathcal{S}_1(K)\|_{G,h,\Gamma_p} \le \mathcal{O}(1)L^{-2}\|K\|_{G_L,h_L,\Gamma_{p-q}} \ . \tag{111}$$

Under the hypotheses of theorem 10 :

$$\|\mathcal{S}_1\mathcal{F}_1 K\|_{G(\kappa+\delta\kappa),h-\delta h,\Gamma} \le \mathcal{O}(1)L^{-2}\|K\|_{G(\kappa),h,\Gamma} \ . \tag{112}$$

Proof. The first bound follows by following the proof of Theorem 9 for the linear terms only, but replacing (95) by the stronger inequality

$$\Gamma_p(L^{-1}\bar{X}^L) \le \mathcal{O}(1)L^{-d-2}\Gamma_{p-q}(X) \tag{113}$$

which is valid for large sets X. This inequality is proved in [11] and [6], Lemma 1.

For the second bound we note that if K is supported on large sets then so is $\mathcal{F}_1 K$. Thus we can use the first bound followed by our bound on $\mathcal{F}_1$. □

Remark. The second bound gives a good bound on $\mathcal{R}_1(K,F) = \mathcal{S}_1\mathcal{E}_1(\mathcal{F}_1K,F)$ since we will use it in a situation where $\mathcal{E}_1(K,F) = K$ and hence $\mathcal{R}_1(K,F) = \mathcal{S}_1\mathcal{F}_1K$.

III.3 Small sets

For small sets the usual strategy would be to extract the fastest growing terms (the relevant variables) and get good bounds on the remainder. This generally works when the canonical scaling dimension of the field is positive. However in d=2 the field has dimension zero and any polynomial in the field is relevant, rendering the strategy intractable. For sine-Gordon we use the fact that the interaction is periodic under translations $\phi \to \phi + 2\pi$ in field space. This allows a Fourier analysis in this translation variable and a new contraction mechanism for the non-zero Fourier modes. The remaining zero modes depend only on $\partial\phi$ which has a positive dimension and thus these terms can be handled by extraction. We now give the details.

Let K be a polymer activity which satisfies $K(X,\phi+2\pi) = K(X,\phi)$. Expand $K(X,\Phi+\phi)$ in a Fourier series in the real variable Φ

$$K(X,\Phi+\phi) = k_0(X,\phi) + \sum_{q\neq 0} e^{iq\Phi}k_q(X,\phi) \tag{114}$$

where

$$k_q(X,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iq\Phi}K(X,\Phi+\phi)d\Phi. \tag{115}$$

Then

$$K(X,\phi) = k_0(X,\phi) + \sum_{q\neq 0} k_q(X,\phi). \tag{116}$$

The terms with $q \neq 0$ are called the *charged* terms and the $q = 0$ term is called the *neutral* term. The terminology is consistent with the Coulomb gas interpretation of the model. We sometimes also use the notation $\bar{K}(X,\phi) = k_0(X,\phi)$.

Note that for a constant shift c of the field

$$k_q(X,\phi+c) = e^{iqc}k_q(X,\phi). \tag{117}$$

Also using $G(\kappa,X,\phi) = G(\kappa,X,\phi+\Phi)$ one can show

$$\|k_q\|_{G(\kappa),h,\Gamma} \leq \|K\|_{G(\kappa),h,\Gamma}. \tag{118}$$

III.3.1 Charged sector

Now we show how in dimension two only, the charged terms exhibit significantly improved behaviour under the fluctuation step.

Lemma 13 *Let $K(X,\phi)$ be supported on small sets, and be periodic in ϕ with Fourier coefficients $k_q(X,\phi)$ as above. Then for $q \neq 0$*

$$\|\mu_C * k_q\|_{G_\ell(\kappa),h,\Gamma_{-1}} \leq m_q \; \|k_q\|_{G(\kappa),h+N_C,\Gamma} \tag{119}$$

where

$$\begin{aligned} N_C &= \sup_{X \text{ small}} \inf_{x\in X} \|C(\cdot - x) - C(0)\|_X \\ m_q &= \exp[-(|q|-1/2)C(0)]. \end{aligned} \tag{120}$$

Remark. The right side of (119) can also be bounded by $m_q\|K\|_{G(\kappa),h+N_C,\Gamma}$. Then if $k_0 = 0$ so that $K(X,\phi) = \sum_{q\neq 0} k_q(X,\phi)$ we have

$$\|\mu_C * K\|_{G_\ell(\kappa),h,\Gamma_{-1}} \leq (\sum_{q\neq 0} m_q) \; \|K\|_{G(\kappa),h+N_C,\Gamma} \leq \mathcal{O}(1)e^{-C(0)/2}\|K\|_{G(\kappa),h+N_C,\Gamma}. \tag{121}$$

In $d = 2$, $C(0) = \mathcal{O}(\log L)$, giving a significant decay factor for L large. In $d > 2$, $C(0) = \mathcal{O}(1)$, and the decay factor is not significant.

Proof. We have

$$(\mu_C * k_q)(X,\phi) = \int k_q(X,\phi+\zeta)d\mu_C(\zeta)\;. \tag{122}$$

Now let f be any function and shift the integral by $\zeta \to \zeta + i\sigma_q f$ where σ_q is the sign of q. We find our expression is

$$e^{(f,C^{-1}f)/2} \int e^{-i\sigma_q(\zeta,C^{-1}f)} k_q(X, \phi + \zeta + i\sigma_q f)\ d\mu_C(\zeta)\ . \tag{123}$$

Taking $f(y) = C(y-x)$ where x is an arbitrary point of X gives

$$e^{C(0)/2} \int e^{-i\sigma_q \zeta(x)} k_q(X, \phi + \zeta + i\sigma_q C(\cdot - x)) d\mu_C(\zeta)\ . \tag{124}$$

Now use (117) with $c = i\sigma_q C(0)$ to obtain

$$(\mu_C * k_q)(X,\phi) = m_q \int e^{-i\sigma_q\zeta(x)} k_{q,x}(X, \phi+\zeta) d\mu_C(\zeta) \tag{125}$$

where

$$k_{q,x}(X,\phi) = k_q(X, \phi + i\sigma_q(C(\cdot - x) - C(0))) \tag{126}$$

is a translation of k_q.

Taking derivatives and norms :

$$\|(\mu_C * k_q)_n(X,\phi)\| \le m_q \int \|(k_{q,x})_n(X, \phi+\zeta)\| d\mu_C(\zeta)\ . \tag{127}$$

By Lemma 5, $\mu_C * G(\kappa, X) \le G_\ell(\kappa, X) 2^{|X|}$ and hence

$$\|(\mu_C * k_q)(X)\|_{G_\ell,h} \le m_q \|k_{q,x}(X)\|_{G,h}\ 2^{|X|} \tag{128}$$

(still for any $x \in X$). Now in general we can estimate translations by

$$\|K(X, \cdot + f)\|_{G,h} \le \|K(X)\|_{G,h+\|f\|_X} \tag{129}$$

where $\|f\|_X$ is defined in (23). This can be seen by making a power series expansion in f. We apply this to $k_{q,x}$ and choose $x \in X$ to minimize $\|C(\cdot - x) - C(0)\|_X$, and find

$$\|k_{q,x}(X)\|_{G,h} \le \|k_q(X)\|_{G,h+N_C}\ . \tag{130}$$

Combining (128) and (130) gives the result.

□

Remark. The price we have paid for the strong contraction factor is a slight loss in the region of analyticity $h + N_C \to h$ or $h \to h - N_C$. Iterating this is a problem in $d = 2$ since we do not recover analyticity in the scaling step. For the UV problem this could be overcome by taking h very large at the start. However for the IR problem we just have to do better.

Lemma 14 *Let the hypotheses of Lemma 13 hold. For* $0 \le \eta \le 1$ *, and any* $p, r \ge 0$*,*

$$\|\mathcal{S}_1 k_q\|_{G(\kappa),h,\Gamma_p} \le \mathcal{O}(1) L^d e^{\eta h_L |q|}\ \|k_q\|_{G_L(\kappa),h_L(1-\eta/2),\Gamma_{p-r}}\ . \tag{131}$$

Remark. Suppose $d = 2$ so that $h_L = h$. The point of the lemma is that we have traded a slightly worse bound (the factor $e^{|q|\eta h}$) for better analyticity (the improvement from $h(1-\eta/2)$ to h). If we combine Lemma 13 and Lemma 14 with the choice $\eta = 2h^{-1}N_C$ (assumed less than 1) we find

$$\begin{aligned} \|\mathcal{S}_1\mathcal{F}_1 k_q\|_{G(\kappa),h,\Gamma} &= \|\mathcal{S}_1(\mu_C * k_q)\|_{G(\kappa),h,\Gamma} \\ &\leq \mathcal{O}(1)L^2 e^{\eta h|q|}\|(\mu_C * k_q)\|_{G_L(\kappa),h(1-\eta/2),\Gamma_{-1}} \\ &\leq \mathcal{O}(1)L^2 e^{2N_C|q|} m_q \|k_q\|_{G(\kappa),h,\Gamma} \; . \end{aligned} \tag{132}$$

Since $C(0) = \mathcal{O}(\log L)$ and $N_C = \mathcal{O}(1)$ the factor $m_q = \exp(-(|q|-1/2)C(0))$ is stronger than the factor $e^{2N_C|q|}$. Hence we have accomplished the goal of finding a strong contraction factor without losing analyticity. (Of course we still have to see if it is strong enough to beat the factor L^2.)

Before embarking on the proof of the lemma we note a preliminary result which exhibits improved scaling behavior when a function vanishes at a point.

Lemma 15 *Let Y be a small set in LX and suppose $f_L(y) = L^{-\alpha}f(x/L)$ vanishes at some point $y_* \in Y$. Then*

$$\|f_L\|_Y \leq \mathcal{O}(1)L^{-1-\alpha}\|f\|_X \; . \tag{133}$$

Proof. First observe that $\partial^\beta(f_L(x)) = L^{-|\beta|-\alpha}(\partial^\beta f)(L^{-1}x)$, so we need only look at the nonderivative term in the norm. Now note that for any $y \in Y$ the length of the shortest rectilinear path within Y from y to y_*, is less than $\mathcal{O}(1)$. Therefore since f_L vanishes at y_*

$$\begin{aligned} |f_L(y)| &= |f_L(y) - f_L(y_*)| \\ &\leq \mathcal{O}(1) \sup_{z\in Y, |\beta|=1} |\partial^\beta f_L(z)| \\ &\leq \mathcal{O}(1)L^{-1-\alpha}\; \|f\|_X \; . \end{aligned} \tag{134}$$

□

Proof (of Lemma 14). With $k'_q = \mathcal{S}_1 k_q$ we have

$$k'_q(X,\phi) = \sum_{Y:\bar{Y}^L = LX} k_q(Y,\phi_L) \tag{135}$$

where the sum is over small sets. For each term of (135) we use (117) to shift ϕ_L by a constant $\eta\phi_L(y_*)$ where y_* is an arbitrary point of Y. Then we have

$$k'_q(X,\phi) = \sum_{\bar{Y}^L = LX} e^{iq\eta\phi_L(y_*)} k_q(Y,(1-\eta)\phi_L + \eta\tilde{\phi}_L) \; . \tag{136}$$

Here we have defined $\tilde{f}(x) = f(x) - f(y^*/L)$ so that $\tilde{f}_L(y) = f_L(y) - f_L(y^*)$. Lemma 15 implies

$$\|(1-\eta)f_L + \eta\tilde{f}_L\|_Y \leq L^{-\alpha}[(1-\eta) + (\mathcal{O}(1)/L)\eta] \leq L^{-\alpha}[1-\eta/2] \tag{137}$$

whenever $\|f\|_X \leq 1$ and so when computing derivatives we obtain

$$\|(k'_q)_n(X,\phi)\| \leq \sum_{\bar{Y}^L=LX} \sum_{a+b=n} \frac{n!}{a!b!} L^{-n\alpha}(|q|\eta)^a(1-\eta/2)^b \|(k_q)_b(Y,(1-\eta)\phi_L+\eta\tilde{\phi}_L)\| \tag{138}$$

We also have by (92)

$$G_L(\kappa, Y, (1-\eta)\phi_L + \eta\tilde{\phi}_L) = G(\kappa, L^{-1}Y, \phi) \leq G(\kappa, X, \phi) \tag{139}$$

and so

$$\|(k'_q)_n(X)\|_G \leq \sum_{\bar{Y}^L=LX} \sum_{a+b=n} \frac{n!}{a!b!} L^{-n\alpha}(|q|\eta)^a(1-\eta/2)^b \|(k_q)_b(Y)\|_{G_L} \tag{140}$$

and so

$$\|k'_q(X)\|_{G,h} \leq e^{\eta h_L|q|} \sum_{\bar{Y}^L=LX} \|k_q(Y)\|_{G_L,h_L(1-\eta/2)} \ . \tag{141}$$

The rest of the proof follows as in theorem 9.

□

III.3.2 Neutral sector

Improved bounds can be arranged for general activities defined on small sets by extracting a finite number of terms characterised by low "scaling dimension". As in [6] we define the *scaling dimension* $\dim K$ of any polymer activity K by

$$\begin{aligned} \dim(K_n) &= r_n + n\dim\phi; \\ \dim(K) &= \inf_n \dim(K_n) \end{aligned} \tag{142}$$

where the infimum is taken over n such that $K_n(X,0) \neq 0$. Here r_n is defined to be the largest integer satisfying $r_n \leq r$ and $K_n(X,\phi=0;p^{\times n}) = 0$ whenever $p^{\times n} = (p_1,\ldots,p_n)$ is an n–tuple of polynomials of total degree less than r_n. One can interpret r_n as the number of derivatives present in the ϕ^n part of K (up to a maximum r).

For comparison purposes we quote the following result from [6]:

Theorem 16 *Suppose $d \geq 3$,, $K(X,\phi)$ is supported on small sets, and $\kappa h^2 \geq \mathcal{O}(1)$. Then for any $p,q \geq 0$ there is a constant $\mathcal{O}(1)$ such that*

$$\|\mathcal{S}_1(K)\|_{G,h,\Gamma_p} \leq \mathcal{O}(1)L^{d-\dim(K)}\|K\|_{G_L,h,\Gamma_{p-q}} \ . \tag{143}$$

The proof needs $\dim\phi > 0$ and fails for $d = 2$. However we can obtain a similar result for $d = 2$ if we restrict to the neutral sector.

Lemma 17 *Suppose $d = 2$, $K(X,\phi)$ is supported on small sets and satisfies the neutrality condition $K(X,\phi + c) = K(X,\phi)$ for any real c, and that $\kappa h^2 \geq \mathcal{O}(1)$. Then for any $p, q \geq 0$ there is a constant $\mathcal{O}(1)$ such that*

$$\|\mathcal{S}_1(K)\|_{G,h,\Gamma_p} \leq \mathcal{O}(1) L^{2-\dim(K)} \|K\|_{G_L,h,\Gamma_{p-q}} \ . \tag{144}$$

Remark. The neutrality condition implies $K_n(X,\phi; f_1, ..., f_n)$ vanishes if any f_i is a constant. Hence $\dim K_n = r_n \geq n$ for $n < r$ and $\dim K_n = r_n = r$ for $n \geq r$.

Proof. Starting from the definition (97) we have

$$(\mathcal{S}_1 K)_n(X,\phi) = \sum_{Y:\bar{Y}^L = LX} (K_{L^{-1}})_n(L^{-1}Y,\phi) \ . \tag{145}$$

Thus we need to estimate

$$\|(K_{L^{-1}})_n(L^{-1}Y,\phi)\| = \sup_{\|f_i\|_X \leq 1} |K_n(Y,\phi_L; f_{1,L}, ..., f_{n,L})| \ . \tag{146}$$

By the remark above the supremum can be taken over fields f_i such that $f_{i,L}$ vanishes at a point in Y. For such fields Lemma 15 applies again giving $\|f_{i,L}\|_Y \leq \mathcal{O}(1)L^{-1}\|f_i\|_X$ and it follows that

$$\|(\mathcal{S}_1 K)_n(X,\phi)\| \leq \sum_Y \|K_n(Y,\phi_L)\| (\mathcal{O}(1)L^{-1})^n \ . \tag{147}$$

We proceed as in the proof of Theorem 9, first summing only over $n \geq \dim(K)$ so we can gain a factor $L^{-\dim(K)}$. With $\dim(K) = k$ we have

$$\sum_{n \geq k} h^n/n! \|(\mathcal{S}_1 K)_n(X)\|_G \leq \mathcal{O}(1) L^{-k} \sum_Y \|K(Y)\|_{G_L,h} \ . \tag{148}$$

We do something different for derivatives K_n with $n < k$. We have the representation

$$\begin{aligned} K_n(Y,\phi_L; f_L^{\times n}) \ &= \ \sum_{m=n}^{k-1} \frac{1}{(m-n)!} K_m(Y,0;\phi_L^{\times(m-n)} \times f_L^{\times n}) \\ &+ \ \int_0^1 dt \ \frac{(1-t)^{k-n-1}}{(k-n-1)!} \ K_k(Y,t\phi_L;\phi_L^{\times(k-n)} \times f_L^{\times n}) \ . \end{aligned} \tag{149}$$

Again by the neutrality condition we can replace ϕ_L by $\tilde{\phi}_L(y) = \phi_L(y) - \phi_L(y_*)$ for some $y_* \in Y$, and similarly for f_L. Now in [6], Lemma 15, it is proved that

$$|K_n(Y,0;f_L^{\times n})| \leq (\mathcal{O}(1))^n L^{-\dim K_n} \|K_n(Y,0)\| \prod_{j=1}^{n} \|f_j\|_X \ . \tag{150}$$

Use this bound on the terms in the sum. The remainder is estimated using $\|\tilde{\phi}_L\|_Y \leq \mathcal{O}(1)L^{-1}\|\tilde{\phi}\|_X$ from Lemma 15. We find

$$\|(\mathcal{S}_1 K)_n(X,\phi)\| \leq \mathcal{O}(1)L^{-k}\sum_Y \left\{\sum_{m=n}^{k-1} \|K_m(Y,0)\| \ \|\tilde{\phi}\|_X^{m-n} + \int_0^1 \ dt \ (1-t)^{k-n-1} \ \|K_k(Y,t\phi_L)\| \ \|\tilde{\phi}\|_X^{k-n}\right\} \tag{151}$$

Now multiply by $G(\kappa,X,\phi)^{-1}$. For the remainder term we use

$$\begin{aligned} G(\kappa,X,\phi)^{-1} &= G(\kappa t^2,X,\phi)^{-1}G(\kappa(1-t^2),X,\phi)^{-1} \\ &\leq G_L(\kappa t^2,Y,\phi_L)^{-1}G(\kappa(1-t^2),X,\phi)^{-1} \end{aligned} \tag{152}$$

where we have used (92) again. We next use

$$\sup_\phi \ \|\tilde{\phi}\|_X^a \ G(\kappa(1-t^2),X,\phi)^{-1} \leq \mathcal{O}(1)(\kappa(1-t^2))^{-a/2} \ . \tag{153}$$

This is a Sobolev inequality on derivatives of order up to r and needs $s > d/2 + r$. For the zeroeth derivative we dominate $\tilde{\phi}$ by a first derivative and then use the Sobolev inequality. Here we use the fact that X is necessarily small and so has diameter $\mathcal{O}(1)$. Now the integral over t can be estimated by $\mathcal{O}(1)\|K_k(Y)\|_{G_L}$ $\kappa^{-(k-n)/2}$. The terms in the sum over m are treated similarly and we end up with

$$\|(\mathcal{S}_1 K)_n(X)\|_G \leq \mathcal{O}(1)L^{-k}\sum_Y \sum_{m=n}^{k} \|K_m(Y)\|_{G_L} \ \kappa^{-(m-n)/2} \ . \tag{154}$$

Since $\kappa^{-1/2} \leq \mathcal{O}(1)h$, this leads for $n < k$ to

$$\frac{h^n}{n!}\|(\mathcal{S}_1 K)_n(X)\|_G \leq \mathcal{O}(1)L^{-k}\sum_Y \|K(Y)\|_{G_L,h} \ . \tag{155}$$

Combining this with (148) we find

$$\|(\mathcal{S}_1 K)(X)\|_{G,h} \leq \mathcal{O}(1)L^{-k}\sum_Y \|K(Y)\|_{G_L,h} \tag{156}$$

and the result follows as before.

□

IV The infrared problem

We return to the sine-Gordon model in $d = 2$. The infrared problem for $\beta > 8\pi$ is to study the partition function

$$Z = \int \exp\left(\zeta \int_{\Lambda_M} \cos(\phi(x))dx\right) \; d\mu_{\beta v_0^M}(\phi). \tag{157}$$

in the limit $M \to \infty$. In particular we want to prove Theorem 1.

We shall use a family of polymer activity norms defined for $j = 0, 1, 2, ...$ by

$$\|K\|_j = \| \cdot \|_{G(\kappa_j),h_j,\Gamma} \tag{158}$$

where the underlying ϕ–norms in (23) are taken with $r = 4, s = 6$. The large field regulator is $G(\kappa_j)$ defined by (25) with

$$\kappa_j = \kappa_0 \left(\sum_{k=0}^{j} 2^{-k}\right) . \tag{159}$$

We choose $c = (8Lc_s)^{-1}$ and $\kappa_0 c^{-1} L^2$ sufficiently small that Lemma 5 holds (thus $\kappa_0 \leq \mathcal{O}(L^{-3})$). Note that κ_j increases slowly in j. The domain of analyticity is defined by

$$h_j = h_\infty \left(1 + \sum_{k=j+1}^{\infty} 2^{-k}\right) \tag{160}$$

with $h_\infty = \kappa_0^{-1/2}$ (so $h_\infty \geq \mathcal{O}(L^{3/2})$). Note that h_j decreases slowly in j and that $\kappa_j^{1/2} h_{j'} \geq \kappa_0^{1/2} h_\infty = 1$. Finally Γ is defined as in (27). We restate Theorem 1 as follows :

Theorem 18 *Let β be chosen from a compact subset of $(8\pi, \infty)$, let $0 < \epsilon < 1$, and let L be chosen sufficiently large. Then there is a number $\bar{\zeta}$ such that for all ζ real with $|\zeta| \leq \bar{\zeta}$ and any $0 \leq j \leq M$ the partition function has the form*

$$Z = e^{\mathcal{E}_j} \int \mathcal{E}xp(\Box + K_j)(\Lambda_{M-j}, \phi) d\mu_{\beta v_0^{M-j}(\sigma_j)}(\phi) \tag{161}$$

where the polymer activities K_j are translation invariant, and even and 2π–periodic in ϕ. They satisfy the bounds

$$\|K_j\|_j \leq \delta^j |\zeta|^{1-\epsilon} \tag{162}$$

where $\delta = \mathcal{O}(1) \max\{L^{-2}, L^{2-\beta/4\pi}\} < 1/4$. Furthermore the energy density and the field strength have the form

$$\begin{aligned} \mathcal{E}_j &= \sum_{k=0}^{j-1} \delta\mathcal{E}_k \\ \sigma_j &= \sum_{k=0}^{j-1} \delta\sigma_k \end{aligned} \tag{163}$$

and satisfy the bounds

$$\begin{aligned} |\delta\mathcal{E}_k| &\leq \mathcal{O}(1)\ \delta^k\ |\zeta|^{1-\epsilon}|\Lambda_{M-k}| \\ |\delta\sigma_k| &\leq \mathcal{O}(1)\ h_\infty^{-2}\ \delta^k\ |\zeta|^{1-\epsilon}\ . \end{aligned} \tag{164}$$

Remark. Since $\|K_j\|_\infty \leq \|K_j\|_j$ the version stated in Theorem 1 follows as well.

Proof. The proof is by induction on j. For $j = 0$ we write the interaction as a sum over unit blocks, make a Mayer expansion, and then group together into connected components to obtain

$$\exp(\sum_{\Delta\subset\Lambda_M} \zeta V(\Delta)) = \sum_{\{\Delta_i\}} \prod_i (e^{\zeta V(\Delta_i)} - 1) = \mathcal{E}xp(\Box + K_0)(\Lambda_M)\ . \tag{165}$$

Here K_0 is supported on connected polymers and is given by

$$K_0(X) = \prod_{\Delta\subset X} (e^{\zeta V(\Delta)} - 1)\ . \tag{166}$$

However by Lemma 20 in the appendix we have the estimate for $|\zeta|$ sufficiently small

$$\|e^{\zeta V(\Delta)} - 1\|_{1,h_0} \leq |\zeta|^{1-\epsilon/2}\ . \tag{167}$$

Hence $\|K_0(X)\|_{1,h_0} \leq (|\zeta|^{1-\epsilon/2})^{|X|}$ and it follows by a standard bound [13] that $\|K_0\|_0 \leq |\zeta|^{1-\epsilon}$. Thus the representation and the bound hold for $j = 0$.

Before proceeding to the general step of the induction we specify the extractions we want to make. For an expression $\mathcal{E}xp(\Box + K)(\Lambda, \phi)$, the extracted part $F = F(K)$ is taken from the neutral sector $\bar{K}(X, \phi) = (2\pi)^{-1} \int_{-\pi}^{\pi} K(X, \Phi + \phi)d\Phi$ for small sets. It is chosen satisfying $F(X, \phi+c) = F(X, \phi)$ and so that $\dim(\bar{K}-F)$ is larger than zero. In fact we want to choose F so that $\dim(\bar{K} - F) \geq 4$ (this is why we need $r = 4$). These conditions are more than sufficient to beat the factor L^2 in the scaling step. As noted earlier the neutrality condition implies $\dim(\bar{K}_n) \geq \min(n, 4)$, and hence we may take $F_n = 0$ for $n \geq 4$. Also note that $\bar{K}_n(X, 0) = 0$ for n odd, and hence we may take $F_1, F_3 = 0$. The remaining conditions are for small sets X :

$$\begin{aligned} (\bar{K} - F)_0(X, 0) &= 0 \\ (\bar{K} - F)_2(X, 0; x_\mu, x_\nu) &= 0 \\ (\bar{K} - F)_2(X, 0; x_\mu, x_\nu x_\rho) &= 0\ . \end{aligned} \tag{168}$$

If we define the extracted part by $F(X) = \sum_\Delta F(X, \Delta)$ and

$$F(X, \Delta, \phi) = \alpha^{(0)}(X) + \sum_{\mu,\nu} \alpha^{(2)}_{\mu,\nu}(X) \int_\Delta (\partial_\mu\phi)(\partial_\nu\phi) + \sum_{\mu,\nu\rho} \alpha^{(2)}_{\mu,\nu\rho}(X) \int_\Delta (\partial_\mu\phi)(\partial^2_{\nu\rho}\phi) \tag{169}$$

then the conditions (168) determine

$$\begin{aligned} \alpha^{(0)}(X) &= |X|^{-1}\ \bar{K}_0(X,0)\ 1_{\mathcal{S}}(X) \\ \alpha^{(2)}_{\mu,\nu}(X) &= (2|X|)^{-1}\bar{K}_2(X,0;x_\mu,x_\nu)\ 1_{\mathcal{S}}(X) \\ \alpha^{(2)}_{\mu,\nu\rho}(X) &= |X|^{-1}\bar{K}_2(X,0;x_\mu,x_\nu x_\rho)\ 1_{\mathcal{S}}(X) \end{aligned} \tag{170}$$

where $1_{\mathcal{S}}$ is the characteristic function of small sets. The last two equations define $F = F(K)$.

Now we continue with the induction, supposing the theorem is true for j and proving it for $j+1$. The RG applied to $\mathcal{E}xp(\Box + K_j)(\Lambda_{M-j},\phi)$ starts with a fluctuation integral with the measure $\mu_{\beta C_j}$ where

$$C_j(x-y) = v_0^{M-j}(\sigma_j, x-y) - v_0^{M-j-1}(\sigma_j,(x-y)/L)\ . \tag{171}$$

Let $\mathcal{F}_j$ be the map on polymer activities associated with this operation, so the new activities are $K_j^{\#} = \mathcal{F}_j(K_j)$. Next we extract $F_j = F(K_j^{\#})$ with coefficients α_j as specified above. Finally we scale to the volume Λ_{M-j-1}. Thus as in Theorem 10 :

$$\begin{aligned} &\left(\mu_{\beta C_j} * \mathcal{E}xp(\Box + K_j)(\Lambda_{M-j})\right)(\phi_L) \\ &= \exp\left(\sum_{X\subset\Lambda_{M-j}} F_j(X,\phi_L)\right)\mathcal{E}xp(\Box + K_{j+1})(\Lambda_{M-j-1},\phi) \end{aligned} \tag{172}$$

where

$$K_{j+1} = \mathcal{R}_j(K_j) \equiv \mathcal{S}(\mathcal{E}(K_j^{\#}, F(K_j^{\#}))\ . \tag{173}$$

Using the lattice invariances one can prove that

$$\begin{aligned} \sum_{X\supset\Delta}\alpha_j^{(0)}(X) &= \delta E_j \\ \sum_{X\supset\Delta}\alpha^{(2)}_{j,\mu,\nu}(X) &= -(2\beta)^{-1}\ \delta_{\mu\nu}\ \delta\sigma_j \\ \sum_{X\supset\Delta}\alpha^{(2)}_{j,\mu,\nu\rho}(X) &= 0 \end{aligned} \tag{174}$$

for some constants $\delta E_j, \delta\sigma_j$. Now (172) becomes

$$(\mu_{\beta C_j} * \mathcal{E}xp(\Box + K_j)(\Lambda_{M-j}))(\phi_L) \tag{175}$$

$$= \exp\left(\delta E_j|\Lambda_{M-j}| - \frac{\delta\sigma_j}{2\beta}\int_{\Lambda_{M-j-1}}(\partial\phi)^2\right)\mathcal{E}xp(\Box + K_{j+1})(\Lambda_{M-j-1},\phi). \tag{176}$$

The partition function Z is the integral of this with respect to $\mu_{\beta v_0^{M-j-1}(\sigma_j)}$. Absorbing the $\int(\partial\phi)^2$ term into this measure changes $v(\sigma_j)$ to $v(\sigma_{j+1})$ with

$$\sigma_{j+1} = \sigma_j + \delta\sigma_j \tag{177}$$

and we have

$$Z = e^{\mathcal{E}_{j+1}} \int \mathcal{E}xp(\Box + K_{j+1})(\Lambda_{M-j-1}, \phi) d\mu_{\beta v_0^{M-j-1}(\sigma_{j+1})}(\phi) \tag{178}$$

where

$$\mathcal{E}_{j+1} = \mathcal{E}_j + \delta E_j |\Lambda_{M-j}| + \log \left[\int \exp \left(\frac{-\delta\sigma_j}{2\beta} \int_{\Lambda_{M-j-1}} (\partial\phi)^2 \right) d\mu_{\beta v_0^{M-j-1}(\sigma_j)}(\phi) \right] \tag{179}$$

This establishes the required form (161) for $j+1$.

Theorem 10 will be used to obtain a crude bound on $\|K_{j+1}\|_{j+1}$. With $\delta h_j = h_j - h_{j+1} = 2^{-j-1}h_\infty$ and $\delta\kappa_j = \kappa_{j+1} - \kappa_j = 2^{-j-1}\kappa_0$ we check the hypotheses of this theorem.

1. This is true by the inductive assumption on K_j for $\bar{\zeta}$ sufficiently small.
2. True by our choice of κ_0, c.
3. First note from Lemma 23 in the appendix that $\|\beta C_j\|_*$ is bounded uniformly in j. Also $\delta^j(\delta h_j)^{-2}$ is bounded uniformly in j for L sufficiently large, and therefore
$$\|K_j\|_j \le \delta^j |\zeta|^{1-\epsilon} \le (8\gamma^2 \|\beta C_j\|_*)^{-1} (\delta h_j)^2 \tag{180}$$
holds for all j provided $\bar{\zeta}$ is small enough.
4. The stability conditions will be verified by using Lemma 21 in the appendix which involves
$$\|\alpha(X)\|_a = |\alpha^{(0)}(X)| + a^2 \sum_{\mu\nu} |\alpha^{(2)}_{\mu\nu}(X)| + a^2 \sum_{\mu\nu\rho} |(\alpha^{(2)}_{\mu\nu\rho}(X)| \,. \tag{181}$$
By this lemma F_j is stable for $(G'_\ell(\kappa_j), h_{j+1}, f_j(X))$ if we take the definition $f_j(X) = \mathcal{O}(1)\|\alpha_j(X)\|_{h_{j+1}}$. We need $\|f_j\|_{\Gamma_{-1}}$ small and $\|f_j\|_{\Gamma_{-1}} \le \mathcal{O}(1)\|K_j\|_j$ and it suffices to show the latter. Now in the definition of $\alpha_j(X)$ replace x by $x - x_*$ where x_* is some point in X. Then we obtain the estimates for $n = 0, 2$
$$|\alpha_j^{(n)}(X)| \le \mathcal{O}(1)\|\bar{K}_n^{\#}(X,0)\| \le \mathcal{O}(1)\|K_n^{\#}(X,0)\| \le \mathcal{O}(1)\|K_n^{\#}(X)\|_{G_\ell(\kappa_j)}. \tag{182}$$
It follows that
$$\|\alpha_j(X)\|_{h_{j+1}} \le \mathcal{O}(1)\|K^{\#}(X)\|_{G_\ell(\kappa_j), h_{j+1}} \,. \tag{183}$$

and hence

$$\|f_j\|_{\Gamma_{-3}} \leq \mathcal{O}(1)\|K^{\#}\|_{G_\ell(\kappa_j),h_{j+1},\Gamma_{-3}} \leq \mathcal{O}(1)\|K_j\|_j \tag{184}$$

Since f is supported on small sets the same bound holds for $\|f_j\|_{\Gamma_{-1}}$.

Lemma 21 also says that F is stable for $(G_\ell(\delta\kappa_j), h_{j+1}, \delta f_j(X))$ if we define $\delta f_j(X) = \mathcal{O}(1)\|\alpha_j(X)\|_{\delta\kappa_j^{-1/2}}$. We must show that $\|\delta f_j\|_{\Gamma_{-3}}$ is sufficiently small under our hypotheses. We have that $1 \leq \delta\kappa_j^{-1}h_{j+1}^{-2} \leq 2^{j+1}$ and hence $|\delta f_j(X)| \leq \mathcal{O}(1)2^j|f_j(X)|$. Therefore

$$|\delta f_j|_{\Gamma_{-3}} \leq \mathcal{O}(1)2^j|f_j|_{\Gamma_{-3}} \leq \mathcal{O}(1)2^j\|K_j\|_j \leq \mathcal{O}(1)(2\delta)^j|\zeta|^{1-\epsilon} \tag{185}$$

which is small for ζ small.

This verifies the hypotheses of Theorem 10 and we conclude

$$\|K_{j+1}\|_{j+1} = \|R_j(K_j)\|_{j+1} \leq \mathcal{O}(1)L^2\|K_j\|_j \ . \tag{186}$$

It remains to improve the crude bound on K_{j+1} to $\|K_{j+1}\|_{j+1} \leq \delta\|K_j\|_j$ so we get the required $\|K_{j+1}\|_{j+1} \leq \delta^{j+1}|\zeta|^{1-\epsilon}$. To accomplish this let $1_{\mathcal{S}}$ (respectively $1_{\bar{\mathcal{S}}}$) be the characteristic function of small (large) sets, write $K_j = \sum_q k_q$ as in (116), and make the decomposition

$$K_{j+1} = \mathcal{R}_{\geq 2}(K_j) + \mathcal{R}_1(K_j 1_{\bar{\mathcal{S}}}) + \mathcal{R}_1(\sum_{q\neq 0} k_q 1_{\mathcal{S}}) + \mathcal{R}_1(k_0 1_{\mathcal{S}}) \ . \tag{187}$$

We will show that each of the four terms on the right can be bounded by $(\delta/4)\|K_j\|_j$.

1. As above one can check that Theorem 10 holds for sK_j, sF_j with $|s| \leq L^4$. Then by Lemma 11 with $D = L^4$

$$\|\mathcal{R}_{\geq 2}(K_j)\|_{j+1} \leq \mathcal{O}(1)L^{-2}\|K_j\|_j \leq \frac{\delta}{4}\|K_j\|_j \ . \tag{188}$$

2. The extraction is zero on large sets and so by Lemma 12

$$\|\mathcal{R}_1(K_j 1_{\bar{\mathcal{S}}})\|_{j+1} = \|\mathcal{S}_1\mathcal{F}_1(K_j 1_{\bar{\mathcal{S}}})\|_{j+1} \leq \mathcal{O}(1)L^{-2}\|K_j\|_j \leq \frac{\delta}{4}\|K_j\|_j \ . \tag{189}$$

3. There is no extraction in $\mathcal{R}_1(k_q 1_{\mathcal{S}})$ since the extraction is based on $\overline{\mathcal{F}_1(k_q 1_{\mathcal{S}})} = \mathcal{F}_1(\bar{k}_q 1_{\mathcal{S}}) = 0$. Hence the third term is $\sum_{q\neq 0}\mathcal{S}_1\mathcal{F}_1(k_q\ 1_{\mathcal{S}})$ which we bound by putting together Lemmas 13, 14. As in (132) we have

$$\|\mathcal{S}_1\mathcal{F}_1(k_q 1_{\mathcal{S}})\|_{j+1} \leq \mathcal{O}(1)L^2 e^{2N_{\beta C_j}|q|}e^{-(|q|-1/2)\beta C_j(0)}\|k_q\|_j \ . \tag{190}$$

However by estimates on C_j in Lemma 23 in the Appendix we have

$$N_{\beta C_j} \leq \beta \|\partial C_j\|_\infty \leq \mathcal{O}(1) \tag{191}$$

and

$$C_j(0) = \frac{\log L}{2\pi(1+\sigma_j)} + \mathcal{O}(e^{-L^{M-j-1}/2}). \tag{192}$$

Using also $\|k_q\|_j \leq \|K_j\|_j$ and the bound on σ_j we have for L sufficiently large :

$$\begin{aligned} \|\mathcal{R}_1\left(\sum_{q\neq 0} k_q 1_{\mathcal{S}}\right)\|_{j+1} &\leq \mathcal{O}(1)L^2 \sum_{q\neq 0}\left(e^{-|q|(\beta C_j(0)-2N_{\beta C_j})+\beta C_j(0)/2}\right)\|K_j\|_j \\ &\leq \mathcal{O}(1)L^{2-\beta/4\pi}\|K_j\|_j \\ &\leq \frac{\delta}{4}\|K_j\|_j \ . \end{aligned} \tag{193}$$

4. This term has the desired bound because of the extraction. Let $K^\dagger = \mathcal{F}_1(k_0 1_{\mathcal{S}})$. Then we have $\mathcal{R}_1(k_0 1_{\mathcal{S}}) = \mathcal{S}_1(K^\dagger - F(K^\dagger))$. The extraction F is defined so that $\dim(\bar{K}^\dagger - F(K^\dagger)) \geq 4$, but we have $\bar{K}^\dagger = K^\dagger$ (since the same is true of k_0) and hence $\dim(K^\dagger - F(K^\dagger)) \geq 4$. Then Lemma 17 applies (note $\kappa_{j+1}h_{j+1}^2 \geq 1$) and gives

$$\|\mathcal{R}_1(k_0 1_{\mathcal{S}})\|_{j+1} \leq \mathcal{O}(1)L^{-2}\|K^\dagger - F(K^\dagger)\|_{G_\ell(\kappa_{j+1}),h_{j+1},\Gamma_{-3}} \ . \tag{194}$$

Now $\|K^\dagger\|_{G_\ell(\kappa_{j+1}),h_{j+1},\Gamma_{-3}} \leq \mathcal{O}(1)\|K_j\|_j$. Furthermore the same bound holds for $F(K^\dagger)$. To see this extend the argument of Lemma 21 in the appendix. If $\alpha^\dagger$ is defined from $K^\dagger$ we argue as in (239) and (183) and find

$$\begin{aligned} \|(F(K^\dagger))(X)\|_{G_\ell(\kappa_{j+1}),h_{j+1}} &\leq \mathcal{O}(1)\ \|\alpha^\dagger(X)\|_{h_{j+1}} \\ &\leq \mathcal{O}(1)\|K^\dagger(X)\|_{G_\ell(\kappa_{j+1}),h_{j+1}} \end{aligned} \tag{195}$$

which is enough. Thus

$$\|\mathcal{R}_1(k_0 1_{\mathcal{S}})\|_{j+1} \leq \mathcal{O}(1)L^{-2}\|K_j\|_j \leq \frac{\delta}{4}\|K_j\|_j \ . \tag{196}$$

This completes the bound on $\|K_{j+1}\|_{j+1}$. The last step is to establish the bounds (164). Using (182) we have

$$\begin{aligned} |\delta E_j| &\leq \mathcal{O}(1)\|K_0^\#\|_{G_\ell(\kappa_j),\Gamma_{-3}} \leq \mathcal{O}(1)\|K_j\|_j \leq \mathcal{O}(1)\delta^j|\zeta|^{1-\epsilon} \\ |\delta\sigma_j| &\leq \mathcal{O}(1)\beta\|K_2^\#\|_{G_\ell(\kappa_j),\Gamma_{-3}} \leq \mathcal{O}(1)h_{j+1}^{-2}\|K_j\|_j \leq \mathcal{O}(1)h_\infty^{-2}\delta^j|\zeta|^{1-\epsilon} \end{aligned} \tag{197}$$

We also need to bound $\delta\mathcal{E}_j$. Let $v = v_0^{M-j-1}(\sigma_j)$ and let $T = v^{1/2}\Delta v^{1/2}$, a positive self-adjoint operator. Doing the integral in (179) we find

$$\begin{aligned} \delta\mathcal{E}_j &= \delta E_j|\Lambda_{M-j}| + \log\left(\det(1+\delta\sigma_j T)^{-1/2}\right) \\ &= \delta E_j|\Lambda_{M-j}| - \frac{1}{2}tr\left(\log(1+\delta\sigma_j T)\right) . \end{aligned} \tag{198}$$

But $\|T\| \leq 2$ and $|\delta\sigma_j|$ is small so the spectrum of $\delta\sigma_j T$ is confined to a small neighborhood of the origin. Hence $|\log(1+\lambda)| \leq \mathcal{O}(1)|\lambda|$ for any eigenvalue λ and hence

$$|tr\left(\log(1+\delta\sigma_j T)\right)| \leq \mathcal{O}(1)tr(|\delta\sigma_j T|) = \mathcal{O}(1)|\delta\sigma_j|tr(T) \leq \mathcal{O}(1)|\delta\sigma_j||\Lambda^{M-j-1}| \tag{199}$$

where the last step is an explicit computation. Now the bounds on δE_j and $\delta\sigma_j$ yield the bound $|\delta\mathcal{E}_j| \leq \mathcal{O}(1)\delta^j|\zeta|^{1-\epsilon}|\Lambda_{M-j}|$. This completes the proof of the infrared theorem.

□

V The ultraviolet problem

The ultraviolet problem on the unit torus Λ_0 for $\beta < 8\pi$ is equivalent to a scaling limit for unit cutoff theories. Thus we study the $N \to \infty$ limit of the partition function

$$Z = \int \exp\left(\zeta_{-N}\int_{\Lambda_N}\cos(\phi(x))dx\right) d\mu_{\beta v_0^N}(\phi). \tag{200}$$

After a number of RG transformations the RG index will increase from $-N$ to a value $j \leq 0$ and we will be on a volume $\Lambda_{|j|}$ with a coupling constant which will have grown from the ultra small ζ_{-N} to

$$\zeta_j = L^{-2|j|}e^{\beta v_0^{|j|}(0)/2}\zeta. \tag{201}$$

At this point polymer activities are estimated with a norm essentially the same as for the IR problem, but with relaxed smoothness in ϕ characterized by $r=2, s=4$ in (23). We take

$$\|\cdot\|_j = \|\cdot\|_{G(\kappa),h_j,\Gamma} \tag{202}$$

with $c = (8Lc_s)^{-1}$, $\kappa = \mathcal{O}(L^{-3})$ sufficiently small so that Lemma 5 holds, and

$$h_j = h_0\left[1 + \sum_{k=1}^{|j|}2^{-k}\right] \tag{203}$$

(which decreases in j) with $h_0 = \kappa^{-1/2} = \mathcal{O}(L^{3/2})$, and Γ as in (27).

Our aim is now to prove the following refinement of Theorem 2.

Theorem 19 *Let β be chosen from a compact subset of $(0, 8\pi)$, let $0 < \epsilon < 1/4$, and let L be chosen sufficiently large. Then there is a number $\bar{\zeta}$ such that for all ζ complex with $|\zeta| \leq \bar{\zeta}$ and any $-N \leq j \leq 0$ the partition function has the form*

$$Z = e^{\mathcal{E}_j} \int \mathcal{E}xp(\Box + K_j)(\Lambda_{|j|}, \phi) d\mu_{\beta v_0^{|j|}}(\phi) \; . \tag{204}$$

The polymer activities K_j are translation invariant, even and 2π–periodic in ϕ, analytic in ζ and have the form

$$K_j = \zeta_j V + \tilde{K}_j \tag{205}$$

where V is given by (18). We have the estimates

$$\begin{aligned} \|\zeta_j V\|_j &\leq |\zeta_j|^{1-\epsilon} \\ \|\tilde{K}_j\|_j &\leq |\zeta_j|^{2-4\epsilon} \; . \end{aligned} \tag{206}$$

Furthermore, the energy density has the form

$$\mathcal{E}_j = \sum_{k=-N}^{j-1} \delta E_k |\Lambda_{|k|}| \tag{207}$$

where

$$|\delta E_k| \leq \mathcal{O}(1) |\zeta_k|^{2-4\epsilon} \; . \tag{208}$$

Proof. The proof is by induction on $j = -N, ... - 1$. For $j = -N$ the initial density can be written just as in the IR problem as

$$\exp(\zeta_{-N} \int_{\Lambda_N} \cos\phi) = \mathcal{E}xp(\Box + K_{-N})(\Lambda_N, \phi) \tag{209}$$

where K_{-N} is supported on connected polymers and given by

$$K_{-N}(X) = \prod_{\Delta \subset X} (e^{\zeta_{-N} V(\Delta)} - 1) \; . \tag{210}$$

If $X = \Delta$ we write $K_{-N}(\Delta) = \zeta_{-N} V(\Delta) + \tilde{K}_{-N}(\Delta)$ where

$$\tilde{K}_{-N}(\Delta) = e^{\zeta_{-N} V(\Delta)} - \zeta_{-N} V(\Delta) - 1 \; . \tag{211}$$

The bound $\|\tilde{K}_{-N}(\Delta)\|_{1,h_{-N}} \leq |\zeta_{-N}|^{2-\epsilon}$ now follows from Lemma 20 in the appendix. Also for $|X| \geq 2$ we have $\|\tilde{K}_{-N}(X)\|_{1,h_{-N}} = \|K_{-N}(X)\|_{1,h_{-N}} \leq \mathcal{O}(1)(|\zeta_{-N}|^{1-\epsilon})^{|X|}$. From these two bounds we can deduce that for $j = -N$

$$\|\tilde{K}_{-N}(X)\|_{G(\kappa), h_{-N}, \Gamma} = |\zeta_{-N}|^{2-2\epsilon} \; . \tag{212}$$

Thus the theorem is established for $j = -N$.

Next we specify the extractions $F(K)$ from a polymer activity K in the general step. Again the extraction is from the neutral part on small sets, but now we only need $\dim(\bar{K} - F(K)) \geq 2$. Thus we extract only the constant

$$(F(K))(X) = \alpha(X)|X| = \bar{K}(X,0)\ 1_{\mathcal{S}}(X)\ . \tag{213}$$

Now we suppose the theorem is true for j and prove it for $j+1$. Starting with $\mathcal{E}xp(\Box + K_j)(\Lambda_{|j|}, \phi)$ we do a fluctuation integral with the measure $\mu_{\beta C_j}$ where

$$C_j(x-y) = v_0^{|j|}(0, x-y) - v_0^{|j+1|}(0, (x-y)/L). \tag{214}$$

This produces new polymer activities $K_j^{\#} = \mathcal{F}(K_j)$. Then we extract $F_j(X) = \alpha_j(X)|X| = \bar{K}_j^{\#}(X,0)1_{\mathcal{S}}$ as above. Finally we scale down to the volume $\Lambda_{|j+1|}$. Thus we have as in Theorem 10

$$\begin{aligned} & (\mu_{\beta C} * \mathcal{E}xp(\Box + K_j)(\Lambda_{|j|}))(\phi_L) \\ = \quad & \exp\left(\sum_{X \subset \Lambda_{|j|}} F_j(X)\right) \mathcal{E}xp(\Box + K_{j+1})(\Lambda_{|j+1|}, \phi) \\ = \quad & \exp\left(\delta E_j |\Lambda_{|j|}|\right) \mathcal{E}xp(\Box + K_{j+1})(\Lambda_{|j+1|}, \phi) \end{aligned} \tag{215}$$

where

$$\begin{aligned} \delta E_j &= \sum_{X \supset \Delta} \alpha_j(X) \\ K_{j+1} &= \mathcal{R}(K_j) = \mathcal{S}(\mathcal{E}(K_j^{\#}, F(K_j^{\#})))\ . \end{aligned} \tag{216}$$

The partition function is obtained from (215) by multiplying by $e^{\mathcal{E}_j}$ and integrating with respect to $\mu_{\beta v_0^{|j+1|}}$ and has the required form

$$Z = e^{\mathcal{E}_{j+1}} \int \mathcal{E}xp(\Box + K_{j+1})(\Lambda_{|j+1|}, \phi) d\mu_{\beta v_0^{|j+1|}}(\phi) \tag{217}$$

where

$$\mathcal{E}_{j+1} = \mathcal{E}_j + \delta E_j |\Lambda_{|j|}|. \tag{218}$$

Next we check the hypotheses of Theorem 10 with $\delta h_j = h_j - h_{j+1}$ and $\delta\kappa = 0$. It is easier than before since only constants are extracted. A degenerate version of Lemma 21 with $\alpha^{(2)} = 0$ implies that F_j is stable for $(G_\ell(\kappa), h_{j+1}, f(X))$ and $(1, h_{j+1}, \delta f(X))$ with $f(X) = \delta f(X) = \mathcal{O}(1)|\alpha_j(X)|$. Since $|\alpha_j|_{\Gamma_{-3}} \leq \mathcal{O}(1)\|K_j\|_j$ is certainly small enough, the stability assumption of Theorem 10 holds. The other assumptions are easily checked and we conclude

$$\|K_{j+1}\|_{j+1} \leq \mathcal{O}(1) L^2 \|K_j\|_j\ . \tag{219}$$

The leading behaviour of the RG is given by noting that

$$\mathcal{R}_1(\zeta_j V) = \zeta_{j+1} V \ . \tag{220}$$

Indeed simple computations give $\mathcal{F}_1(V) = e^{-\beta C(0)/2} V$ and $\mathcal{E}_1(V, F(V)) = V - F(V) = V$ and $\mathcal{S}_1 V = L^2 V$. Thus $\mathcal{R}_1(\zeta_j V) = L^2 e^{-\beta C(0)/2} \zeta_j V$ and since $L^2 e^{-\beta C(0)/2} \zeta_j = \zeta_{j+1}$ the claim is verified. Because of this we now have :

$$\tilde{K}_{j+1} = \mathcal{R}_1(\tilde{K}_j) + \mathcal{R}_{\geq 2}(K_j) \ . \tag{221}$$

If we expand $\tilde{K}_j = \sum_q k_q$ on small sets as before this can be written

$$\tilde{K}_{j+1} = \mathcal{R}_{\geq 2}(K_j) + \mathcal{R}_1(\tilde{K}_j 1_{\bar{\mathcal{S}}}) + \mathcal{R}_1(\sum_{q \neq 0} k_q 1_{\mathcal{S}}) + \mathcal{R}_1(k_0 1_{\mathcal{S}}) \ . \tag{222}$$

To show $\|\tilde{K}_{j+1}\|_{j+1} \leq |\zeta_{j+1}|^{2-4\epsilon}$ we show that each of the four terms on the right of (222) can be bounded by $|\zeta_{j+1}|^{2-4\epsilon}/4$.

1. One checks that Theorem 10 holds for sK_j, sF_j with $s \leq |\zeta_j|^{-1+2\epsilon}$. Then by Lemma 11 with $D = |\zeta_j|^{-1+2\epsilon}$ we have

$$\|\mathcal{R}_{\geq 2}(K_j)\|_{j+1} \leq \mathcal{O}(1) L^2 |\zeta_j|^{1-2\epsilon} \|K_j\|_j \leq |\zeta_j|^{2-4\epsilon}/4 \ . \tag{223}$$

2. No extractions are taken from large sets so $\mathcal{R}_1(\tilde{K}_j 1_{\bar{\mathcal{S}}}) = \mathcal{S}_1 \mathcal{F}_1(\tilde{K}_j 1_{\bar{\mathcal{S}}})$. Therefore we can use Lemma 12 and find

$$\|\mathcal{R}_1(\tilde{K}_j 1_{\bar{\mathcal{S}}})\|_{j+1} \leq \mathcal{O}(1) L^{-2} \|\tilde{K}_j 1_{\bar{\mathcal{S}}}\|_j \leq |\zeta_j|^{2-4\epsilon}/4 \leq |\zeta_{j+1}|^{2-4\epsilon}/4 \ . \tag{224}$$

3. The third term is bounded using Lemmas 13,14 just as in the infrared section, and we gain a factor $e^{-\beta C(0)/2 + 2N_{\beta C}} = \mathcal{O}(1) L^{-\beta/4\pi}$. We have

$$\begin{aligned} \|\mathcal{R}_1\left(\sum_{q \neq 0} k_q\right) 1_{\mathcal{S}}\|_{j+1} &\leq \mathcal{O}(1) L^2 \sum_{q \neq 0} \left(e^{-|q|(\beta C(0) - 2N_{\beta C}) + \beta C(0)/2}\right) \|\tilde{K}_j\|_j \\ &\leq \mathcal{O}(1) L^{2-\beta/4\pi} |\zeta_j|^{2-4\epsilon} \\ &\leq \mathcal{O}(1) L^{(2-\beta/4\pi)(1-(2-4\epsilon))} |\zeta_{j+1}|^{2-4\epsilon} \\ &\leq |\zeta_{j+1}|^{2-4\epsilon}/4 \ . \end{aligned} \tag{225}$$

Here we have used $|\zeta_j| \leq \mathcal{O}(1) L^{-(2-\beta/4\pi)} |\zeta_{j+1}|$.

4. Letting $K^\dagger = \mathcal{F}_1(k_0 1_{\mathcal{S}})$ we have $\mathcal{R}_1(k_0 1_{\mathcal{S}}) = \mathcal{S}_1(K^\dagger - F(K^\dagger))$. The extraction F is now defined so that $\dim(K^\dagger - F(K^\dagger)) \geq 2$ and Lemma 17 gives

$$\|\mathcal{R}_1(k_0 1_{\mathcal{S}})\|_{j+1} \leq \mathcal{O}(1) \|K^\dagger - F(K^\dagger)\|_{G_\ell, h_{j+1}, \Gamma_{-3}} \ . \tag{226}$$

This is bounded by $\mathcal{O}(1)\ \|K^\dagger\|_{G_\ell, h_{j+1}, \Gamma_{-3}} \leq \mathcal{O}(1) \|\tilde{K}_j\|_j$ and thus

$$\|\mathcal{R}_1(k_0 1_{\mathcal{S}})\|_{j+1} \leq \mathcal{O}(1) |\zeta_j|^{2-\epsilon} \leq |\zeta_{j+1}|^{2-\epsilon}/4 \ . \tag{227}$$

Now the bound on $\|\tilde{K}_{j+1}\|_{j+1}$ is complete. Next we need the bound on δE_j. We have as before

$$|\delta E_j| \leq \mathcal{O}(1)\|K_j^{\#}\|_{G_\ell,h_{j+1},\Gamma_{-3}} \leq \mathcal{O}(1)\|K_j\|_j \leq \mathcal{O}(1)|\zeta_j|^{1-\epsilon} . \tag{228}$$

But we are claiming more, namely that the bound is actually $\mathcal{O}(1)|\zeta_j|^{2-4\epsilon}$. To see the improvement note that δE_j depends on $\bar{K}_j^{\#}$ where $K_j^{\#} = \mathcal{F}(K_j) = \mathcal{F}_1(K_j) + \mathcal{F}_{\geq 2}(K_j)$ Since $\overline{\mathcal{F}_1(K_j)} = \mathcal{F}_1(\bar{K}_j)$ and since $\bar{V} = 0$ this term only depends of $\tilde{K}_j$. Thus both terms are $\mathcal{O}(|\zeta_j|^{2-4\epsilon})$. We omit the details of this estimate.

The analyticity of $K_j(X,\phi)$ in ζ follows by observing that $K_{-N}(X,\phi)$ is analytic for complex $|\zeta| \leq \bar{\zeta}$ and that each RG transformation preserves this property. This completes the proof of the ultraviolet theorem.

□

A Appendix

A.1 Estimates on potentials

Lemma 20 *Let $V(\Delta,\phi) = \int_\Delta \cos(\phi(x))dx$ for a unit block Δ. Then for any complex ζ.*

$$\begin{aligned} \|V(\Delta)\|_{G=1,h} &\leq e^h \\ \|e^{\zeta V(\Delta)}\|_{G=1,h} &\leq 2\exp(|\zeta|e^{2h}) . \end{aligned} \tag{229}$$

Furthermore for $0 < \epsilon < 1$ and $|\zeta|$ sufficiently small (depending on h,ϵ)

$$\begin{aligned} \|e^{\zeta V(\Delta)} - 1\|_{G=1,h} &\leq |\zeta|^{1-\epsilon} \\ \|e^{\zeta V(\Delta)} - \zeta V(\Delta) - 1\|_{G=1,h} &\leq |\zeta|^{2-\epsilon} . \end{aligned} \tag{230}$$

Proof. A computation shows that $\|V_n(\Delta,\phi)\| \leq 1$ and the first bound follows. For the second bound we compute the n^{th} derivative and resum as in [5] and find

$$\frac{(2h)^n}{n!}\|(e^{\zeta V(\Delta)})_n(\phi)\| \leq \exp\left(\sum_{n=0}^{\infty}\frac{(2h)^n}{n!}|\zeta|\|V_n(\Delta,\phi)\|\right) . \tag{231}$$

Again we use $\|V_n(\Delta,\phi)\| \leq 1$ and then take the supremum over ϕ to obtain

$$\frac{(2h)^n}{n!}\|(e^{\zeta V(\Delta)})_n\|_{G=1} \leq \exp\left(|\zeta|e^{2h}\right) . \tag{232}$$

Now multiply by 2^{-n} and sum over n to get the result.

For the third bound we write

$$e^{\zeta V(\Delta)} - 1 = \frac{1}{2\pi i}\int \frac{e^{z\zeta V(\Delta)}}{z(z-1)}dz \tag{233}$$

where the contour is the circle $|z| = |\zeta|^{-1+\epsilon/2} \geq 2$. Since $\|e^{z\zeta V(\Delta)}\|_{1,h} \leq \mathcal{O}(1)$ for $|\zeta|$ small by the second bound we get a bound $\mathcal{O}(1)|\zeta|^{1-\epsilon/2} \leq |\zeta|^{1-\epsilon}$. The fourth bound is similar. This completes the proof.

□

The next lemma is useful in verifying the stability hypothesis. Fix a unit square Δ and consider a family of quadratic polynomials $F(X,\Delta)$ defined for small sets $X \supset \Delta$ which have the form

$$F(X,\Delta) = \alpha^{(0)}(X) + \sum_{1\leq |a|,|b|\leq r} \alpha^{(2)}_{ab}(X)\int_\Delta \partial^a\phi(x)\partial^b\phi(x)dx \tag{234}$$

where a, b are multi-indices. (We could as well include a term linear in $\partial\phi$.) We also define

$$\|\alpha(X)\|_a = |\alpha^{(0)}(X)| + a^2|\alpha^{(2)}(X)| \equiv |\alpha^{(0)}(X)| + a^2\sum_{ab}|\alpha^{(2)}_{ab}(X)|\,. \tag{235}$$

Lemma 21 *Let $\alpha(X)$ be supported on small sets and let $a = \max\{\kappa^{-1/2}, h\}$ for $\kappa \leq 1$ and $h \geq 1$. Also let $k = \mathcal{O}(1)$ be the number of small sets containing a unit block Δ. Then for all complex $z(X)$ satisfying*

$$40k|z(X)|\|\alpha(X)\|_a \leq 1 \tag{236}$$

we have

$$\|\exp\left(\sum_{X\supset\Delta} z(X)F(X,\Delta)\right)\|_{G'(\kappa),h} \leq 2\,. \tag{237}$$

Thus F is stable for $(G'(\kappa), h, 40k\|\alpha(X)\|_a)$,

Remark. Similarly F is stable for $(G'_\ell(\kappa), h, \mathcal{O}(1)\|\alpha(X)\|_a)$ with a larger constant $\mathcal{O}(1)$.

Proof. We have as above

$$\begin{aligned}&\tfrac{(3h)^n}{n!}\|\left(\exp(\textstyle\sum_{X\supset\Delta} z(X)F(X,\Delta))\right)_n(\phi)\| \leq \\ &\qquad \exp\left(\textstyle\sum_{X\supset\Delta}|z(X)|\sum_{n=0}^2 \tfrac{(3h)^n}{n!}\|F_n(X,\Delta,\phi)\|\right)\,.\end{aligned} \tag{238}$$

Now compute the derivatives and estimate them by

$$\begin{aligned}|F_0(X,\Delta,\phi)| &\leq |\alpha^{(0)}(X)| + |\alpha^{(2)}(X)|\|\partial\phi\|^2_{s,\Delta}\\ \|F_1(X,\Delta,\phi)\| &\leq 2|\alpha^{(2)}(X)|\|\partial\phi\|_{s,\Delta}\\ \|F_2(X,\Delta,\phi)\| &\leq 2|\alpha^{(2)}(X)|\,.\end{aligned} \tag{239}$$

Then estimate

$$\begin{aligned}\sum_{n=0}^{2}\frac{(3h)^n}{n!}\|F_n(X,\Delta,\phi)\| &\leq |\alpha^{(0)}(X)| + \left(\|\partial\phi\|_{s,\Delta}^2 + 6h\|\partial\phi\|_{s,\Delta} + 9h^2\right)|\alpha^{(2)}(X)| \\ &\leq |\alpha^{(0)}(X)| + \left(10\|\partial\phi\|_{s,\Delta}^2 + 10h^2\right)|\alpha^{(2)}(X)| \\ &\leq |\alpha^{(0)}(X)| + 10a^2(1+\kappa\|\partial\phi\|_{s,\Delta}^2)|\alpha^{(2)}(X)| \\ &\leq 40(1/4+\kappa\|\partial\phi\|_{s,\Delta}^2)\|\alpha(X)\|_a \ . \end{aligned} \tag{240}$$

Now since $40|z(X)|\|\alpha(X)\|_a \leq k^{-1}$ we find

$$\sum_{X\supset\Delta}|z(X)|\sum_{n=0}^{2}\frac{(3h)^n}{n!}\|F_n(X,\Delta,\phi)\| \leq 1/4+\kappa\|\partial\phi\|_{\Delta,s}^2 \ . \tag{241}$$

Using this in (238) yields

$$\frac{(3h)^n}{n!}\|\left(\exp(\sum_{X\supset\Delta}z(X)F(X,\Delta))\right)_n\|_{G'(\kappa)} \leq e^{\frac{1}{4}} \ . \tag{242}$$

Now multiply by 3^{-n} and sum over n to obtain the result.

□

A.2 Estimates on covariances

Let $C_\infty(\sigma,x)$ be the covariance on $\mathbf{R}^d,\ d\geq 2$ defined by

$$C_\infty(\sigma,x) = (2\pi)^{-d}\int_{\mathbf{R}^d} dp\ \frac{e^{ipx}}{p^2}\ [(e^{p^4}+\sigma)^{-1} - (e^{L^4p^4}+\sigma)^{-1}] \ . \tag{243}$$

Lemma 22 *1. There is $\sigma_0 = \mathcal{O}(1)$ such that for $|\sigma|\leq\sigma_0$ and any multi-index β there are constants c_1, c_2 such that*

$$\begin{aligned}|\partial^\beta C_\infty(\sigma,x)| &\leq c_1\exp(-|x|/L) \\ \int|\partial^\beta C_\infty(\sigma,x)|dx &\leq c_2 \ . \end{aligned} \tag{244}$$

The constant $c_1 = \mathcal{O}(1)\log L$ for $d=2,\beta=0$, but may be chosen independent of L otherwise. We also have $c_2 \leq \mathcal{O}(1)\int_1^L s^{1-|\beta|}ds$.

2. In $d=2$,

$$C_\infty(\sigma,0) = \frac{\log L}{2\pi(1+\sigma)} \ . \tag{245}$$

Proof. We rewrite the covariance and its derivatives as

$$\begin{aligned}
\partial^\beta C_\infty(\sigma,x) &= (2\pi)^{-d}\int_1^L ds \int_{\mathbf{R}^d} dp\ \frac{e^{ipx}}{p^2}(ip)^\beta\ (-\frac{\partial}{\partial s})(e^{s^4p^4}+\sigma)^{-1} \\
&= (2\pi)^{-d}\int_1^L ds \int_{\mathbf{R}^d} dp\ \frac{e^{ipx}}{p^2}(ip)^\beta \frac{(4s^3p^4e^{s^4p^4})}{(e^{s^4p^4}+\sigma)^2} \\
&= 4(2\pi)^{-d}\int_1^L \frac{ds}{s^{d-1+|\beta|}} \int_{\mathbf{R}^d} dp\ e^{is^{-1}px}\left[(ip)^\beta p^2\ \frac{e^{p^4}}{(e^{p^4}+\sigma)^2}\right] \quad (246)
\end{aligned}$$

The function in brackets is analytic, bounded and integrable in the strip $|\mathrm{Im}(p)| \le 1$ around the real axis when $|\sigma|$ is small. Therefore we can shift the p integral one unit in an imaginary direction and exhibit the exponential decay in x. We find

$$|\partial^\beta C_\infty(\sigma,x)| \le \mathcal{O}(1)\int_1^L \frac{ds}{s^{d-1+|\beta|}}\ e^{-s^{-1}|x|} \quad (247)$$

and the bounds (244) follow. In $d=2$ we compute

$$\begin{aligned}
C_\infty(\sigma,0) &= \pi^{-2}\int_1^L \frac{ds}{s}\int_0^\infty 2\pi r dr\ \frac{r^2e^{r^4}}{(e^{r^4}+\sigma)^2} \quad (248)\\
&= \frac{\log L}{2\pi(1+\sigma)}\ . \quad (249)
\end{aligned}$$

This completes the proof.

□

Now let $C^M(\sigma,x)$ be the covariance on Λ_M as defined in (22).

Lemma 23 *Let* $|\sigma| \le \sigma_0$.

1. *For any multi-index* β *and* $|x| \le L^M/2$

$$\begin{aligned}
|\partial^\beta C^M(\sigma,x)| &\le \mathcal{O}(1)c_1\exp(-|x|/L) \\
\int |\partial^\beta C^M(\sigma,x)|dx &\le \mathcal{O}(1)c_2\ . \quad (250)
\end{aligned}$$

2. *In* $d=2$,

$$C^M(\sigma,0) = \frac{\log L}{2\pi(1+\sigma)} + \mathcal{O}(1)e^{-L^{M-1}/2}\ . \quad (251)$$

Proof. We have the representation

$$C^M(\sigma,x) = \sum_{n\in\mathbf{Z}^2} C_\infty(\sigma,x+nL^M)\ . \quad (252)$$

This follows since both sides are doubly periodic with period L^M, and they have the same Fourier coefficients, namely $p^{-2}((e^{p^4}+\sigma)^{-1} - (e^{L^4p^4}+\sigma)^{-1})$ for $p \ne 0$ and 0 for $p=0$. The terms in the sum are estimated by the previous lemma and we obtain all the stated results. □

Acknowledgement

We thank David Brydges for his encouragement and many helpful conversations.

References

[1] A. Abdesselam, V. Rivasseau, Trees, forests, and jungles: a botanical garden for cluster expansions, in V. Rivasseau, editor, *Constructive Physics*, 311–326, Berlin, 1995, Springer.

[2] G. Benfatto, G. Gallavotti, F. Nicolo, On the massive sine-Gordon equation in the first few regions of collapse. *Commun. Math. Phys.* **83**, 387–410 (1982)

[3] D.Brydges, J.Dimock, T.R. Hurd, Weak perturbations of Gaussian measures. In J.Feldman, R.Froese, and L.Rosen, editors, *Mathematical Quantum Theory I: Field Theory and Many–body Theory.* AMS, 1994.

[4] D.Brydges, J.Dimock, T.R. Hurd, Applications of the renormalization group. In J.Feldman, R.Froese, and L.Rosen, editors, *Mathematical Quantum Theory I: Field Theory and Many–body Theory.* AMS, 1994.

[5] D.C. Brydges, J.Dimock, T.R. Hurd, The short distance behavior of ϕ^4_3. *Commun. Math. Phys.*, **172**, 143–186, 1995.

[6] D.C. Brydges, J.Dimock, T.R. Hurd, Estimates on renormalization group transformations, *Can. J. Math.* **50**, 756–793, (1998)

[7] D.C. Brydges, J.Dimock, T.R. Hurd,, A non-Gaussian fixed point for ϕ^4 in $4 - \epsilon$ dimensions, *Commun. Math. Phys.* **198**, 111–156, (1998)

[8] D. Brydges and G. Keller, Correlation functions of general observables in dipole-type systems, *Helv. Phys. Acta***67**, 43–116 (1994).

[9] D. Brydges, T. Kennedy, Mayer expansions and the Hamilton-Jacobi equation, *J. Stat. Phys.* **48**, 19 (1987).

[10] D. Brydges, P. Martin, Coulomb systems at low density, cond-mat/9904122

[11] D.C. Brydges, H.T. Yau, Grad ϕ perturbations of massless Gaussian fields. *Commun. Math. Phys.*, **129**, 351–392, (1990).

[12] J. Dimock, Bosonization of Massive Fermions *Commun. Math. Phys.* **198**, 247–281, (1998).

[13] J.Dimock, T.R. Hurd, A renormalization group analysis of the Kosterlitz-Thouless phase. *Commun. Math. Phys.*, **137**, 263–287, (1991).

[14] J. Dimock, T. R. Hurd, A renormalization group analysis of correlation functions for the dipole gas. *J. Stat. Phys.*, **66**, 1277–1318, (1992).

[15] J.Dimock, T.R.Hurd, Construction of the two-dimensional sine-Gordon model for $\beta < 8\pi$. *Commun. Math. Phys.*, **156**, 547–580, (1993).

[16] J.Fröhlich, E.Seiler, The massive Thirring-Schwinger model (QED_2): convergence of perturbation theory and particle structure. *Helv. Phys. Acta*, **49**, 889–, (1976).

[17] J. Fröhlich, T. Spencer, The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and Coulomb gas, *Commun. Math. Phys.* **81**, 527–602, (1981).

[18] J. Glimm, A. Jaffe, T. Spencer, The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions. in G. Velo., A. Wightman, eds, *Constructive Quantum field theory*, Springer-Verlag, New York, 1973

[19] T.R. Hurd, Charge correlations for the two-dimensional Coulomb gas. in V. Rivasseau, editor, *Constructive Physics*, 311-326, Berlin, 1995, Springer.

[20] J.M. Kosterlitz, D.J. Thouless: Ordering, metastability and phase transitions in two-dimensional systems. *J. Phys. C* **6**, 1181–1203 (1973)

[21] D.H.U. Marchetti, A. Klein, Power-law falloff in two- dimensional Coulomb gases at inverse temperature $\beta > 8\pi$. *J. Stat. Phys.* **64**, 135–162 (1991)

[22] F. Nicolo, P. Perfetti, The sine-Gordon field theory model at $\alpha^2 = 8\pi$, the non-superrenormalizable theory. *Commun. Math. Phys.* **123**, 425–452 (1989)

[23] F. Nicolo, J. Renn, A. Steinmann: On the massive sine-Gordon equation in all regions of collapse. *Commun. Math. Phys.* **105**, 291–326 (1986)

J. Dimock
Research supported by
NSF Grant PHY9722045
Dept. of Mathematics
SUNY at Buffalo
Buffalo, N.Y. 14214

T.R. Hurd
Research supported
by Natural Sciences and
the Engineering Research Council of Canada
Dept. of Mathematics and Statistics
McMaster University
Hamilton, Ontario, L8S 4K1
e-mail : hurdt@mcmail.cis.McMaster.CA

Communicated by V. Rivasseau
submitted 08/09/99, accepted 01/12/99