Ann. Henri Poincaré 1 (2000) 443 – 459

1424-0637/00/030443-17 $ 1.50+0.20/0

Annales Henri Poincaré

On the Existence of Ground States for Massless Pauli-Fierz Hamiltonians

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I Introduction

We consider in this paper the problem of the existence of a ground state for a class of Hamiltonians used in physics to describe a confined quantum system (”matter”) interacting with a massless bosonic field. These Hamiltonians were called *Pauli-Fierz Hamiltonians* in [DG]. Examples, like the spin-boson model or a simplified model of a confined atom interacting with a bosonic field are given in [DG, Sect. 3.3].

Pauli-Fierz Hamiltonians can be described as follows: Let $\mathcal{K}$ and K be respectively the Hilbert space and the Hamiltonian describing the matter. The assumption that the matter is confined is expressed mathematically by the fact that $(K + \mathrm{i})^{-1}$ is *compact* on $\mathcal{K}$.

The bosonic field is described by the Fock space $\Gamma(\mathfrak{h})$ with the one-particle space $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, where $\mathbb{R}^d$ is the momentum space, and the free Hamiltonian

$$\mathrm{d}\Gamma(\omega(k)) = \int \omega(k) a^*(k) a(k) \mathrm{d}k.$$

The positive function $\omega(k)$ is called the *dispersion relation*. The constant $m := \inf \omega$ can be called the *mass* of the bosons, and we will consider here the case of *massless* bosons , ie we assume that $m = 0$.

The interaction of the “matter” and the bosons is described by the operator

$$V = \int v(k) \otimes a^*(k) + v^*(k) \otimes a(k) \mathrm{d}k,$$

where $\mathbb{R}^d \ni k \to v(k)$ is a function with values in operators on $\mathcal{K}$. Thus, the system is described by the Hilbert space $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$ and the Hamiltonian

$$H = K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)) + gV, \tag{I.1}$$

g being a coupling constant.

If $\mathcal{K} = \mathbb{C}$, the Hamiltonian H is solvable (see eg [Be, Sect. 7]) and H is defined as a selfadjoint operator if

$$\int \frac{1}{\omega(k)} |v(k)|^2 \mathrm{d}k < \infty,$$

and admits a ground state in $\mathcal{H}$ if and only if

$$\int \frac{1}{\omega(k)^2}|v(k)|^2 \mathrm{d}k < \infty.$$

In this paper we show that H admits a ground state in $\mathcal{H}$ for all values of the coupling constant under corresponding assumptions in the general case.

The existence of a ground state of H in the Hilbert space $\mathcal{H}$ is an important physical property of the system described by H. For example it has the following consequence for the scattering theory of H: assume that $\omega \in C^\infty(\{k|\omega(k) > 0\})$ and $\nabla\omega(k) \neq 0$ in $\{k|\omega(k) > 0\}$. Assume also that

$$\mathbb{R}^d \ni k \mapsto \|v(k)(K+1)^{-\frac{1}{2}}\|_{B(\mathcal{K})}$$

is locally in the Sobolev space H^s in $\{k|\omega(k > 0\}$ for some $s > 1$ (a short-range condition on the interaction). Then under the conditions *(H0)*, *(H1)*, *(I1)* below, it is easy to prove the existence of the limits

$$W^{\pm}(h) := \text{s-}\lim_{t\to\pm\infty} e^{\mathrm{i}tH} e^{\mathrm{i}\phi(h_t)} e^{-\mathrm{i}tH}$$

for $h \in \mathfrak{h}_0 := \{h \in \mathfrak{h}|\omega^{-\frac{1}{2}}h \in \mathfrak{h}\}$ and $h_t = e^{-\mathrm{i}t\omega}h$. The operators $W^{\pm}(h)$ are called *asymptotic Weyl operators*. They satisfy

$$W^{\pm}(h)W^{\pm}(g) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}Im(h|g)}W(h+g),\ h,g \in \mathfrak{h}_0,$$

and

$$\mathrm{e}^{\mathrm{i}tH}W^{\pm}(h)\mathrm{e}^{-\mathrm{i}tH} = W^{\pm}(h_{-t}).$$

In particular they form two regular CCR representations over the preHilbert space $\mathfrak{h}_0$. It is easy to show that the space of bound states $\mathcal{H}_{\mathrm{pp}}(H)$ of H is included into the space of vacua for these representations (see for example [DG]). Hence the existence of a ground state for H implies that the CCR representations defined by the asymptotic Weyl operators admit Fock subrepresentations. As a consequence wave operators can be defined.

When the Hamiltonian H admits no ground state in the Hilbert space $\mathcal{H}$, the ground state of H has to be interpreted as a state ω on some C^*-algebra of field observables. Similarly the scattering theory for H has to be significantly modified. These phenomena have been extensively studied by Froehlich [Fr]. In particular the arguments in the proof of Lemma IV.5 are inspired by [Fr, Sect. 2.3], where it is shown that the state ω is locally normal.

Let us end the introduction by making some comments on related works. In [AH], the existence of a ground state is shown under rather similar conditions, if the coupling constant g is sufficiently small. In [Sp], the same problem is considered in the case the small system described by $(\mathcal{K}, K)$ is a confined atom, and the coupling function $k \mapsto v(k)$ is a *real multiplication operator* in the atomic variables (ie $v^*(k) = v(-k)$ is a multiplication operator on $\mathcal{K}$). Using functional integral methods and Perron-Frobenius arguments, the existence of a ground state is shown for all values of the coupling constant.

Our result is hence a generalization of the results both of [AH] and [Sp].

If we drop the assumption that the small system is confined (mathematically this amounts to drop the hypothesis *(H0)* below), then the only result is the one of [BFS], where the existence of a ground state is shown for small coupling constant if K is an atomic Hamiltonian and assumptions similar to *(I1)*, *(I2)* are made.

II Result

II.1 Introduction

In this section we introduce the class of Hamiltonians that we will study in this paper. We have stated our result under rather general hypotheses, allowing for a mild UV divergency of the interaction. Clearly the behavior of the interaction for large momenta should not be important for the existence of a ground state, which essentially depends only on the low momentum behavior of the interaction. Therefore the reader wishing to avoid some technicalities can for example assume that the operator K is bounded and that the function $\mathbb{R}^d \ni k \mapsto v(k)$ is compactly supported.

II.2 Hamiltonian

Let $\mathcal{K}$ be a separable Hilbert space representing the degrees of freedom of the atomic system. The Hamiltonian describing the atomic system is denoted by K. We assume that K is selfadjoint on $\mathcal{D}(K) \subset \mathcal{K}$ and bounded below. Without loss of generality we can assume that K is positive. We assume

$$(H0) \quad (K + \mathrm{i})^{-1} \text{ is compact.}$$

The physical interpretation is that the atomic system is confined.

Let $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$ be the 1−particle Hilbert space in the momentum representation and let $\Gamma(\mathfrak{h})$ be the bosonic Fock space over $\mathfrak{h}$, representing the field degrees of freedom. We will denote by k the momentum operator of multiplication by k on $L^2(\mathbb{R}^d, \mathrm{d}k)$, and by $x = \mathrm{i}\nabla_k$ the position operator on $L^2(\mathbb{R}^d, \mathrm{d}k)$. Let $\omega \in C(\mathbb{R}^d, \mathbb{R})$ be the boson dispersion relation. We assume

$$(H1) \quad \begin{cases} \nabla\omega \in L^\infty(\mathbb{R}^d), \\ \lim_{|k|\to\infty} \omega(k) = +\infty, \\ \inf \omega(k) = 0. \end{cases}$$

To stay close to the usual physical situation, we will also assume that $\omega(0) = 0, \omega(k) \neq 0$ for $k \neq 0$, although the results below hold also in the general case. The typical example is of course the massless relativistic dispersion relation $\omega(k) = |k|$.

The Hamiltonian describing the field is equal to $\mathrm{d}\Gamma(\omega)$. The Hilbert space of the interacting system is

$$\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h}).$$

The Hamiltonian $H_0 := K \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega)$ of the non-interacting system is associated with the quadratic form

$$Q_0(u,u) := (K^{\frac{1}{2}} \otimes 1\!\!1 u, K^{\frac{1}{2}} \otimes 1\!\!1 u) + \int \omega(k)(1\!\!1 \otimes a(k)u, 1\!\!1 \otimes a(k)u)\mathrm{d}k,$$

with $D(Q_0) = D((K + \mathrm{d}\Gamma(\omega))^{\frac{1}{2}})$.

The interaction between the atom and the boson field is described with a coupling function v

$$\mathbb{R}^d \ni k \mapsto v(k),$$

such that for a.e. $k \in \mathbb{R}^d$, $v(k)$ is a bounded operator from $D(K^{\frac{1}{2}})$ into $\mathcal{K}$ and from $\mathcal{K}$ into $D(K^{\frac{1}{2}})^*$. We associate to the coupling function v the quadratic form

$$V(u,u) = \int (1\!\!1 \otimes a(k)u, v(k) \otimes 1\!\!1 u) + (v(k) \otimes 1\!\!1 u, 1\!\!1 \otimes a(k)u)\mathrm{d}k, \tag{II.1}$$

A rather minimal assumption under which the quadratic form $Q = Q_0 + V$ gives rise to a selfadjoint operator is

$$(I1)\quad \begin{array}{l} \text{for a.e. } k \in \mathbb{R}^d \ v(k)(K+1)^{-\frac{1}{2}},\ (K+1)^{-\frac{1}{2}}v(k) \in B(\mathcal{K}), \\ \forall u_1, u_2 \in \mathcal{K},\ k \mapsto (u_2, v(k)(K+1)^{-\frac{1}{2}}u_1),\ k \mapsto (u_2, (K+1)^{-\frac{1}{2}}v(k)u_1) \\ \text{are measurable,} \\ C(R) := \int \frac{1}{\omega(k)}(\|v(k)(K+R)^{-\frac{1}{2}}\|^2 + \|(K+R)^{-\frac{1}{2}}v(k)\|^2)\mathrm{d}k < \infty, \\ \lim_{R\to+\infty} C(R) = 0. \end{array}$$

Note that it follows from the results quoted in the Appendix that the functions $k \mapsto \|v(k)(K+R)^{-\frac{1}{2}}\|$, $k \mapsto \|(K+R)^{-\frac{1}{2}}v(k)\|$ are measurable, and hence the last condition in $(I1)$ has a meaning.

Proposition II.1 *Assume hypothesis (I1). Then the quadratic form V is Q_0-form bounded with relative bound* 0*. Consequently one can associate with the quadratic form $Q = Q_0 + V$ a unique bounded below selfadjoint operator H with $D(H^{\frac{1}{2}}) = D(H_0^{\frac{1}{2}})$.*

The Hamiltonian H is called a *Pauli-Fierz Hamiltonian*.

Proof. We apply the estimate (A.1) in Lemma A.1 with $B = K$, $m = \omega$. □

II.3 Results

Under assumption *(I1)*, one can associate a bounded below, selfadjoint Hamiltonian H to the quadratic form Q. Let us introduce the following assumption on the behavior of $v(k)$ near $\{k|\omega(k) = 0\}$:

$$(I2) \int \frac{1}{\omega(k)^2} \|v(k)(K+1)^{-\frac{1}{2}}\|^2 \mathrm{d}k < \infty.$$

Theorem 1 *Assume hypotheses (H0), (H1), (I1), (I2). Then* $\inf \operatorname{spec}(H)$ *is an eigenvalue of* H*. In other words* H *admits a ground state in* $\mathcal{H}$*.*

III The cut-off Hamiltonians

III.1 Operator bounds

Let us introduce the following assumption:

$$(I1') \quad \begin{array}{l} C'(R) := \int (1 + \frac{1}{\omega(k)})(\|v(k)(K+R)^{-\frac{1}{2}}\|^2 + \|(K+1)^{-\frac{1}{2}} v(k)\|^2)\mathrm{d}k < \infty, \\ \lim_{R\to+\infty} C'(R) = 0. \end{array}$$

Proposition III.1 *Assume (I1), (I1'). Then the operator*

$$V = a^*(v) + a(v) = \int v(k) \otimes a^*(k) + v^*(k) \otimes a(k) d\, k$$

is H_0-*bounded with relative bound* 0*. Consequently* $H = H_0 + V$ *is a bounded below selfadjoint operator with* $D(H) = D(H_0)$*.*

Proof. We apply the estimates (A.2), (A.3) in Lemma A.1 with $B = K$, $m = \omega$.□

III.2 Cut-off Hamiltonians

In the sequel we will need to introduce various cut-off Hamiltonians. For $0 < \sigma \ll 1$ an infrared cutoff parameter and $\tau \gg 1$ an ultraviolet cutoff parameter, we denote by V_σ, $V_{\sigma,\tau}$ the quadratic forms defined as V in (II.1) with the coupling function v replaced respectively by v_σ, $v_{\sigma,\tau}$ for

$$v_\sigma = \mathbb{1}_{\{\sigma\le\omega\}}(k)v,\ v_{\sigma,\tau} = \mathbb{1}_{\{\sigma\le\omega\le\tau\}}(k)v.$$

We denote by $H_\sigma, H_{\sigma,\tau}$ the selfadjoint operators associated with the quadratic forms $Q_0 + V_\sigma, Q_0 + V_{\sigma,\tau}$. Note that since $v_{\sigma,\tau}$ satisfies *(I1')*, we have $D(H_{\sigma,\tau}) = D(H_0)$.

Applying Lemma A.2 in the Appendix and the fact that $D(H^{\frac{1}{2}}) = D(H_0^{\frac{1}{2}})$ we obtain

$$\begin{aligned} &\lim_{\tau\to+\infty}(H_{\sigma,\tau} - \lambda)^{-1} = (H_\sigma - \lambda)^{-1}, \\ &\lim_{\sigma\to 0}(H_\sigma - \lambda)^{-1} = (H - \lambda)^{-1}, \end{aligned} \tag{III.1}$$

for $\lambda \in \mathbb{R}, \lambda \ll -1$, and

$$\begin{aligned} &\|\big((H_{\sigma,\tau} - z)^{-1} - (H_\sigma - z)^{-1}\big)(H_0+1)^{\frac{1}{2}}\| \in o(1)|Imz|^{-1}\ \tau \to +\infty, \\ &\|\big((H_\sigma - z)^{-1} - (H - z)^{-1}\big)(H_0+1)^{\frac{1}{2}}\| \in o(1)|Imz|^{-1}\ \sigma \to 0, \end{aligned} \tag{III.2}$$

for $z \in \mathbb{C}\backslash\mathbb{R}$.

III.3 Existence of ground states for the cut-off Hamiltonians

Let $\tilde{\omega}_\sigma : \mathbb{R}^d \to \mathbb{R}$ be a dispersion relation satisfying

$$\begin{cases} \nabla\tilde{\omega}_\sigma \in L^\infty(\mathbb{R}^d), \\ \tilde{\omega}_\sigma(k) = \omega(k) \text{ if } \omega(k) \geq \sigma, \\ \tilde{\omega}_\sigma(k) \geq \sigma/2. \end{cases} \tag{III.3}$$

Let $\tilde{H}_\sigma$ be the operator associated to the quadratic form $\|K^{\frac{1}{2}}u\|^2 + \int \tilde{\omega}_\sigma(k)$ $\|a(k)u\|^2 \mathrm{d}k + V_\sigma(u,u)$.

Lemma III.2 *H_σ admits a ground state in $\mathcal{H}$ if and only if $\tilde{H}_\sigma$ admits a ground state in $\mathcal{H}$.*

Proof. Let $\mathfrak{h}_\sigma := L^2(\{k|\omega(k) < \sigma\}, \mathrm{d}k)$, $\mathfrak{h}_\sigma^\perp = L^2(\{k|\omega(k) \geq \sigma\}, \mathrm{d}k)$. Let U be the canonical unitary map

$$U : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}_\sigma^\perp) \otimes \Gamma(\mathfrak{h}_\sigma)$$

(see for example [DG, Sect. 2.7]). Let us still denote by U the unitary map $\mathbb{1}_\mathcal{K} \otimes U$ from $\mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h})$ into $\mathcal{K} \otimes \Gamma(\mathfrak{h}_\sigma^\perp) \otimes \Gamma(\mathfrak{h}_\sigma)$. By [DG, Sect. 2.7], the operator $UH_\sigma U^*$ is equal to

$$\mathbb{1}_{\mathcal{K}\otimes\Gamma(\mathfrak{h}_\sigma^\perp)} \otimes \mathrm{d}\Gamma(\omega_{\sigma,1}) + H_\sigma^2 \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\sigma)},$$

where $\omega_{\sigma,1} = \omega_{|\mathfrak{h}_\sigma}$ and H_σ^2 is the operator associated with the quadratic form $\|K^{\frac{1}{2}}u\|^2 + \int_{\{\omega(k)\geq\sigma\}} \omega_\sigma(k)\|a(k)u\|^2 \mathrm{d}k + V_\sigma(u,u)$. Similarly $U\tilde{H}_\sigma U^*$ is equal to

$$\mathbb{1}_{\mathcal{K}\otimes\Gamma(\mathfrak{h}_\sigma^\perp)} \otimes \mathrm{d}\Gamma(\tilde{\omega}_{\sigma,1}) + H_\sigma^2 \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\sigma)},$$

where $\tilde{\omega}_{\sigma,1} = \tilde{\omega}_{\sigma|\mathfrak{h}_\sigma}$. Now H_σ^2 has a ground state ψ if and only if $U\tilde{H}_\sigma U^*$ or $UH_\sigma U^*$ have a ground state (equal to $\psi \otimes \Omega$, where $\Omega \in \Gamma(\mathfrak{h}_\sigma)$ is the vacuum vector). This proves the lemma. □

The following result is essentially well known (see [AH], [BFS]) and rather easy to show.

Proposition III.3 *Assume hypotheses (H0), (H1), (I1). Then for any $\sigma > 0$ H_σ admits a ground state.*

Proof. By Lemma III.2 it suffices to show that $\tilde{H}_\sigma$ admits a ground state. Let for $\tau \in \mathbb{N}$ $\tilde{H}_{\sigma,\tau}$ be the Hamiltonian associated with the quadratic form $\|K^{\frac{1}{2}}u\|^2 + \int \tilde{\omega}_\sigma(k)\|a(k)u\|^2 \mathrm{d}k + V_{\sigma,\tau}(u,u)$. Let

$$\tilde{E}_{\sigma,\tau} = \inf \operatorname{spec}(\tilde{H}_{\sigma,\tau}),\ \tilde{E}_\sigma = \inf \operatorname{spec}(\tilde{H}_\sigma).$$

Applying Lemma A.2, we have for $z \in \mathbb{C}\backslash\mathbb{R}$

$$(z - \tilde{H}_\sigma)^{-1} = \lim_{n\to+\infty}(z - \tilde{H}_{\sigma,n})^{-1}. \tag{III.4}$$

On the other hand applying the bounds in Lemma A.1 we have $D(\tilde{H}_{\sigma,\tau}) = D(K + \mathrm{d}\Gamma(\tilde{\omega}_\sigma))$. The Hamiltonian $\tilde{H}_{\sigma,\tau}$ is very similar to the class of massive Pauli-Fierz Hamiltonians studied in [DG]. It is easy to see that the arguments of [DG] extend to $\tilde{H}_{\sigma,\tau}$. In particular, following the proofs of [DG, Lemma 3.4], [DG, Thm. 4.1], we obtain that $\chi(\tilde{H}_{\sigma,\tau})$ is compact if $\chi \in C_0^\infty(]-\infty, \tilde{E}_{\sigma,\tau} + \sigma/2[)$. Using (III.4) and the fact that $\tilde{E}_\sigma = \lim_{n\to+\infty} \tilde{E}_{\sigma,\tau}$, we obtain that $\chi(\tilde{H}_\sigma)$ is compact if $\chi \in C_0^\infty(]-\infty, \tilde{E}_\sigma + \sigma/2[)$. This implies that $\tilde{H}_\sigma$ and hence H_σ admit a ground state. □

III.4 The pullthrough formula

As in [BFS], we shall use the pullthrough formula to get control on the ground states of H_σ. Since the domain H_σ is not explicitely known under assumption *(I1)*, some care is needed to prove the pullthrough formula in our situation.

Proposition III.4 *As an identity on $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}, \mathrm{d}k; \mathcal{H})$, we have:*

$$\begin{aligned}&(H_\sigma + \omega(k) - z)^{-1}a(k)\psi = \\ &a(k)(H_\sigma - z)^{-1}\psi + (H_\sigma + \omega(k) - z)^{-1}v_\sigma(k)(H_\sigma - z)^{-1}\psi,\ \psi \in \mathcal{H}.\end{aligned}$$

Proof. For $u_1, u_2 \in D(H_0)$, the following identity makes sense as an identity on $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}, \mathrm{d}k)$:

$$(a^*(k)u_1, (H_{\sigma,\tau} - z)u_2) = ((H_{\sigma,\tau} + \omega(k) - \overline{z})u_1, a(k)u_2) + (u_1, v_{\sigma,\tau}(k)u_2).$$

Setting $u_2 = (H_{\sigma,\tau} - z)^{-1}v_2$, we obtain that for $v_2 \in \mathcal{H}$, $a(k)v_2 \in L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}, \mathrm{d}k; D(H_0)^*)$ and

$$a(k)v_2 = (H_{\sigma,\tau} + \omega(k) - z)a(k)(H_{\sigma,\tau} - z)^{-1}v_2 + v_{\sigma,\tau}(k)(H_{\sigma,\tau} - z)^{-1}v_2.$$

Hence for $\psi \in \mathcal{H}$, $(H_\sigma + \omega(k) - z)^{-1}a(k)\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}, \mathrm{d}k; \mathcal{H})$ and

$$\begin{aligned}&(H_{\sigma,\tau} + \omega(k) - z)^{-1}a(k)\psi \\ &= a(k)(H_{\sigma,\tau} - z)^{-1}\psi + (H_{\sigma,\tau} + \omega(k) - z)^{-1}v_{\sigma,\tau}(k)(H_{\sigma,\tau} - z)^{-1}\psi,\end{aligned} \tag{III.5}$$

holds as an identity in $L^2_{\rm loc}(\mathbb{R}^d\backslash\{0\}, {\rm d}k; \mathcal{H})$.

By *(I1)*, $(v_{\sigma,\tau}(k) - v_\sigma(k))(H_0+1)^{-\frac{1}{2}}$ tends to 0 in $L^2(\mathbb{R}^d\backslash\{0\}, {\rm d}k; B(\mathcal{K}))$ when $\tau \to +\infty$. Using also (III.2) and letting $\tau \to +\infty$ we obtain

$$(H_\sigma+\omega(k)-z)^{-1}a(k)\psi = a(k)(H_\sigma - z)^{-1}\psi + (H_\sigma+\omega(k)-z)^{-1}v_\sigma(k)(H_\sigma - z)^{-1}\psi,$$

as claimed. □

IV Proof of Thm. 1

Let

$$E_\sigma := \inf {\rm spec}(H_\sigma),\ E := \inf {\rm spec}(H).$$

We denote by ψ_σ, $\sigma > 0$ a normalized ground state of H_σ. Applying the pullthrough formula to ψ_σ, we obtain easily the following identity on $L^2(\mathbb{R}^d, {\rm d}k; \mathcal{H})$:

$$a(k)\psi_\sigma = (E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma. \tag{IV.1}$$

The first rather obvious bound on the family of ground states ψ_σ is the following.

Lemma IV.1 *Assume hypotheses (H0), (H1), (I1). Then*

$$(\psi_\sigma, H_0\psi_\sigma) \le C,\ \textit{uniformly in } \sigma > 0. \tag{IV.2}$$

The bound (IV.2) follows immediately from the fact that the quadratic forms Q_σ are equivalent to Q_0, uniformly in σ. The following lemma is also well-known (see eg [BFS, Thm. II.5], [AH, Lemma 4.3]). We denote by N the number operator on $\Gamma(\mathfrak{h})$.

Lemma IV.2 *Assume hypotheses (H0), (H1), (I1), (I2). Then*

$$(\psi_\sigma, N\psi_\sigma) \le C,\ \textit{uniformly in } \sigma > 0. \tag{IV.3}$$

Proof. We have using (IV.1)

$$\begin{aligned}(\psi_\sigma, N\psi_\sigma) &= \int \|a(k)\psi_\sigma\|^2 {\rm d}k \\ &= \int \|(E_\sigma - H_\sigma(k) - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\|^2 {\rm d}k \\ &\le \|(H_0+1)^{\frac{1}{2}}\psi_\sigma\|^2 \int \tfrac{1}{\omega(k)^2}\|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2 {\rm d}k \\ &\le C,\end{aligned}$$

uniformly in $\sigma > 0$ using *(I2)* and (IV.2). □

Lemma IV.3 *Assume hypotheses (H0), (H1), (I1), (I2). Then*

$$E - E_\sigma \in o(\sigma). \tag{IV.4}$$

Proof. Let $0 < \sigma' < \sigma$. We have

$$\begin{aligned} E_{\sigma'} - E_\sigma &\le (Q_{\sigma'} - Q_\sigma)(\psi_\sigma, \psi_\sigma) = (V_{\sigma'} - V_\sigma)(\psi_\sigma, \psi_\sigma), \\ E_\sigma - E_{\sigma'} &\le (Q_\sigma - Q_{\sigma'})(\psi_{\sigma'}, \psi_{\sigma'}) = (V_\sigma - V_{\sigma'})(\psi_{\sigma'}, \psi_{\sigma'}), \end{aligned} \tag{IV.5}$$

Applying (A.1) with $m(k) = 1$, we obtain

$$|(V_{\sigma'} - V_\sigma)(u, u)| \le C(\sigma', \sigma)(u, Nu)^{\frac{1}{2}}(u, (K+1)u)^{\frac{1}{2}}, \tag{IV.6}$$

for

$$C(\sigma', \sigma) = \Big(\int_{\{\sigma' < \omega(k) \le \sigma\}} \|v(k)(K+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k\Big)^{\frac{1}{2}}$$

Using (IV.6) for $u = \psi_\sigma$ or $\psi_{\sigma'}$, the right hand side of (IV.5) is bounded by $C_0 C(\sigma', \sigma)$, uniformly in σ, σ', using (IV.2) and (IV.3). We note that by (III.1) $E = \lim_{\sigma' \to 0} E_{\sigma'}$. Hence letting σ' tend to 0 we get $|E - E_\sigma| \le C_0 C(0, \sigma) \in o(\sigma)$, using hypothesis *(I2)*. □

Proposition IV.4 *Assume hypotheses (H0), (H1), (I1), (I2). Then*

$$a(k)\psi_\sigma - (E - H - \omega(k))^{-1} v(k)\psi_\sigma \to 0$$

when $\sigma \to 0$ *in* $L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})$.

Proof. We have, using (IV.1)

$$\begin{aligned} & a(k)\psi_\sigma - (E - H - \omega(k))^{-1} v(k)\psi_\sigma \\ = \; & (E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma - (E - H - \omega(k))^{-1} v(k)\psi_\sigma \\ = \; & -1\!\!1_{\{\omega(k) \le \sigma\}}(k)(E - H - \omega(k))^{-1} v(k)\psi_\sigma \\ & + (E - H - \omega(k))^{-1}(H - H_\sigma)(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma \\ & + (E_\sigma - E)(E - H - \omega(k))^{-1}(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma \\ =: \; & R_{\sigma,1}(k) + R_{\sigma,2}(k) + R_{\sigma,3}(k). \end{aligned}$$

We will estimate separately $R_{\sigma,i}$, $1 \le i \le 3$. First

$$\|R_{\sigma,1}(k)\|_{\mathcal{H}} \le 1\!\!1_{\{\omega(k) \le \sigma\}}(k) \frac{1}{\omega(k)} \|v(k)(K+1)^{-\frac{1}{2}}\|_{B(\mathcal{K})} \|(K+1)^{\frac{1}{2}}\psi_\sigma\|_{\mathcal{H}},$$

which shows using hypothesis *(I2)* and (IV.2) that

$$R_{\sigma,1} \in o(\sigma^0) \text{ in } L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H}). \tag{IV.7}$$

Let us next estimate $R_{\sigma,2}$. Using the fact that $(v - v_\sigma)(k)(K+1)^{-\frac{1}{2}}$ belongs to $L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})$, it is easy to verify that

$$\begin{aligned} & (E - H - \omega(k))^{-1}(H - H_\sigma)(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma \\ = \; & (E - H - \omega(k))^{-1}(a^*(v - v_\sigma) + a(v - v_\sigma))(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma. \end{aligned}$$

Note that it follows from functional calculus that

$$\|(E - H - \omega(k))^{-1}(H + b)^{\frac{1}{2}}\| \le C \sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}). \tag{IV.8}$$

Using also the fact that $(K+1)^{\frac{1}{2}}(H+b)^{-\frac{1}{2}}$ is bounded, we have:

$$\begin{aligned} & \|(E - H - \omega(k))^{-1}(a^*(v - v_\sigma) + a(v - v_\sigma))(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\| \\ \le \; & C \sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}) \|(K+1)^{-\frac{1}{2}}(a^*(v - v_\sigma) + a(v - v_\sigma)) \\ & \qquad (E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\| \\ \le \; & C \sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}) \Big(\int_{\{\omega(k) \le \sigma\}} \|v(k)(K+1)^{-\frac{1}{2}}\|^2 + \|(K+1)^{-\frac{1}{2}} \\ & \qquad v(k)\|^2 \mathrm{d}k \Big)^{\frac{1}{2}} \times \|(N+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\|, \end{aligned}$$

applying the estimates (A.2), (A.3) in Lemma A.1 to $B = \mathbb{1}$, $m = 1$, $v(k) = (K+1)^{-\frac{1}{2}}(v - v_\sigma)(k)$.

To bound $\|(N+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\|$, we write using again the pullthrough formula (IV.1):

$$\begin{aligned} & a(k')(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma \\ = \; & (E_\sigma - H_\sigma - \omega(k) - \omega(k'))^{-1} a(k') v_\sigma(k)\psi_\sigma \\ & + (E_\sigma - H_\sigma - \omega(k'))^{-1} v_\sigma(k')(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma \\ = \; & (E_\sigma - H_\sigma - \omega(k) - \omega(k'))^{-1} v_\sigma(k)(E_\sigma - H_\sigma - \omega(k'))^{-1} v_\sigma(k')\psi_\sigma \\ & + (E_\sigma - H_\sigma - \omega(k'))^{-1} v_\sigma(k')(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma. \end{aligned}$$

This gives

$$\begin{array}{rl} & \|N^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\|^2 \\ = & \int \|a(k')(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\|^2 \mathrm{d}k' \\ \leq & 2\int \|(E_\sigma - H_\sigma - \omega(k) - \omega(k'))^{-1}v_\sigma(k)(E_\sigma - H_\sigma - \omega(k'))^{-1}v_\sigma(k')\psi_\sigma\|^2 \mathrm{d}k' \\ & +2\int \|(E_\sigma - H_\sigma - \omega(k'))^{-1}v_\sigma(k')(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\|^2 \mathrm{d}k' \\ \leq & C\int \frac{1}{\omega(k)^2}\|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2\|(K+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k'))^{-1}\|^2\times \\ & \|v_\sigma(k')(K+1)^{-\frac{1}{2}}\|^2\|(K+1)^{\frac{1}{2}}\psi_\sigma\|^2 \mathrm{d}k' \\ & +C\int \frac{1}{\omega(k')^2}\|v_\sigma(k')(K+1)^{-\frac{1}{2}}\|^2\|(K+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1}\|^2\times \\ & \|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2\|(K+1)^{\frac{1}{2}}\psi_\sigma\|^2 \mathrm{d}k'. \end{array}$$

We use the bound (IV.8) and we obtain

$$\begin{array}{rl} & \|N^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\|^2 \\ \leq & C(\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}))^2\|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2\times \\ & \int(\sup(\omega(k')^{-1}, \omega(k')^{-\frac{1}{2}}))^2\|v_\sigma(k')(K+1)^{-\frac{1}{2}}\|^2 \mathrm{d}k'\times \\ & \|(K+1)^{\frac{1}{2}}\psi_\sigma\|^2 \\ \leq & C(\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}))^2\|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2, \end{array}$$

using (IV.2) and hypothesis *(I2)*. Hence

$$\begin{array}{l} \|R_{\sigma,2}(k)\|_{\mathcal{H}} \leq \\ C(\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}))^2\|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|(\int_{\{\omega(k)\leq\sigma\}}\|(K+1)^{-\frac{1}{2}}v(k)\|^2\mathrm{d}k)^{\frac{1}{2}}. \end{array}$$

By *(I2)*,

$$(\int_{\{\omega(k)\leq\sigma\}}\|(K+1)^{-\frac{1}{2}}v(k)\|^2\mathrm{d}k)^{\frac{1}{2}} \in o(\sigma),$$

and since $\mathrm{supp} v_\sigma \subset \{\omega(k) \geq \sigma\}$, we obtain

$$\|R_{\sigma,2}(k)\| \leq o(\sigma^0)\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}})\|v(k)(K+1)^{-\frac{1}{2}}\|. \tag{IV.9}$$

Finally using Lemma IV.3, (IV.2) and the fact that $\mathrm{supp} v_\sigma \subset \{\omega(k) \geq \sigma\}$, we obtain

$$\|R_{3,\sigma}(k)\| \leq o(\sigma^0)\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}})\|v(k)(K+1)^{-\frac{1}{2}}\|. \tag{IV.10}$$

Combining (IV.7), (IV.9), (IV.10) and using *(I2)* we obtain the proposition. □

As a consequence of Prop. IV.4, we have the following lemma, which is the main part of the proof of Thm. 1. We recall that $x := \mathrm{i}\nabla_k$ is the position operator on $L^2(\mathbb{R}^d, \mathrm{d}k)$.

Lemma IV.5 *Let $F \in C_0^\infty(\mathbb{R})$ be a cutoff function with $0 \le F \le 1$, $F(s) = 1$ for $|s| \le \frac{1}{2}$, $F(s) = 0$ for $|s| \ge 1$. Let $F_R(x) = F(\frac{|x|}{R})$. Then*

$$\lim_{\sigma\to 0, R\to+\infty} (\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) = 0. \tag{IV.11}$$

Proof. Recall that if B is a bounded operator on $\mathfrak{h}$ with distribution kernel $b(k, k')$, we have

$$(u, \mathrm{d}\Gamma(B)u) = \int\int b(k, k')(a(k)u, a(k')u)\mathrm{d}k\mathrm{d}k',\ u \in D(N^{\frac{1}{2}}).$$

Using this identity, we obtain

$$(\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) = (a(\cdot)\psi_\sigma, (1 - F(\frac{|D_k|}{R}))a(\cdot)\psi_\sigma)_{L^2(\mathbb{R}^d,\mathrm{d}k;\mathcal{H})}.$$

By Prop. IV.4, we have

$$\begin{aligned}&(\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) =\\ &((E - H - \omega(\cdot))^{-1}v(\cdot)\psi_\sigma, (1 - F(\tfrac{|D_k|}{R}))(E - H - \omega(\cdot))^{-1}v(\cdot)\psi_\sigma) + o(\sigma^0),\end{aligned}$$

uniformly in R. This yields

$$\begin{aligned}(\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) \le\ & \|(E - H - \omega(\cdot))^{-1}v(\cdot)\|_{L^2(\mathbb{R}^d,\mathrm{d}k,B(\mathcal{H}))}\times\\ & \|(1 - F(\tfrac{|D_k|}{R}))(E - H - \omega(\cdot))^{-1}v(\cdot)\|_{L^2(\mathbb{R}^d,\mathrm{d}k,B(\mathcal{H}))} + o(\sigma^0).\end{aligned}$$

Now it follows from hypothesis *(I2)* and (IV.8) that $(E - H - \omega(\cdot))^{-1}v(\cdot)$ belongs to $L^2(\mathbb{R}^d, \mathrm{d}k, B(\mathcal{H}))$, and hence

$$\|(1 - F(\frac{|D_k|}{R}))(E - H - \omega(\cdot))^{-1}v(\cdot)\|_{L^2(\mathbb{R}^d,\mathrm{d}k,B(\mathcal{H}))} \in o(R^0).$$

This proves (IV.11). □

We can now prove Theorem 1.

Proof of Theorem 1. Let us first recall the a priori bounds on the family of ground states $\{\psi_\sigma\}$. From (IV.2), (IV.3), we have

$$\|N^{\frac{1}{2}}\psi_\sigma\| \le C,\ \|H_0^{\frac{1}{2}}\psi_\sigma\| \le C,\ \text{uniformly in } \sigma. \tag{IV.12}$$

Let also F be a cutoff function as in Lemma IV.5. Then it is easy to verify, using the fact that $0 \le F \le 1$, that

$$(1 - \Gamma(F_R))^2 \le (1 - \Gamma(F_R)) \le \mathrm{d}\Gamma(1 - F_R).$$

Using Lemma IV.5, we obtain

$$\lim_{\sigma\to 0, R\to\infty} \|1 - \Gamma(F_R)\psi_\sigma\| = 0. \tag{IV.13}$$

Let us denote by $\chi(s \leq s_0)$ a cutoff function supported in $\{|s| \leq s_0\}$, equal to 1 in $\{|s| \leq s_0/2\}$.

Since the unit ball in $\mathcal{H}$ is compact for the weak topology, there exist a sequence $\sigma_n \to 0$ and a vector $\psi \in \mathcal{H}$ such that ψ_{σ_n} tends weakly to ψ. By Lemma A.3 in the Appendix, it suffices to show that $\psi \neq 0$ to prove the theorem.

Assume that $\psi = 0$. Note using hypotheses *(H0)*, *(H1)*, that for any λ, R the operator $\chi(N \leq \lambda)\chi(H_0 \leq \lambda)\Gamma(F_R)$ is compact on $\mathcal{H}$. Then

$$\lim_{n\to\infty} \chi(N \leq \lambda)\chi(H_0 \leq \lambda)\Gamma(F_R)\psi_{\sigma_n} = 0, \tag{IV.14}$$

for any λ, R. By (IV.13), we can pick R large enough such that for $n \geq n_0$

$$\|(1 - \Gamma(F_R))\psi_{\sigma_n}\| \leq 10^{-2}. \tag{IV.15}$$

Since $(1 - \chi(s \leq s_0)) \leq s_0^{-\frac{1}{2}} s^{\frac{1}{2}}$, we can using (IV.12) pick λ large enough such that

$$\|(1 - \chi(N \leq \lambda))\psi_{\sigma_n}\| \leq 10^{-2},\ \|(1 - \chi(H_0 \leq \lambda))\psi_{\sigma_n}\| \leq 10^{-2}. \tag{IV.16}$$

But (IV.15), (IV.16) and (IV.14) imply that for n large enough $\|\psi_{\sigma_n}\| \leq 10^{-1}$ which is a contradiction. Hence $\psi \neq 0$ and the theorem is proved.

A Appendix

We use the notations of Sect. II. The following lemma is well known if the coupling function $v(k)$ is of the form $v\lambda(k)$ for v a fixed linear operator on $\mathcal{K}$ and $k \mapsto \lambda(k)$ a scalar function. In our general setting it seems not to be in the literature.

Let us first recall some terminology and results about measurability of vector and operator-valued functions. Let $\mathcal{K}$ be a Hilbert space. A map $k \mapsto \psi(k) \in \mathcal{K}$ is said measurable if it is measurable if we equip $\mathcal{K}$ with the norm topology. Let now $\mathbb{R}^d \ni k \mapsto T(k) \in B(\mathcal{K})$ be defined for a.e. k. The map $k \mapsto T(k)$ is said weakly measurable if for all $\psi_1, \psi_2 \in \mathcal{K}$ the map $k \mapsto (\psi_2, T(k)\psi_1)$ is measurable. If $\mathcal{K}$ is *separable* the following facts are true (see eg [Di, Chap. II §2]):

i) the function $k \mapsto \|T(k)\|$ is measurable,

ii) for any $k \mapsto \psi(k) \in \mathcal{K}$ measurable, the function $k \mapsto T(k)\psi(k)$ is measurable.

In particular for $\psi \in \mathcal{K}$ the function $k \mapsto T(k)\psi$ is measurable. These facts will be used in the proof of Lemma A.1 below.

Lemma A.1 *Let $B \geq 0$ be a selfadjoint operator on the separable Hilbert space $\mathcal{K}$, $v : \mathbb{R}^d \ni k \mapsto v(k)$ a function such that for a.e. $k \in \mathbb{R}^d$, $v(k)(B+1)^{-\frac{1}{2}} \in B(\mathcal{K})$,*

$\mathbb{R}^d \ni k \mapsto v(k)(B+1)^{-\frac{1}{2}} \in B(\mathcal{K})$ *is weakly measurable and* $m : \mathbb{R}^d \ni k \mapsto m(k) \in \mathbb{R}^+$ *be a measurable function. Then*

$$|\int (v(k)u, a(k)u)\mathrm{d}k| \leq C(R)(u, \mathrm{d}\Gamma(m)u)^{\frac{1}{2}}(u, (B+R)u)^{\frac{1}{2}}, \tag{A.1}$$

for

$$C(R) = (\int \frac{1}{m(k)} \|v(k)(B+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k)^{\frac{1}{2}}.$$

If moreover for a.e. $k \in \mathbb{R}^d$, $(B+1)^{-\frac{1}{2}}v(k) \in B(\mathcal{K})$ *and* $\mathbb{R}^d \ni k \mapsto (B+1)^{-\frac{1}{2}}v(k) \in B(\mathcal{K})$ *is weakly measurable, then*

$$\| \int v^*(k) \otimes a(k)u \,\mathrm{d}k\| \leq C_1(R)\|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}}u\|, \tag{A.2}$$

for

$$C_1(R) = (\int \frac{1}{m(k)} \|(B+R)^{-\frac{1}{2}}v(k)\|^2 \mathrm{d}k)^{\frac{1}{2}},$$

and

$$\| \int v(k) \otimes a^*(k)u \,\mathrm{d}k\| \leq C_2(R)\|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}}u\| + C_3(R)\|u\|, \tag{A.3}$$

for

$$C_2(R) = (\int \frac{1}{m(k)} \|v(k)(B+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k)^{\frac{1}{2}},$$

$$C_3(R) = (\int \|v(k)(B+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k)^{\frac{1}{2}}.$$

Proof. The estimate (A.1) follows directly from Cauchy-Schwarz inequality. (We use the fact that for $u \in \mathcal{K} \otimes D(N^{\frac{1}{2}}) \cap D(\mathrm{d}\Gamma(m)^{\frac{1}{2}})$ the map $k \mapsto a(k)u \in \mathcal{H}$ is measurable). To prove (A.2), we consider the operator

$$w_R : \mathcal{K} \ni u \mapsto w_R(k)u := m(k)^{-\frac{1}{2}}(B+R)^{-\frac{1}{2}}v(k)u \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{K}) = \mathcal{K} \otimes \mathfrak{h}.$$

Clearly $\|w_R\|_{B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})} \leq C_1(R)$ and hence $\|w_R w_R^*\|_{B(\mathcal{K}\otimes\mathfrak{h})} \leq C_1(R)^2$. This gives

$$|\int\int (w_R^*(k)\psi(k), w_R^*(k')\psi(k'))_{\mathcal{K}} \mathrm{d}k\mathrm{d}k'| \leq C_1(R)^2 \int \|\psi(k)\|_{\mathcal{K}}^2 \mathrm{d}k, \tag{A.4}$$

for $\psi \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{K})$. The bound (A.4) still holds for $\psi \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})$ if we replace the scalar product $(.,.)_{\mathcal{K}}$ by the scalar product $(.,.)_{\mathcal{H}}$. We have:

$$\begin{aligned}
\|a(v)u\|^2 = \;& \| \textstyle\int v^*(k)a(k)u \,\mathrm{d}k\|^2 \\
= \;& \textstyle\int\int (v^*(k)a(k)u, v^*(k')a(k')u)_{\mathcal{H}} \mathrm{d}k\mathrm{d}k' \\
= \;& \textstyle\int\int (w_R^*(k)\psi(k), w_R^*(k')\psi(k'))_{\mathcal{H}} \mathrm{d}k\mathrm{d}k,'
\end{aligned}$$

for $\psi(k) = m(k)^{\frac{1}{2}} a(k)(B+R)^{\frac{1}{2}} u$. Using (A.4) we obtain

$$\|a(v)u\|^2 \le C_1(R)^2 \int \omega(k)\|a(k)(B+R)^{\frac{1}{2}}u\|^2 \mathrm{d}k$$

$$= C_1(R)^2 \|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}} u\|^2.$$

This proves (A.2).

Similarly, introducing the operator

$$\tilde{w}_R : \mathcal{K} \ni u \mapsto \tilde{w}_R(k)u = m(k)^{-\frac{1}{2}} v(k)(B+R)^{-\frac{1}{2}} \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{K}) = \mathcal{K} \otimes \mathfrak{h},$$

we have $\|\tilde{w}_R\|_{B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})} \le C_2(R)$ and hence $\|\tilde{w}_R^* \tilde{w}_R\|_{B(\mathcal{K})} \le C_2(R)^2$. This yields

$$\| \int \tilde{w}_R^*(k) \tilde{w}_R(k) \mathrm{d}k \|_{B(\mathcal{K})} \le C_2(R)^2. \tag{A.5}$$

(The integral in (A.5) should be considered in the weak sense on $B(\mathcal{K})$, ie as a quadratic form on $\mathcal{K}$). We have

$$\begin{aligned} \|a^*(v)u\|^2 = & \int\int (v(k)a^*(k)u, v(k')a^*(k')u)_{\mathcal{H}} \mathrm{d}k \mathrm{d}k' \\ = & \int\int (v(k)a(k')u, v(k')a(k)u)_{\mathcal{H}} \mathrm{d}k \mathrm{d}k' \\ & + \int (v(k)u, v(k)u) \mathrm{d}k. \end{aligned}$$

The second term in the r.h.s. is bounded by

$$\begin{aligned} & \int \|v(k)(B+R)^{-\frac{1}{2}}\|^2 \|(B+R)^{\frac{1}{2}}u\|^2 \mathrm{d}k \\ & \le C_3^2(R) \|(B+R)^{\frac{1}{2}}u\|^2. \end{aligned}$$

We write then the first term as

$$\begin{aligned} & \int\int (\tilde{w}_R(k)\psi(k'), \tilde{w}_R(k')\psi(k))_{\mathcal{H}} \mathrm{d}k \mathrm{d}k' \\ & \le \int\int \|\tilde{w}_R(k)\psi(k')\|_{\mathcal{H}}^2 \mathrm{d}k \mathrm{d}k' \\ & \le \| \int \tilde{w}_R^*(k)\tilde{w}_R(k) \mathrm{d}k \| \int \|\psi(k')\|_{\mathcal{H}}^2 \mathrm{d}k' \\ & \le C_2(R)^2 \|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}} u\|^2, \end{aligned}$$

which proves (A.3). □

Lemma A.2 *Let Q be a closed, positive quadratic form, Q_n be closed quadratic forms on $D(Q)$ such that Q_n converges to Q when $n \to +\infty$ in the topology of $D(Q)$. Let H, H_n be the associated selfadjoint operators. Then for z in a bounded set $U \subset \mathbb{C}\backslash\mathbb{R}$, we have:*

$$\|((H-z)^{-1} - (H_n - z)^{-1})(H+1)^{-\frac{1}{2}}\| \in o(1)|Imz|^{-1}, \ \textit{when } n \to +\infty.$$

and for $\lambda \in \mathbb{R}, \lambda \ll -1$

$$\|((H-\lambda)^{-1} - (H_n-\lambda)^{-1})(H+1)^{-\frac{1}{2}}\| \in o(1) \text{ when } n \to +\infty.$$

Proof. Let for $z \in \mathbb{C}, u \in \mathcal{H}$, $R_n(z) = (H_n - z)^{-1}, R(z) = (H-z)^{-1}$, $r = R_n(z)u - R(z)u$. We have for $v \in D(Q)$:

$$\begin{aligned}(v,u) &= Q(v, R(z)u) - z(v, R(z)u)\\ &= Q_n(v, R_n(z)u) - z(v, R_n(z)u).\end{aligned}$$

Hence for $v = r$ we obtain

$$Q(r, R(z)u) - Q_n(r, R_n(z)u) + z\|r\|^2 = 0,$$

or

$$Q(r,r) - z\|r\|^2 = (Q - Q_n)(r, R(z)u). \tag{A.6}$$

If $\lambda \in \mathbb{R}, \lambda \ll -1$, we deduce from (A.6) that

$$(Q+1)(r,r) \in o(1)(Q+1)(r,r)^{\frac{1}{2}}(Q+1)(R(\lambda)u, R(\lambda)u)^{\frac{1}{2}}.$$

This implies that $(Q+1)(r,r)$ is $o(1)\|u\|$, as claimed.

Let now $z \in U \subset \mathbb{C}\backslash\mathbb{R}$. Taking the imaginary part of (A.6) we obtain

$$\begin{aligned}\|r\|^2 &\in o(1)|Imz|^{-1}(Q+1)(r,r)^{\frac{1}{2}}(Q+1)(R(z)u, R(z)u)^{\frac{1}{2}}\\ &\in o(1)|Imz|^{-2}(Q+1)(r,r)^{\frac{1}{2}}\|u\|^2,\end{aligned}$$

since $(Q+1)(R(z)u, R(z)u)$ is bounded by $|Imz|^{-2}\|u\|^2$ for $z \in U$. Taking then the real part of (A.6) we obtain

$$\begin{aligned}|Q(r,r)| &\in o(1)(Q+1)(r,r)^{\frac{1}{2}}(Q+1)(R(z)u, R(z)u)^{\frac{1}{2}} + o(1)|Imz|^{-2}\\ &\qquad (Q+1)(r,r)^{\frac{1}{2}}\|u\|^2\\ &\in o(1)|Imz|^{-2}(Q+1)^{\frac{1}{2}}(r,r)\|u\|^2.\end{aligned}$$

This implies that $(Q+1)(r,r)^{\frac{1}{2}} \in o(1)|Imz|^{-1}\|u\|$ as claimed. □

The following result is shown in [AH, Lemma 4.9]

Lemma A.3 *Let* H, H_n *for* $n \in \mathbb{N}$ *be selfadjoint operators on a Hilbert space* $\mathcal{H}$. *Let* ψ_n *be a normalized eigenvector of* H_n *with eigenvalue* E_n. *Assume that*

i) $H_n \to H$ *when* $n \to \infty$ *in strong resolvent sense,*

ii) $\lim_{n\to\infty} E_n = E$,

iii) $\text{w-}\lim_{n\to\infty} \psi_n = \psi \neq 0$.

Then ψ *is an eigenvector of* H *with eigenvalue* E.

Acknowledgement

We would like to thank J.M. Bony, J. Fröhlich and G.M. Graf for useful discussions.

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Communicated by Gian Michele Graf
submitted 25/06/99, accepted 14/12/99