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Classical and Quantised Resolvent Algebras for the Cylinder

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Abstract. Buchholz and Grundling (Commun Math Phys 272:699–750, 2007) introduced a C*-algebra called the resolvent algebra as a canonical quantisation of a symplectic vector space and demonstrated that this algebra has several desirable features. We define an analogue of their resolvent algebra on the cotangent bundle $T^*\mathbb{T}^n$ of an n-torus by first generalising the classical analogue of the resolvent algebra defined by the first author of this paper in earlier work (van Nuland in J Funct Anal 277:2815–2838, 2019) and subsequently applying Weyl quantisation. We prove that this quantisation is almost strict in the sense of Rieffel and show that our resolvent algebra shares many features with the original resolvent algebra. We demonstrate that both our classical and quantised algebras are closed under the time evolutions corresponding to large classes of potentials. Finally, we discuss their relevance to lattice gauge theory.

1. Introduction

Much of modern physics concerns the search for and examination of quantum versions of known classical theories. Examples include quantum statistical mechanics, quantum field theory, and quantum gravity. Showing that a classical theory is indeed the limit of the quantum theory at hand can be done at various levels of rigour. The most precise way to establish this limit is by strict deformation quantisation, where one 'quantises' a classical (commutative) Poisson algebra into a quantum (noncommutative) C*-algebra [15, 23] (cf. [12, p. 5] for an overview of various definitions in the literature).

Only few pairs of a classical and a quantum C*-algebra are known to connect in this rigorous fashion [2, 15, 24, 25], and each has its merits and drawbacks. In particular, when taking the torus as a configuration space, we found the known examples too limited in certain respects. Hence, in this paper, we define a quantum observable algebra on the torus, i.e. a C*-algebra $A_\hbar \subseteq B(L^2(\mathbb{T}^n))$, which satisfies the following properties:

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- P1 : The algebra $A_\hbar$ has a classical counterpart A_0 and can be obtained from this commutative algebra through (strict) quantisation.
- P2 : The algebra $A_\hbar$ is closed under the time evolution associated with the potential V for each $V \in C(\mathbb{T}^n)$. The classical analogue A_0 satisfies a similar condition.
- P3 : The classical and quantum algebras associated with a given system are both sufficiently large to accommodate natural embeddings of the respective algebras corresponding to their subsystems.
- P4 : The algebras A_0 and $A_\hbar$ contain the algebra $C_0(T^*\mathbb{T}^n)$ and its quantisation $\mathcal{K}(L^2(\mathbb{T}^n))$, respectively, without being larger than necessary.

An observable algebra satisfying only P1, P2 and P4 has long been known, namely $\mathcal{K}(L^2(\mathbb{T}^n))$, the compact operators on $L^2(\mathbb{T}^n)$, with $C_0(T^*\mathbb{T}^n)$ as its classical limit (cf. [15], in particular sections II.3.4, III.3.6 and III.3.11). We now sketch how the need for P3 arises in quantum lattice gauge theory. Although a significant portion of this introduction is dedicated to this argument, it is presented here in condensed form; more details are found in [29, Sect. 5.1].

Lattice Gauge Theory. In the Hamiltonian lattice gauge theory by Kogut and Susskind [14], one approximates a time slice of spacetime by a finite 'lattice', or more accurately, an oriented graph Λ. The elements of the set of vertices Λ^0 are points in the time slice, while the set of oriented edges Λ^1 are paths between these points. A gauge field corresponding to some connection on a principal fibre bundle over spacetime with structure group some Lie group G is approximated by the parallel transport maps along the edges of Λ. After choosing a trivialisation of the restriction of the principal fibre bundle to Λ^0, the set of all possible parallel transporters can be identified with G^{Λ^1}; this is the configuration space of the Hamiltonian lattice gauge theory, and it carries a natural action of G^{Λ^0} (endowed with the obvious group structure). This group is the analogue of the set of gauge transformations.

Let us take a brief moment to comment on the choice of the structure group G of the gauge theory. Lattice gauge theory was originally introduced by Wilson [33] to explain the phenomenon of quark confinement in the context of the gauge theory known as *quantum chromodynamics (QCD)*. The underlying structure group of QCD is SU(3); hence, the corresponding configuration space is evidently not a torus, and therefore, lattice QCD is outside of the scope of this article. However, it is worth noting that the structure group of electromagnetism is $\mathbb{T}$, so the results in this paper may be applied to corresponding lattice gauge theories or perhaps serve as a stepping stone towards an analogous construction that can be applied to lattice QCD.

We now return to the argument. The Hilbert space of the corresponding quantum lattice gauge theory is $\mathcal{H} = L^2(G^{\Lambda^1})$, where G^{Λ^1} is endowed with the normalised Haar measure. The field algebra of the system is some C*-algebra A_Λ that is represented on $\mathcal{H}$, from which the observable algebra can be obtained by applying a reduction procedure with respect to the gauge group (cf. [13,30]). The observable algebra is accordingly represented on the set of elements of $\mathcal{H}$ that are invariant under gauge transformations. Since the distinction between field and observable algebras is irrelevant with regard to the issue that motivates the present investigation—the embedding maps take the same form in both cases—we will continue to refer to A_Λ as the observable algebra in what follows.

In the context of lattice gauge theory, one is interested in constructing an algebra of the continuum system from the above algebras A_Λ. This is done by considering direct systems of lattices, and we are naturally led to consider the following situation. Suppose that Λ_1 and Λ_2 are both lattices approximating a time slice and that Λ_2 is a better approximation than Λ_1, i.e. $\Lambda_1^0 \subseteq \Lambda_2^0$, the graph Λ_2 contains more paths than Λ_1, and each edge in Λ^1 can be written as a concatenation of paths in Λ^2; for a precise definition, we refer to [1]. We should then be able to find a corresponding embedding map $A_{\Lambda_1} \hookrightarrow A_{\Lambda_2}$. The embedding map takes a simple form if Λ_2 is obtained from Λ_1 by only adding edges: in that case, we have $\mathcal{H}_2 = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_1^c$, where $\mathcal{H}_1^c = L^2(G^{\Lambda_2^1 \setminus \Lambda_1^1})$, and the embedding is given by the restriction of the map

$$B(\mathcal{H}_1) \to B(\mathcal{H}_2) \cong B(\mathcal{H}_1) \hat{\otimes} B(\mathcal{H}_1^c)\,, \qquad a \mapsto a \otimes \mathbf{1}\,,$$

to A_{Λ_1}, where $\mathbf{1}$ denotes the identity on $\mathcal{H}_1^c$, and $\hat{\otimes}$ denotes the von Neumann algebraic tensor product.

A first guess for the observable algebras of the two quantum systems could be $\mathcal{K}(\mathcal{H}_1)$ and $\mathcal{K}(\mathcal{H}_2)$, the algebras of compacts. However, except in trivial cases, the Hilbert space $\mathcal{H}_1^c$ will be infinite-dimensional, which means that $a \otimes \mathbf{1}$ will not be a compact operator. Thus, the algebra $\mathcal{K}(\mathcal{H}_2)$ is too small to accommodate these embeddings. This problem was already noticed by Stottmeister and Thiemann in [31]. In an earlier paper [1] on Hamiltonian lattice gauge theory coauthored by one of the authors of the present article, the above problem was not encountered since different embedding maps were used. There are nevertheless good reasons to believe that the embedding maps used in [31] are the correct ones, though we will not elaborate on them here, and refer to [29, Chapter 8] instead. The argument presented there is not specific to lattice gauge theory, but can be made for any physical system that is comprised of smaller subsystems.

Another guess for the observable algebra of the composite system could be the one generated by the embedded algebras of all subgraphs, as is done in [11]. However, this raises questions about regulator independence of this procedure in situations where one takes limits corresponding to an infinite volume or continuum limit of a collection of systems parametrised by a cutoff. As this problem is beyond the scope of the present article, we will refer the reader to the discussion in [29, Sect. 5.1]. The main point is that there is ample reason to try to solve the problem through an appropriate choice of algebras, i.e. algebras that satisfy P3.

The Resolvent Algebra on. $\mathbb{R}^n$ In the case where the configuration space is $\mathbb{R}^n$, there already exists an algebra satisfying P1, P2, P3 and P4, namely the resolvent algebra $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$. The resolvent algebra $\mathcal{R}(X, \sigma)$ on a symplectic vector space (X, σ) is a C*-algebra that was originally introduced by Buchholz and Grundling in [8], and subsequently studied in greater detail in [9] and [5] by the same authors. Before we adapt this algebra to the case of $T^*\mathbb{T}^n$ instead of $\mathbb{R}^{2n}$ as its underlying phase space, let us recall the main idea behind the construction of the resolvent algebra.

The resolvent algebra is constructed as the completion of a *-algebra with respect to a certain C*-seminorm [9, Definition 3.4]; the *-algebra is defined in terms of generators and relations. To each pair $(\lambda, f) \in (\mathbb{R}\backslash\{0\}) \times X$, a generator $\mathcal{R}(\lambda, f)$ is associated. Such a generator is thought of as the resolvent (depending on λ) corresponding to some unbounded operator $\phi(f)$ associated with the vector f, where ϕ denotes a linear map from X to a space of operators on a dense subspace of a Hilbert space on which $\mathcal{R}(X, \sigma)$ can be represented faithfully.

For example, suppose that (X, σ) is $\mathbb{R}^2$ endowed with the standard symplectic form. Then $\mathcal{R}(X, \sigma)$ admits a faithful representation on $L^2(\mathbb{R})$ such that the unbounded operators corresponding to the vectors $(1, 0)$ and $(0, 1)$ are the standard position and momentum operators, respectively (up to a factor of $\hbar$ in the latter case), see [9, Corollary 4.4 and Theorem 4.10]. Both of these unbounded operators can be defined on the (invariant) dense subspace $C_c^\infty(\mathbb{R})$, on which they are essentially self-adjoint.

For each $f \in X$, the generator $\mathcal{R}(\lambda, f)$ is mapped to the bounded operator $(i\lambda\mathbf{1} - \phi(f))^{-1}$; in particular, taking $f = 0$, we see that $\mathcal{R}(X, \sigma)$ is unital. The relations defining the *-algebra from which the resolvent algebra is constructed serve to encode the fact that $\mathcal{R}(\lambda, f)$ behaves like the resolvent of the unbounded operator $\phi(f)$, as well as the linearity of ϕ. Last but not least, the canonical commutation relations (CCR) are introduced by the defining relations of $\mathcal{R}(X, \sigma)$ in which the symplectic form appears, thereby justifying the term "canonical quantum systems" in the title of [9].

The resolvent algebra is not the only approach to the reformulation of the CCR in a framework based on bounded operators; another is obtained through exponentiation of the unbounded operators of interest, leading to the Weyl form of the CCR and the Weyl algebra. There is a bijection between certain classes of representations of these two algebras [9, Corollary 4.4]. In particular, generators of the resolvent algebras can be expressed in terms of generators of the Weyl algebra by means of the Laplace transform, as is done in [8]. By changing the representation in that definition to the usual representation on $L^2(\mathbb{R})$ of the Weyl algebra on $\mathbb{R}^2$, one obtains the representation mentioned earlier.

Buchholz and Grundling note that their resolvent algebra has some desirable qualities not shared by the Weyl algebra, such as the presence of observables corresponding to bounded functions in regular representations. Furthermore—and this is particularly relevant for this paper—the resolvent algebra associated with $\mathbb{R}^2$ endowed with the standard symplectic form is closed under (quantum) time evolution for a much larger class of Hamiltonians than the Weyl algebra (cf. [9, Proposition 6.1]). The authors explain this as a consequence of the fact that their resolvent algebra contains resolvents of many Hamiltonians. Moreover, Buchholz has shown that the resolvent algebra is stable under dynamics in the context of oscillating lattice systems [6] and nonrelativistic Bose fields [7].

According to [9, Theorem 5.1], for any symplectic vector space (X,σ) and any decomposition $S\oplus S^{\perp}$ of X into subspaces that are nondegenerate with respect to σ (and where $S^{\perp}$ denotes the complement of S with respect to σ), the resolvent algebra $\mathcal{R}(X,\sigma)$ naturally contains a copy of $\mathcal{R}(S,\sigma)\hat{\otimes}\mathcal{R}(S^{\perp},\sigma)$; with respect to corresponding faithful representations of these three resolvent algebras, the embeddings of $\mathcal{R}(S,\sigma)$ and $\mathcal{R}(S^{\perp},\sigma)$ are given by the analogues of the aforementioned embedding map for lattice gauge theory. Here, $\hat{\otimes}$ denotes any C*-tensor product (nuclearity of the resolvent algebra is shown in [5]), and σ by abuse of notation denotes the symplectic form on X, as well as its restrictions to S and $S^{\perp}$.

We have seen how properties P3 and P2 were shown by Buchholz and Grundling to hold for the resolvent algebra. Now for P1 the question is whether it arises as the strict quantisation of an algebra that can be considered the observable algebra of a classical system in the sense of Landsman, i.e. the C*-algebra generated by the image of a dense Poisson subalgebra of the classical algebra under a quantisation map [15]. This question was answered affirmatively by one of the authors of this paper in [19], where it is shown that in the case where (X,σ) is $\mathbb{R}^{2n}$ endowed with the standard symplectic form, there is a corresponding commutative C*-algebra $C_{\mathcal{R}}(\mathbb{R}^{2n})$, which is the C*-subalgebra of $C_b(\mathbb{R}^{2n})$ generated by functions of the form

$$x\mapsto(i\lambda-x\cdot v)^{-1},\quad \lambda\in\mathbb{R}\backslash\{0\},\ v\in\mathbb{R}^{2n},$$

where $\cdot$ denotes the standard inner product. Similar to the way in which the algebra $C_0(\mathbb{R}^{2n})$ may be quantised into the compact operators on $L^2(\mathbb{R}^n)$ by considering the dense Poisson subalgebra $\mathcal{S}(\mathbb{R}^{2n})$ of Schwartz functions and defining Weyl or Berezin quantisation on them, an analogue of the space of Schwartz functions for $C_{\mathcal{R}}(\mathbb{R}^{2n})$ is identified as follows. First, for every linear subspace $V\subseteq\mathbb{R}^{2n}$, let P_V denote the orthogonal projection onto V. Then the space

$$\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n}):=\operatorname{span}_{\mathbb{C}}\{g\circ P_V\colon V\subseteq\mathbb{R}^{2n}\text{ is a subspace, }g\in\mathcal{S}(V)\},$$

is defined. It is readily seen that this is a dense Poisson subalgebra of $C_{\mathcal{R}}(\mathbb{R}^{2n})$ that is closed under the *-operation of complex conjugation. The Weyl quantisation of $g\circ P_V$ is defined using the Fourier transform of g as a function on V [19, Sect. 3.2], but is otherwise equal to the definition of the Weyl quantisation of ordinary Schwartz functions on $\mathbb{R}^{2n}$. It is then argued that the Weyl quantisation map admits a (unique) linear extension to $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$. Furthermore, it is shown that the images of $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$ under Weyl and Berezin quantisation are both dense subspaces of $\mathcal{R}(\mathbb{R}^{2n},\sigma)$. The resulting algebra $C_{\mathcal{R}}(\mathbb{R}^{2n})$ is accordingly referred to as the *classical resolvent algebra on* $\mathbb{R}^{2n}$. As is shown in [19], these definitions are easily extended to spaces of functions whose domain is an inner product space of infinite dimension.

In addition to being the classical counterpart of the resolvent algebra as defined by Buchholz and Grundling, the classical resolvent algebra offers an interesting perspective on our earlier discussion on embeddings of observable algebras. In some sense, $C_{\mathcal{R}}(\mathbb{R}^{2n})$ is the smallest C*-subalgebra of $C_b(\mathbb{R}^{2n})$ that contains $C_0(\mathbb{R}^{2n})$, whilst also containing its analogues associated with linear subspaces of $\mathbb{R}^{2n}$. This may be formalised as follows. Consider the category whose objects are finite-dimensional real vector spaces and whose morphisms consist of projections of a vector space onto one of its subspaces. Then there is a contravariant functor C_b from this category to the category of C*-algebras that maps an object V to the space $C_b(V)$ and that maps morphisms to their pullbacks between these spaces. It is now consistent with the definition of the classical resolvent algebra to define $C_{\mathcal{R}}$ as the smallest subfunctor of C_b with the property that the image of every object V contains $C_0(V)$. Note that this implies that $C_{\mathcal{R}}(\mathbb{R}^{2n})$ is unital, as it contains the embedding of $C_0(\{0\})$. This makes precise in which sense P4 holds for the resolvent algebras on $\mathbb{R}^{2n}$.

Resolvent Algebras on the Cylinder. In this paper, we define an analogue of the resolvent algebra for the cotangent bundle $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$ of the n-torus $\mathbb{T}^n$. Our approach differs significantly from that of Buchholz and Grundling, in that we do not define it in terms of generators and relations. Rather, we first identify a classical resolvent algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ using ideas from [19] and indicate how this definition may be generalised. We then give a concrete characterisation of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. Namely, identifying $T^*\mathbb{T}^n$ with $\mathbb{T}^n \times \mathbb{R}^n$, we prove that $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ equals $C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, where $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ is the C*-algebra generated by the functions

$$x \mapsto 1/(i + x \cdot v) \quad \text{and} \quad x \mapsto e^{ix\cdot v}\,, \quad \text{for all} \quad v \in \mathbb{R}^n\,. \tag{1}$$

In addition, we identify a dense *-subalgebra $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$ carrying a natural Poisson structure. The algebra is spanned by functions of the form $e_k \otimes h$, where $e_k[x] := e^{2\pi i k\cdot x}$, and h is a smooth function that is a product of an element of $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^n)$ and a function of the form $x \mapsto e^{i\xi\cdot x}$ for some $\xi \in \mathbb{R}^n$. (Here, $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^n)$ is defined analogously to the definition of $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$ above.)

To define a quantum counterpart, we apply Weyl quantisation, making P1 integral to the definition of the (quantum) resolvent algebra on $T^*\mathbb{T}^n$. Our Weyl quantisation map $\mathcal{Q}^W_\hbar : \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n) \to B(L^2(\mathbb{T}^n))$ is defined with an integral formula inspired by [25]. When writing $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ as a tensor product as above, $\mathcal{Q}^W_\hbar$ can be characterised by the formula

$$\mathcal{Q}^W_\hbar(e_k \otimes h)\psi_l = h(\pi\hbar(k + 2l))\psi_{k+l}\,, \tag{2}$$

where $e_k \otimes h \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, and ψ_k is e_k viewed as an element of $L^2(\mathbb{R}^n)$ for each $k \in \mathbb{Z}^n$. The above formula is consistent with the usual Weyl quantisation on (a Poisson *-subalgebra of) the smaller classical algebra $C_0(T^*\mathbb{T}^n)$, see e.g. [15, Sect. II.3.4], as $\mathbb{T}^n$ is in particular a Riemannian manifold with its corresponding Levi-Civita connection. Although this consistency already justifies (2) as a reasonable extension of Weyl quantisation, we start Sect. 4 with a systematic way to arrive at (2). Thereafter, we define the (quantum) resolvent algebra on the torus as

$$A_\hbar := C^*(\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))) \subseteq B(L^2(\mathbb{T}^n)),$$

before remarking that $A_\hbar \cong A_{\hbar'}$ for all $\hbar, \hbar' \in (0, \infty)$. We check P3 by using this explicit description of $\mathcal{Q}_\hbar^W$ and the fact that P3 holds for $C_\mathcal{R}(T^*\mathbb{T}^n)$, which is readily seen. P4 is satisfied by definition of $C_\mathcal{R}(T^*\mathbb{T}^n)$. In addition, we show that an analogue of P2 holds for our algebras, both the classical and the quantum one, in the following very strong sense: our classical resolvent algebra $C_\mathcal{R}(T^*\mathbb{T}^n)$ is closed under the classical time evolution associated with the potential V for each $V \in C^1(\mathbb{T}^n)$ with Lipschitz continuous derivative. Our quantum resolvent algebra is closed under the quantum time evolution associated with the potential V for each $V \in C(\mathbb{T}^n)$. (In both cases, the free part of the Hamiltonian is the usual one.) Unlike the analogous result in [9] in which a similar result is established only for $\mathbb{R}^{2n}$ with $n = 1$, we give proofs of these statements for arbitrary $n \in \mathbb{N}$.

The paper is structured as follows. In Sect. 2, we first give a well-motivated definition of the classical resolvent algebra $C_\mathcal{R}(T^*\mathbb{T}^n)$. We proceed by analysing its structure, culminating in an alternative, more practical characterisation of $C_\mathcal{R}(T^*\mathbb{T}^n)$, namely as the tensor product $C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_\mathcal{R}(\mathbb{R}^n)$. Furthermore, we identify a dense Poisson *-subalgebra that serves the same purpose as $\mathcal{S}_\mathcal{R}(\mathbb{R}^n)$ in [19].

Section 3 proves the fact that $C_\mathcal{R}(T^*\mathbb{T}^n)$ is closed under the classical time evolution as mentioned above.

In Sect. 4, we adapt Weyl quantisation to functions on $T^*\mathbb{T}^n$, proving an explicit formula for generators of $C_\mathcal{R}(T^*\mathbb{T}^n)$ in the process. This formula is then used to show that Weyl quantisation is almost a strict quantisation in the sense of Landsman. We say 'almost', because we explicitly show that its norm fails to be continuous with respect to $\hbar$ for $\hbar > 0$. However, the quantisation map is continuous in a weaker sense.

In Sect. 5, we show that our quantised resolvent algebra is closed under the quantum time evolution.

2. Definition and Basic Results

On the phase space $\mathbb{R}^{2n}$, we already have a commutative C*-algebra that satisfies P2, P3 and P4 mentioned in the introduction and forms the classical part of a strict deformation quantisation, namely the commutative resolvent algebra $C_\mathcal{R}(\mathbb{R}^{2n})$ defined in [19]. We begin this section by adapting its definition to $T^*\mathbb{T}^n$. As mentioned in introduction, we identify $T^*\mathbb{T}^n$ with $\mathbb{T}^n \times \mathbb{R}^n$ and note that the latter space carries a natural left action of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ by translation.

Definition 1. For each $(v, w) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, let $(\mathbb{T}^n \times \mathbb{R}^n)/\{(v, w)\}^\perp$ be the space of orbits of the restriction of the action of $\mathbb{R}^{2n}$ to the linear subspace $\{(v, w)\}^\perp := \{x \in \mathbb{R}^{2n} \mid x \cdot (v, w) = 0\}$ of $\mathbb{R}^{2n}$, and let

$$\pi_{(v,w)} \colon \mathbb{T}^n \times \mathbb{R}^n \to (\mathbb{T}^n \times \mathbb{R}^n)/\{(v, w)\}^\perp$$

be the corresponding canonical projection. We then define the *commutative resolvent algebra* $C_\mathcal{R}(T^*\mathbb{T}^n)$ as the smallest C*-subalgebra of $C_b(\mathbb{T}^n \times \mathbb{R}^n)$

generated by the set of functions

$$\left\{ f \circ \pi_{(v,w)} \;\middle|\; (v,w) \in \mathbb{R}^{2n},\ f \in C_0((\mathbb{T}^n \times \mathbb{R}^n)/\{(v,w)\}^\perp) \right\},$$

that is, the set of continuous functions invariant under the action of $\{(v,w)\}^\perp \subseteq \mathbb{R}^{2n}$ for which the induced map on $(\mathbb{T}^n \times \mathbb{R}^n)/\{(v,w)\}^\perp$ vanishes at infinity.

To establish the link with the definition of $C_{\mathcal{R}}(\mathbb{R}^n)$ given in [19], note that there is a straightforward generalisation of the above definition to arbitrary topological spaces M carrying a left action of $\mathbb{R}^m$ for some $m \in \mathbb{N}$. Taking $M = \mathbb{R}^n$ and $m = n$ then yields the definition of $C_{\mathcal{R}}(\mathbb{R}^n)$. Unfortunately, T^*G does not have an appropriate action of $\mathbb{R}^{2n}$ for a nonabelian Lie group G that would enable us to unambiguously generalise this construction.

The definition of the classical resolvent algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is clearly motivated, but very unwieldy in practice. Our first task is therefore to find an alternative, more elementary characterisation of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. To this end, we will use the following elementary facts about the action of $\mathbb{R}^n$ on $\mathbb{T}^n$. Throughout the rest of the text, we let $[x] \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the quotient class of $x \in \mathbb{R}^n$.

Lemma 2. *Let $v \in \mathbb{R}^n \backslash \{0\}$.*

(1) *Exactly one of the following two statements holds true:*
   (i) *The map $\mathbb{R} \to \mathbb{T}^n$, $t \mapsto [tv]$ is periodic.*
   (ii) *The set $H := \{[x] \in \mathbb{T}^n : x \in \mathbb{R}^n,\ v \cdot x = 0\}$ is dense in $\mathbb{T}^n$.*

(2) *Suppose now that $t \mapsto [tv]$ is periodic, with period T. Furthermore, let $\pi_v : \mathbb{T}^n \to \mathbb{T}^n/H$ be the quotient map. Then H is a closed subgroup of $\mathbb{T}^n$, and*

$$\varphi : \mathbb{T}^n/H \to \mathbb{T}, \quad \pi_v([x]) \mapsto [Tv \cdot x],$$

is a well-defined isomorphism of topological groups.

Proof.

(1) We show that at least one of the two statements is true; we postpone the proof that the two statements are mutually exclusive to the proof of the second part of this lemma. The case $n = 1$ is trivial, and the case $n = 2$ is the well-known result that a line in $\mathbb{T}^2$ is dense iff it has irrational slope. We therefore assume that $n > 2$, and we will reduce the problem to the known two-dimensional case.

Suppose that (ii) is false for $n > 2$, and let $U \subseteq \mathbb{T}^n \backslash H$ be a non-empty open subset. Without loss of generality, we may assume that $U = [y] + U$ for all $y \perp v$. As $v \neq 0$ by assumption, we may choose j such that $v_j \neq 0$. Now suppose k is a different index with $v_k \neq 0$. Define

$$T_2 := \{[x] \in \mathbb{T}^n : x \in \mathbb{R}^n,\ x_i = 0 \text{ for all } i \notin \{j,k\}\} \subseteq \mathbb{T}^n,$$

which is a subgroup of $\mathbb{T}^n$ isomorphic to $\mathbb{T}^2$. Note that

$$U \cap T_2 \subseteq T_2 \setminus \{[x] \in T_2 : x \in \mathbb{R}^n,\ v_j x_j + v_k x_k = 0\}.$$

To see that $U \cap T_2$ is non-empty, let us pick $[x] \in U$. Because $v_j \neq 0$, there exists a $y \perp v$ such that $[y + x] \in T_2$ (for instance $y = \frac{v \cdot x}{v_j}\delta_j - x$, where δ_j is the j^{th} standard basis vector). Since $[y+x] \in [y] + U = U$ we have $[y+x] \in U \cap T_2 \neq \emptyset$. Applying the result for $n = 2$, one finds that $[t(v_j, v_k)]$ is periodic in t. Since k (such that $v_k \neq 0$) was arbitrary, every component v_k is a rational multiple of the nonzero component v_j; hence, $[tv]$ is periodic in t.

(2) We first note that the map φ is induced (in two steps) by the continuous group homomorphism

$$\mathbb{R}^n \to \mathbb{T}\,, \quad x \mapsto [Tv \cdot x]\ .$$

Since $Tv \in \mathbb{Z}^n$, this map factors through $\mathbb{T}^n$, thereby inducing a continuous group homomorphism $\varphi_0 \colon \mathbb{T}^n \to \mathbb{T}$. It is readily seen that H is a subgroup of $\mathbb{T}^n$ that is a subset of $\ker \varphi_0$, so φ_0 factors through the quotient $\mathbb{T}^n/H$, thereby inducing the continuous group homomorphism φ.

Next, we argue that φ is in fact a homeomorphism. We prove this by showing that the map

$$\mathbb{T} \to \mathbb{T}^n/H\,, \quad [t] \mapsto \pi_v\left(\left[\frac{tv}{T\,\|v\|^2}\right]\right)\,,$$

is a well-defined inverse; we will tentatively refer to this map as φ^{-1} in what follows.

First we show that φ^{-1} is well defined. This amounts to showing that

$$\frac{v}{T\,\|v\|^2} \cdot \mathbb{Z} \subseteq \{a + x \colon a \in \mathbb{Z}^n,\ x \in \{v\}^\perp\}\,,$$

which is the case exactly when $v/(T\,\|v\|^2)$ is an element of the set on the right-hand side of the inclusion. Note that the components of the vector Tv are coprime; otherwise, T/m would be the period of $t \mapsto [tv]$ for some natural number $m > 1$. By the higher-dimensional Bézout identity, there exists a tuple $a \in \mathbb{Z}^n$ such that $Tv \cdot a = 1$. Now observe that

$$\frac{v}{T\,\|v\|^2} = \frac{v \cdot a}{\|v\|^2} v = a + \left(\frac{v \cdot a}{\|v\|^2} v - a\right)\,,$$

and note that the first and second terms on the right-hand side of this equation are contained in $\mathbb{Z}^n$ and $\{v\}^\perp$, respectively. Thus, φ^{-1} is well defined. It is straightforward to check that φ^{-1} is both a left- and a right-inverse of φ, so φ^{-1} is indeed the inverse of φ. It is readily seen that φ^{-1} is continuous, so φ is both a group isomorphism and a homeomorphism.

To see that φ is in fact an isomorphism of topological groups, note that $H = \ker \varphi_0$ by injectivity of φ, so H is a closed subgroup of $\mathbb{T}^n$, and the quotient $\mathbb{T}^n/H$ naturally inherits the structure of a topological group from $\mathbb{T}^n$; in particular, the quotient is Hausdorff. This concludes the proof of part (2).

Finishing up the proof of part (1), we note that in case (i), the quotient $\mathbb{T}^n/H$ is homeomorphic to $\mathbb{T}$, whereas in case (ii), the quotient is an indiscrete space. Thus, the two cases are mutually exclusive. $\square$

Remark 3. In case (i), H is a Lie subgroup of the Lie group $\mathbb{T}^n$ by the closed subgroup theorem. In fact, φ is an isomorphism of Lie groups from $\mathbb{T}^n/H$ to $\mathbb{T}$ endowed with their respective canonical Lie group structures. See [29, Lemma 5.4] for a Lie-theoretic version of the previous lemma.

Before we characterise $C_{\mathcal{R}}(T^*\mathbb{T}^n)$, we must introduce another algebra, for which it is in turn useful to recall that the algebra of *almost periodic functions on* $\mathbb{R}^n$ is the C*-subalgebra of $C_b(\mathbb{R}^n)$ generated by functions of the form $x \mapsto e^{i\xi\cdot x}$, where $\xi \in \mathbb{R}^n$ is arbitrary. Almost periodic functions were originally introduced by H. Bohr in [3] for $n = 1$ using a different definition, whose equivalence with the one mentioned above he proved in [4]. This algebra will be denoted by $\mathcal{W}^0(\mathbb{R}^n)$.

Definition 4. Let $n \in \mathbb{N}$. We define the algebra $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ as the C*-subalgebra of $C_b(\mathbb{R}^n)$ generated by the classical resolvent algebra $C_{\mathcal{R}}(\mathbb{R}^n)$ and the algebra of almost periodic functions $\mathcal{W}^0(\mathbb{R}^n)$ on $\mathbb{R}^n$.

Next up is the main result of this section, which unveils $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ as a tensor product of two algebras. We regard the algebraic tensor product of two C*-algebras $A \subseteq C_b(X)$ and $B \subseteq C_b(Y)$ as a subset of $C_b(X \times Y)$ via $(f \otimes g)(x, y) = f(x)g(y)$ and denote its corresponding completion by $A\hat{\otimes}B$. Since commutative C*-algebras are nuclear (cf. [18, Theorem 6.4.15]), this is equivalent to any other C*-algebraic tensor product.

Theorem 5. *For each* $n \in \mathbb{N}$*, we have*

$$C_{\mathcal{R}}(T^*\mathbb{T}^n) = C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)\,.$$

Proof. The statement is trivial for $n = 0$, so suppose $n \geq 1$. We first prove the inclusion $C_{\mathcal{R}}(T^*\mathbb{T}^n) \subseteq C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ by showing that the generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ are contained in the right-hand side. Let $(v, w) \in \mathbb{R}^n \times \mathbb{R}^n$, and let $f = g \circ \pi_{(v,w)}$ be one of the generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. As in Lemma 2, let H be the image of $\{v\}^\perp$ under the canonical projection map $\mathbb{R}^n \to \mathbb{T}^n$. Moreover, let H' be the image of $\{(v, w)\}^\perp$ under the canonical projection map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$.

By part (1) of Lemma 2, we may distinguish between the following three cases. In each of these cases, we obtain the general form of f by first giving a characterisation of the quotient space $(\mathbb{T}^n \times \mathbb{R}^n)/H'$:

(i) $v = 0$: in this case, we have $H' = \mathbb{T}^n \times \{w\}^\perp$; in particular, it is a closed subgroup of $\mathbb{T}^n \times \mathbb{R}^n$, and the map

$$(\mathbb{T}^n \times \mathbb{R}^n)/H' \to \mathbb{R}\cdot w\,, \quad \pi_{(v,w)}([x], p) \mapsto (w \cdot p)w\,,$$

is an isomorphism of topological groups. It follows that f is the pullback of a function in $C_0(\mathbb{R} \cdot w)$ along the above map, from which it is readily seen that

$$f \in \mathbb{C}\mathbf{1}_{\mathbb{T}^n}\hat{\otimes}C_{\mathcal{R}}(\mathbb{R}^n) \subseteq C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)\,;$$

In particular, note that f is constant iff $w = 0$.

To handle the remaining two cases in which $v \neq 0$, we introduce the map

$$\theta\colon (\mathbb{T}^n \times \mathbb{R}^n)/H' \to \mathbb{T}^n/H\,,$$
$$\pi_{(v,w)}([x],p) \mapsto \pi_v\left(\left[\frac{v\cdot x + w\cdot p}{\|v\|^2}v\right]\right) = \pi_v\left(\left[x + \frac{w\cdot p}{\|v\|^2}v\right]\right)\,,$$

and show that it is a well-defined group isomorphism and a homeomorphism. To see that it is a well-defined continuous group homomorphism, note that it is induced by a continuous group homomorphism

$$\theta_0\colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n/H\,,$$

which is defined using a similar formula, and whose kernel contains the subgroup H'. To see that θ is a group isomorphism and a homeomorphism, we note that the map

$$\mathbb{T}^n/H \to (\mathbb{T}^n \times \mathbb{R}^n)/H'\,, \quad \pi_v([x]) \mapsto \pi_{(v,w)}([x],0)\,,$$

is a well-defined continuous group homomorphism (by a similar argument as for θ) that can be checked to be the inverse of θ. In particular, $H' = \ker\theta_0$. As we will see below, θ need not be an isomorphism of topological groups if we require such groups to be Hausdorff spaces. We proceed with the remaining two cases:

(ii) $v \neq 0$ and H is dense in $\mathbb{T}^n$: in this case, the quotient topology on $\mathbb{T}^n/H$ is the indiscrete topology; hence, $(\mathbb{T}^n \times \mathbb{R}^n)/H'$ is also indiscrete by our discussion above. It follows that the function f is constant, so $f \in C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

(iii) $v \neq 0$ and the curve $t \mapsto [tv]$ on $\mathbb{T}^n$ is periodic: then the map θ_0 defined above is a continuous surjective group homomorphism; hence, its kernel H' is a closed subgroup of $\mathbb{T}^n \times \mathbb{R}^n$, and the map θ is an isomorphism of topological groups. Composing θ with the map φ from part (2) of Lemma 2, we obtain the isomorphism of topological groups

$$\varphi\circ\theta\colon (\mathbb{T}^n \times \mathbb{R}^n)/H' \to \mathbb{T}\,, \quad \pi_{(v,w)}([x],p) \mapsto [T(v\cdot x + w\cdot p)]\,,$$

with T as defined in Lemma 2. Then $f = g\circ\varphi\circ\theta\circ\pi_{(v,w)}$ for some $g \in C(\mathbb{T})$; let us first assume that $g = e_k$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} f([x],p) &= \exp\left(2\pi i k T(v\cdot x + w\cdot p)\right) \\ &= \exp\left(2\pi i k T v\cdot x\right)\cdot\exp\left(2\pi i k T w\cdot p\right)\,, \end{aligned}$$

which shows that $f \in C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Since the family of exponential functions $(e_k)_{k\in\mathbb{Z}}$ generate $C(\mathbb{T})$, and since pullback along the map

$$\varphi\circ\theta\circ\pi_{(v,w)}\colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}\,,$$

is a homomorphism of C*-algebras, it follows that

$$f = g\circ\varphi\circ\theta\circ\pi_{(v,w)} \in C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)\,,$$

for arbitrary $g \in C(\mathbb{T})$.

This establishes the inclusion $C_{\mathcal{R}}(T^*\mathbb{T}^n) \subseteq C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. The reverse inclusion is a consequence of the following three observations:

- From case (i) in the previous part of this proof, we readily obtain $\mathbb{C}\mathbf{1}_{\mathbb{T}^n}\hat{\otimes}C_{\mathcal{R}}(\mathbb{R}^n) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$.
- From case (iii), setting $w = 0$ and taking v to be a standard basis vector of $\mathbb{R}^n$, we obtain $C(\mathbb{T}^n)\hat{\otimes}\mathbb{C}\mathbf{1}_{\mathbb{R}^n} \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$.
- Finally, by considering case (iii) again, but now with v the first standard basis vector and $w \in \mathbb{R}^n$ arbitrary, we see that $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ contains functions of the form
$$([x], p) \mapsto \exp(2\pi i k x_1)\exp(i\xi \cdot p)\,,$$
where $k \in \mathbb{Z}\backslash\{0\}$, and $\xi \in \mathbb{R}^n$ is arbitrary. The previous point now implies that functions of the form
$$([x], p) \mapsto \exp(i\xi \cdot p)\,,$$
are elements of the resolvent algebra, so $\mathbb{C}\mathbf{1}_{\mathbb{T}^n}\hat{\otimes}\mathcal{W}^0(\mathbb{R}^n) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$.

$\square$

We identify a broad class of embeddings between the classical algebras.

Corollary 6. *Let $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$. For any surjective continuous map $\varphi\colon \mathbb{T}^{n_2} \to \mathbb{T}^{n_1}$ and any surjective linear map $L\colon \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, define the surjection*
$$M := \varphi \times L\colon \mathbb{T}^{n_2} \times \mathbb{R}^{n_2} \to \mathbb{T}^{n_1} \times \mathbb{R}^{n_1}, \quad ([x], p) \mapsto (\varphi([x]), L(p)).$$
Then the pull-back $M^*\colon f \mapsto f \circ M$ restricts to a map $C_{\mathcal{R}}(T^*\mathbb{T}^{n_1}) \to C_{\mathcal{R}}(T^*\mathbb{T}^{n_2})$, and the restriction is an embedding in the sense of C^*-algebras.

Proof. As M is surjective and continuous, $M^*\colon C_b(\mathbb{T}^{n_1}\times\mathbb{R}^{n_1}) \to C_b(\mathbb{T}^{n_2}\times\mathbb{R}^{n_2})$ is automatically an injective *-homomorphism, and therefore an embedding. We note that φ^* sends $C(\mathbb{T}^{n_1})$ to $C(\mathbb{T}^{n_2})$ and L^* sends $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^{n_1})$ to $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^{n_2})$. Hence, M^* sends elementary tensors in $C_{\mathcal{R}}(T^*\mathbb{T}^{n_1}) = C(\mathbb{T}^{n_1})\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^{n_1})$ into $C_{\mathcal{R}}(T^*\mathbb{T}^{n_2}) = C(\mathbb{T}^{n_2})\hat{\otimes}\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^{n_2})$. By linearity and continuity of M^*, the statement follows. $\square$

We finish this section by defining the analogue of the space of Schwartz functions of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. This allows us to introduce the notation $h_{U,\xi,g}$ for the generators of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, which is used in Sect. 4.

Definition 7. For each $k \in \mathbb{Z}^n$, let
$$e_k\colon \mathbb{T}^n \to \mathbb{C}, \quad [x] \mapsto e^{2\pi i k\cdot x}\,.$$
For each subspace $U \subseteq \mathbb{R}^n$, for each $\xi \in U^\perp$, and for each Schwartz function $g \in \mathcal{S}(U)$, let
$$h_{U,\xi,g}\colon \mathbb{R}^n \to \mathbb{C}, \quad p \mapsto e^{i\xi\cdot p} g(P_U(p))\,,$$
where $P_U\colon \mathbb{R}^n \to U$ denotes the orthogonal projection onto U. We define the space $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ as the span of functions of the form $e_k \otimes h_{U,\xi,g} : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}$.

Proposition 8.

(1) *The space $\mathrm{span}\{h_{U,\xi,g}\colon U \subseteq \mathbb{R}^n \text{ linear},\ \xi \in U^\perp,\ g \in \mathcal{S}(U)\}$ is a norm-dense *-subalgebra of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ that is closed under partial differentiation.*

(2) The space $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is a *-subalgebra of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ that is closed under partial differentiation, and is consequently a Poisson subalgebra of $C^\infty(T^*\mathbb{T}^n)$. Moreover, $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is norm-dense in $C_{\mathcal{R}}(T^*\mathbb{T}^n)$.

Proof.

(**1**) Denote $\mathcal{B} := \operatorname{span}\{h_{U,\xi,g}\} \subset \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. For any $h_{U,\xi,g}$ as in Definition 7,
$$h^*_{U,\xi,g} = \overline{h_{U,\xi,g}} = h_{U,-\xi,\overline{g}} \in \mathcal{B}\,,$$
hence $\mathcal{B}$ is closed under the *-operation.

Assume for the moment that $\mathcal{B}$ is closed under multiplication. To see that $\mathcal{B}$ is invariant under partial differentiation, it suffices to show that partial derivatives of functions of the form $h_{U,\xi,g}$ are elements of $\mathcal{B}$. Any partial derivative can be written as a sum of two directional derivatives; one in a direction lying in U, and one in a direction lying in $U^\perp$. It is readily seen that both of these directional derivatives are elements of $\mathcal{B}$, hence so is their sum.

To show that $\mathcal{B}$ is closed under multiplication, it suffices to show that the product of two functions h_{U_1,ξ_1,g_1} and h_{U_2,ξ_2,g_2} as in Definition 7, is an element of $\mathcal{B}$. Let
$$\begin{aligned} U &:= U_1 + U_2\,,\\ \xi &:= \xi_1 + \xi_2 - P_U(\xi_1+\xi_2) \in U^\perp\,,\\ \tilde{g} &:= (g_1 \circ P_{U_1})(g_2 \circ P_{U_2})\,. \end{aligned}$$
Note that the restrictions of $\tilde{g}$ to U and $U^\perp$ are Schwartz and constant, respectively. Setting
$$g\colon U \to \mathbb{C}\,, \quad p \mapsto e^{iP_U(\xi_1+\xi_2)\cdot p}\tilde{g}|_U \circ P_U(p) = e^{i(\xi_1+\xi_2)\cdot p}\tilde{g}|_U \circ P_U(p)\,,$$
we see that $h_{U_1,\xi_1,g_1} \cdot h_{U_2,\xi_2,g_2} = h_{U,\xi,g}$, which establishes that $\mathcal{B}$ is closed under multiplication.

Thus, $\mathcal{B}$ is a *-subalgebra of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. In addition to this fact, the elements of the form $h_{\{0\},\xi,1}$ generate $\mathcal{W}^0(\mathbb{R}^n)$, while the elements of the form $h_{U,0,g}$ generate $C_{\mathcal{R}}(\mathbb{R}^n)$; hence, $\mathcal{B}$ generates $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ as a C*-algebra. We infer that $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ is the closure of $\mathcal{B}$.

(**2**) For each $k \in \mathbb{Z}^n$, define e_k as in Definition 7. It is a trivial matter to check that the linear span of $\{e_k\colon k \in \mathbb{Z}\}$ is a *-subalgebra of $C(\mathbb{T}^n)$ that is closed with respect to partial differentiation, and it is a result from Fourier analysis that this linear subspace is dense in $C(\mathbb{T}^n)$. Using these facts in conjunction with part (1) of this proposition and Theorem 5, it is readily seen that all of the assertions are true.

□

3. Classical Time Evolution

In this section, we prove that $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is preserved under the (time) flow induced by the Hamiltonian
$$H(q,p) = \tfrac{1}{2}p^2 + V(q),$$

for each potential $V \in C^1(\mathbb{T}^n)_{\mathbb{R}}$ such that ∇V is Lipschitz continuous. This is arguably the most natural assumption on V; the Picard–Lindelöf theorem then ensures that the Hamilton equations have unique solutions.

Precisely stated, for every $(q_0, p_0) \in \mathbb{T}^n \times \mathbb{R}^n$, there exist unique functions $q\colon \mathbb{R} \to \mathbb{T}^n$ and $p\colon \mathbb{R} \to \mathbb{R}^n$ that satisfy

$$\begin{cases} (\dot{q}(t), \dot{p}(t)) = (p(t), -\nabla V(q(t))) & t \in \mathbb{R}\,, \\ (q(0), p(0)) = (q_0, p_0)\,. \end{cases} \tag{3}$$

Note that the expression on the right-hand side of the first line of equation (3) is the Hamiltonian vector field X_H corresponding to H evaluated at $(q(t), p(t))$. For each $t \in \mathbb{R}$, the time evolution of the system after time t is the map

$$\Phi_V^t\colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n, \quad (q_0, p_0) \mapsto (q(t), p(t)),$$

which is the flow corresponding to X_H evaluated at time t; it is well known to be a homeomorphism.

Note that we have already made the notation of the flow less cumbersome by writing Φ_V^t instead of $\Phi_{X_H}^t$. In what follows, we restrict our attention to the case $t = 1$, further simplifying the notation by defining $\Phi_V := \Phi_V^1$. The following lemma shows that we may do so without loss of generality:

Lemma 9. *The algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is preserved under the pullback of Φ_V for each V if and only if it is preserved under the pullback of Φ_V^t for each V, for each $t \in \mathbb{R}$.*

Proof. For any $t \neq 0$ (as $t = 0$ is trivial), we make the following transformation on phase space

$$\phi(q, p) := (q, tp).$$

Because the momentum part of ϕ is linear, its pullback preserves the commutative resolvent algebra. Given an integral curve $(q(t), p(t))$ of the vector field X_H corresponding to the potential V, i.e. a solution of equation (3), one can easily check that $s \mapsto \phi(q(ts), p(ts))$ is an integral curve corresponding to the potential $t^2 V$. We therefore conclude that

$$\Phi_V^t(q_0, p_0) = \phi^{-1} \circ \Phi_{t^2 V}^1 \circ \phi(q_0, p_0),$$

which implies the claim. □

We prove our main theorem in three steps: taking $V = 0$; taking V trigonometric; and finally taking general V. In the second and third step we will need the following consequence of Gronwall's inequality. Let d denote the canonical distance function on $\mathbb{T}^n$ as well as on $\mathbb{T}^n \times \mathbb{R}^n$. (Note that these distance functions are the ones induced by the canonical Riemannian metrics on $\mathbb{T}^n$ and $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$, respectively.)

Lemma 10. *Let $f, g\colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}^{2n}$ be Lipschitz continuous functions, let c be the Lipschitz constant of f, and let $y, z\colon [0,1] \to \mathbb{T}^n \times \mathbb{R}^n$ be curves that satisfy $\dot{y}(t) = f(y(t))$ and $\dot{z}(t) = g(z(t))$ for each $t \in [0,1]$. Finally, suppose that $\varepsilon > 0$ is a number such that $\|f - g\|_\infty \leq \varepsilon$. Then we have*

$$d(y(t), z(t)) \leq (d(y(0), z(0)) + t\varepsilon)e^{tc}.$$

Proof. By translation invariance of the metric on $\mathbb{T}^n \times \mathbb{R}^n$, we have

$$\begin{aligned} d(y(t), z(t)) &\leq d((y(t) - y(0)) - (z(t) - z(0)), 0) + d(y(0), z(0)) \\ &\leq \int_0^t \|f(y(s)) - g(z(s))\| \; ds + d(y(0), z(0)) \\ &\leq c \int_0^t d(y(s), z(s)) \, ds + t\varepsilon + d(y(0), z(0)) \,. \end{aligned}$$

With the integral version of Gronwall's inequality, this implies the lemma. □

3.1. Free Time Evolution

For each pair $(q_0, p_0) \in \mathbb{T}^n \times \mathbb{R}^n$, we have $q(t) = q_0 + tp_0$ and $p(t) = p_0$, denoting the usual action of $\mathbb{R}^n$ on $\mathbb{T}^n$ by $+$. The latter notation, explicitly written as $[x] + p = [x + p]$ for $x, p \in \mathbb{R}^n$, will be used in the remainder of this section. We find that $\Phi_0(q_0, p_0) = (q_0 + p_0, p_0)$, and obtain the following preliminary result. Let * denote the pullback.

Lemma 11. *Free time evolution preserves the commutative resolvent algebra, i.e.*

$$\Phi_0^*(C_{\mathcal{R}}(T^*\mathbb{T}^n)) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n).$$

Proof. We have

$$\Phi_0^*(e_k \otimes h_{U,\xi,g})(q_0, p_0) = e_k(q_0) e^{2\pi i k \cdot p_0} e^{i\xi \cdot p_0} g(P_U(p_0)).$$

Defining $\tilde{g} \in C_0(U)$ by $\tilde{g}(p) := e^{2\pi i P_U(k) \cdot p} g(p)$, and $\tilde{\xi} := \xi + 2\pi P_{U^\perp}(k)$, we obtain

$$\Phi_0^*(e_k \otimes h_{U,\xi,g}) = e_k \otimes h_{U,\tilde{\xi},\tilde{g}}.$$

Thus, the generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ are mapped into $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ by Φ_0^*, and since this map is a *-homomorphism, the lemma follows. □

3.2. Trigonometric Potentials

We say that V is a *trigonometric potential* if it is real-valued and of the form $V = \sum_{k \in \mathcal{N}} a_k e_k$, for some coefficients $a_k \in \mathbb{C}$ and a finite subset $\mathcal{N} \subseteq \mathbb{Z}^n$. The main trick used to establish time invariance of the classical resolvent algebra is to use induction on the size of $\mathcal{N}$. The induction basis, $\mathcal{N} = \emptyset$, corresponds to free time evolution. In order to carry out the induction step we fix a vector $k \in \mathcal{N}$, and compare the dynamics corresponding to V with the dynamics corresponding to $V - V_k$, where

$$V_k := a_k e_k + a_{-k} e_{-k}.$$

Similar to the already defined curves $q \colon [0,1] \to \mathbb{T}^n$ and $p \colon [0,1] \to \mathbb{R}^n$, the dynamics corresponding to $V - V_k$ of the point (q_0, p_0) is incapsulated by the curves $\tilde{q} \colon [0,1] \to \mathbb{T}^n$ and $\tilde{p} \colon [0,1] \to \mathbb{R}^n$ satisfying

$$\begin{cases} (\dot{\tilde{q}}(t), \dot{\tilde{p}}(t)) = (\tilde{p}(t), -\nabla(V - V_k)(\tilde{q}(t))) \qquad t \in \mathbb{R}\,, \\ (\tilde{q}(0), \tilde{p}(0)) = (q_0, p_0)\,. \end{cases} \tag{4}$$

We compare the two dynamics in the following proposition.

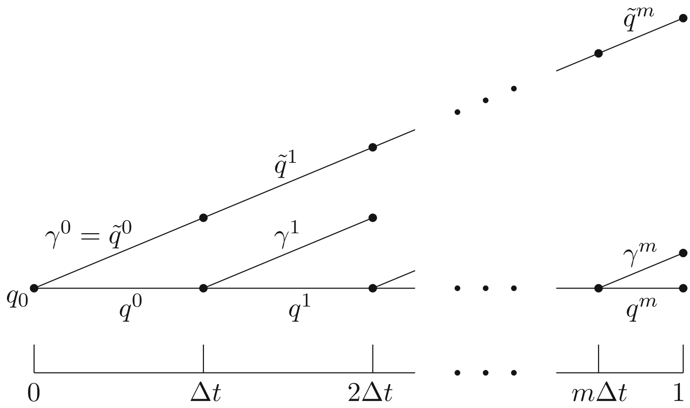

FIGURE 1. The position functions q^j,γ^j and $\tilde{q}^j$. Sloping lines correspond to $V - V_k$, whereas the horizontal line that depicts q corresponds to V

Proposition 12. *Let $k \in \mathbb{Z}^n$ and $\delta > 0$. There exists a $D_k > 0$ such that for each $(q_0, p_0) \in \mathbb{T}^n \times \mathbb{R}^n$ satisfying $|k \cdot p_0| > D_k$, we have*

$$d\left(\Phi_V(q_0, p_0), \Phi_{V-V_k}(q_0, p_0)\right) < \delta.$$

Proof. Note that the statement is vacuously true for any $D_k > 0$ if $k = 0$. We therefore fix a nonzero $k \in \mathbb{Z}^n$. Throughout the proof, we use a variation in big O notation, expanding in the variable $\Delta t := |k \cdot p_0|^{-1}$, uniformly in q_0. That is, we write $f(q_0, p_0) = \mathcal{O}(\Delta t^d)$ if there exist $N, C > 0$ such that for all q_0, p_0 with $|k \cdot p_0| > N$ we have $|f(q_0, p_0)| \leq C|k \cdot p_0|^{-d}$. Therefore, to prove the proposition, it suffices to show that

$$d\left(\begin{pmatrix} q(1) \\ p(1) \end{pmatrix}, \begin{pmatrix} \tilde{q}(1) \\ \tilde{p}(1) \end{pmatrix}\right) = \mathcal{O}(\Delta t)\,. \tag{5}$$

Assume that $\Delta t \in (0, 1)$. We divide the time interval $[0, 1]$ into m intervals of length Δt, where $m := \lfloor \frac{1}{\Delta t} \rfloor$, and a final interval of length $1 - m\Delta t$. For each $t \in [0, \Delta t]$ and each $j \in \{0, \ldots, m\}$ (these will be the assumptions on t and j throughout the rest of the proof), let

$$q^j(t) := q(j\Delta t + t)\,, \quad p^j(t) := p(j\Delta t + t)\,,$$

and define the curves $\tilde{q}^j$ and $\tilde{p}^j$ analogously. Note that (q^j, p^j) and $(\tilde{q}^j, \tilde{p}^j)$ satisfy the differential equations (3) and (4), respectively, but with different initial conditions. Furthermore, for every j, we define the curve $\gamma^j : [0, \Delta t] \to \mathbb{T}^n$ as the unique solution to the initial value problem

$$\begin{cases} (\dot{\gamma}^j(t), \ddot{\gamma}^j(t)) = (\dot{\gamma}^j(t), -\nabla(V - V_k)(\gamma^j(t))) & t \in \mathbb{R}\,, \\ (\gamma^j(0), \dot{\gamma}^j(0)) = (q^j(0), p^j(0))\,. \end{cases} \tag{6}$$

where on the first line, we have emphasised the similarity of this equation with the equations (3) and (4) by including $\dot{\gamma}^j(t)$. We do not introduce any special notation for $\dot{\gamma}^j$, however.

As depicted in Fig. 1, the curve $\gamma^j\colon [0,\Delta t] \to \mathbb{T}^n$ plays a key rôle in comparing q^j with $\tilde{q}^j$; the curve $(\gamma^j, \dot{\gamma}^j)$ is an integral curve along the same Hamiltonian vector field as $(\tilde{q}^j, \tilde{p}^j)$, but with the same initial conditions as (q^j, p^j).

We now expand our expressions in orders of Δt. Using equation (3) and the fundamental theorem of calculus, we obtain

$$\left\|p^j(t) - p^j(0)\right\| \leq \int_0^{\Delta t} \left\|\nabla V(q^j(s))\right\| \, ds \leq \|\nabla V\|_\infty \Delta t = \mathcal{O}(\Delta t)\,. \tag{7}$$

In particular, taking $t = \Delta t$, we get $\left\|p^{j+1}(0) - p^j(0)\right\| = \mathcal{O}(\Delta t)$, and therefore by induction

$$\left\|p^j(0) - p_0\right\| = \mathcal{O}(1)\,, \tag{8}$$

for every $0 \leq j \leq m$. Equations (7) and (8) give us

$$\begin{aligned} d(q^j(t), q^j(0) + tp_0) &\leq \left\|\int_0^t (p^j(s) - p_0)\, ds\right\| \\ &\leq \int_0^{\Delta t} \left\|p^j(s) - p^j(0)\right\| + \left\|p^j(0) - p_0\right\| \, ds \\ &= \mathcal{O}(\Delta t)\,. \end{aligned} \tag{9}$$

A result similar to (7) exists for $\dot{\gamma}^j$ instead of p^j, and hence,

$$\left\|p^j(t) - \dot{\gamma}^j(t)\right\| = \mathcal{O}(\Delta t)\,, \tag{10}$$

which implies

$$d(q^j(t), \gamma^j(t)) = \mathcal{O}(\Delta t^2)\,. \tag{11}$$

Using the definitions of V_k and Δt, we show that the distance between $p^j(\Delta t)$ and $\dot{\gamma}^j(\Delta t)$ is in fact of order Δt^2. We first note that

$$\begin{aligned} \left\|p^j(\Delta t) - \dot{\gamma}^j(\Delta t)\right\| &= \left\|\int_0^{\Delta t} (\nabla V(q^j(s)) - \nabla(V - V_k)(\gamma^j(s)))\, ds\right\| \\ &\leq \int_0^{\Delta t} \left\|\nabla(V - V_k)(q^j(s)) - \nabla(V - V_k)(\gamma^j(s))\right\| \, ds \\ &\quad + \left\|\int_0^{\Delta t} \nabla V_k(q^j(s))\, ds\right\|\,. \end{aligned}$$

By (11), the first term is $\mathcal{O}(\Delta t^3)$. For the second term we can use (9) and the observation that

$$\int_0^{\Delta t} \nabla V_k(q^j(0) + sp_0)\, ds = 0.$$

Hence, the second term is $\mathcal{O}(\Delta t^2)$. All in all, we obtain the estimate

$$\left\|p^j(\Delta t) - \dot{\gamma}^j(\Delta t)\right\| = \mathcal{O}(\Delta t^2)\,.$$

This estimate, together with (11), implies

$$d\left(\begin{pmatrix}\gamma^{j+1}(0)\\ \dot{\gamma}^{j+1}(0)\end{pmatrix}, \begin{pmatrix}\gamma^j(\Delta t)\\ \dot{\gamma}^j(\Delta t)\end{pmatrix}\right) = d\left(\begin{pmatrix}q^j(\Delta t)\\ p^j(\Delta t)\end{pmatrix}, \begin{pmatrix}\gamma^j(\Delta t)\\ \dot{\gamma}^j(\Delta t)\end{pmatrix}\right) = \mathcal{O}(\Delta t^2)\,. \tag{12}$$

Since γ^j and $\tilde{q}^j$ satisfy the same differential equation, say with associated Lipschitz constant c, Lemma 10 (with $f = g : (q,p) \mapsto (p, -\nabla(V - V_k)(q))$) implies that

$$d\left(\begin{pmatrix}\gamma^j(t)\\ \dot\gamma^j(t)\end{pmatrix}, \begin{pmatrix}\tilde q^j(t)\\ \tilde p^j(t)\end{pmatrix}\right) \le e^{ct} d\left(\begin{pmatrix}\gamma^j(0)\\ \dot\gamma^j(0)\end{pmatrix}, \begin{pmatrix}\tilde q^j(0)\\ \tilde p^j(0)\end{pmatrix}\right). \tag{13}$$

Taking $t = \Delta t$, we by definition have

$$d\left(\begin{pmatrix}\gamma^j(\Delta t)\\ \dot\gamma^j(\Delta t)\end{pmatrix}, \begin{pmatrix}\tilde q^{j+1}(0)\\ \tilde p^{j+1}(0)\end{pmatrix}\right) \le e^{c\Delta t} d\left(\begin{pmatrix}\gamma^j(0)\\ \dot\gamma^j(0)\end{pmatrix}, \begin{pmatrix}\tilde q^j(0)\\ \tilde p^j(0)\end{pmatrix}\right). \tag{14}$$

Combining (12) and (14), we find that

$$d\left(\begin{pmatrix}\gamma^{j+1}(0)\\ \dot\gamma^{j+1}(0)\end{pmatrix}, \begin{pmatrix}\tilde q^{j+1}(0)\\ \tilde p^{j+1}(0)\end{pmatrix}\right) \le e^{c\Delta t} d\left(\begin{pmatrix}\gamma^j(0)\\ \dot\gamma^j(0)\end{pmatrix}, \begin{pmatrix}\tilde q^j(0)\\ \tilde p^j(0)\end{pmatrix}\right) + \mathcal{O}(\Delta t^2).$$

Because $e^{jc\Delta t} = \mathcal{O}(1)$, repeated use of the above equation gives

$$d\left(\begin{pmatrix}\gamma^m(0)\\ \dot\gamma^m(0)\end{pmatrix}, \begin{pmatrix}\tilde q^m(0)\\ \tilde p^m(0)\end{pmatrix}\right) = \mathcal{O}(\Delta t). \tag{15}$$

Let $t := 1 - m\Delta t$. Using (13), we find

$$\begin{aligned} d\left(\begin{pmatrix}q(1)\\ p(1)\end{pmatrix}, \begin{pmatrix}\tilde q(1)\\ \tilde p(1)\end{pmatrix}\right) &\le d\left(\begin{pmatrix}q(1)\\ p(1)\end{pmatrix}, \begin{pmatrix}\gamma^m(t)\\ \dot\gamma^m(t)\end{pmatrix}\right) + d\left(\begin{pmatrix}\gamma^m(t)\\ \dot\gamma^m(t)\end{pmatrix}, \begin{pmatrix}\tilde q(1)\\ \tilde p(1)\end{pmatrix}\right)\\ &\le d\left(q(1), \gamma^m(t)\right) + \|p(1) - \dot\gamma^m(t)\|\\ &\quad + e^{ct} d\left(\begin{pmatrix}\gamma^m(0)\\ \dot\gamma^m(0)\end{pmatrix}, \begin{pmatrix}\tilde q^m(0)\\ \tilde p^m(0)\end{pmatrix}\right). \end{aligned}$$

The first term is $\mathcal{O}(\Delta t^2)$ by (11), the second is $\mathcal{O}(\Delta t)$ by (10), and the last term is $\mathcal{O}(\Delta t)$ by (15). This implies (5), and thereby the proposition. □

Proposition 12 expresses a property of the classical time evolution associated with a trigonometric potential in terms of points in phase space. To translate this result to the world of observables, we fix $\varepsilon > 0$ and notice that any $g \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is uniformly continuous. Hence, for every $k \in \mathcal{N}$ we may fix a D_k such that

$$\sup_{x \in U_k} |\Phi_V^* g(x) - \Phi_{V-V_k}^* g(x)| \le \varepsilon, \tag{16}$$

where

$$U_k := \mathbb{T}^n \times \{x \in \mathbb{R}^n : |k \cdot x| > D_k\}.$$

We also define the opens

$$\begin{aligned} W_k &:= \mathbb{T}^n \times \{x \in \mathbb{R}^n : |k \cdot x| > 2D_k\};\\ U_\infty &:= \mathbb{T}^n \times \{x \in \mathbb{R}^n : |k \cdot x| < 4D_k \text{ for all } k \in \mathcal{N}\};\\ W_\infty &:= \mathbb{T}^n \times \{x \in \mathbb{R}^n : |k \cdot x| < 3D_k \text{ for all } k \in \mathcal{N}\}, \end{aligned}$$

and remark that $\{U_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$ are open covers satisfying $\overline{W_i} \subseteq U_i$ for all $i \in I := \mathcal{N} \cup \{\infty\}$. Since we already know how $\Phi_V^* g$ approximately behaves on $\bigcup_{k \in \mathcal{N}} U_k$, let us see how it behaves on U_∞.

Lemma 13. *There exists an* $f_\infty \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$ *that equals* $\Phi_V^* g$ *on* U_∞*.*

Proof. Let $S := \operatorname{span}_{\mathbb{R}} \mathcal{N}$. We write our phase space as a product of topological spaces

$$\mathbb{T}^n \times \mathbb{R}^n = (\mathbb{T}^n \times S) \times S^\perp,$$

and note that

$$C_0(\mathbb{T}^n \times S) \hat{\otimes} \mathcal{W}_{\mathcal{R}}^0(S^\perp),$$

is an ideal in $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. On the other hand, regarding our phase space as a coproduct of abelian Lie groups

$$\mathbb{T}^n \times \mathbb{R}^n = (\mathbb{T}^n \times S) \oplus S^\perp,$$

we define ϕ^t as the restriction of Φ_V^t to $\mathbb{T}^n \times S$ for each $t \in \mathbb{R}$. Because $\nabla V \perp S^\perp$, we have $\dot{p}(t) \perp S^\perp$, and hence,

$$\phi^t \colon \mathbb{T}^n \times S \to \mathbb{T}^n \times S$$

is a well-defined homeomorphism. Moreover, we find the equation

$$\Phi_V^t(q, p_\parallel + p_\perp) = \phi^t(q, p_\parallel) + (tp_\perp, p_\perp)\,, \quad \text{for all} \quad p_\parallel \in S,\ p_\perp \in S^\perp\,,$$

because its two sides solve the same differential equation. Using the above relation in a straightforward calculation on generators, one can show that

$$\Phi_V^*(C_0(\mathbb{T}^n \times S) \otimes \mathcal{W}_{\mathcal{R}}^0(S^\perp)) \subseteq C_0(\mathbb{T}^n \times S) \otimes \mathcal{W}_{\mathcal{R}}^0(S^\perp).$$

Actually, the same holds for Φ_V^{-1}, which implies that Φ_V^* is a *-automorphism of the ideal $C_0(\mathbb{T}^n \times S) \otimes \mathcal{W}_{\mathcal{R}}^0(S^\perp)$. Now note that U_∞ is of the form $K \times S^\perp$ for some compact subset $K \subseteq \mathbb{T}^n \times S$. By Urysohn's lemma, we may choose a function $\tilde{g} \in C_0(\mathbb{T}^n \times S) \otimes \mathcal{W}_{\mathcal{R}}^0(S^\perp)$ that is 1 on U_∞, and define $f_\infty := \tilde{g} \cdot \Phi_V^* g$. We then find that

$$f_\infty = ((\tilde{g} \circ \Phi_V^{-1}) \cdot g) \circ \Phi_V \in C_0(\mathbb{T}^n \times S) \otimes \mathcal{W}_{\mathcal{R}}^0(S^\perp),$$

and therefore $f_\infty \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$. □

We can finally prove that our classical resolvent algebra is invariant under any time evolution corresponding to a trigonometric potential.

Proposition 14. *For every trigonometric potential* $V \colon \mathbb{T}^n \to \mathbb{R}$ *and* $g \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$ *we have* $\Phi_V^* g \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$*.*

Proof. We use induction on the size of $\mathcal{N}$ in $V = \sum_{k \in \mathcal{N}} a_k e_k$ (while assuming that $\mathcal{N}$ is chosen minimally). The induction base is precisely Lemma 11.

We now carry out the induction step. The induction hypothesis says that time evolution with respect to $V - V_k$ preserves $C_{\mathcal{R}}(T^*\mathbb{T}^n)$, for each $k \in \mathcal{N}$. Therefore, writing $f_k := \Phi_{V-V_k}^* g$, we have $f_k \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$. Fixing f_∞ as in Lemma 13, we have $f_i \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$, and equation (16) implies that

$$\| f_i|_{U_i} - \Phi_V^* g|_{U_i} \|_\infty < \varepsilon\,, \tag{17}$$

for each $i \in I = \mathcal{N} \cup \{\infty\}$. We now construct a partition of unity $\{\eta_i\}$ subordinate to the open cover $\{U_i\}$ of $\mathbb{T}^n \times \mathbb{R}^n$, to patch together the functions $\{f_i\}$ and obtain a single function in $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. We start by defining nonnegative functions $\zeta_i \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$ that are 1 on W_i and 0 outside of U_i. Explicitly, for each $k \in \mathcal{N}$, we take $\zeta_k := \mathbf{1}_{\mathbb{T}^n} \otimes (g_k \circ P_{\mathrm{span}(k)})$ for some bump function g_k on $\mathrm{span}(k)$, and we take $\zeta_\infty := \mathbf{1}_{\mathbb{T}^n} \otimes (g_\infty \circ P_S)$ for some bump function g_∞ on S. Because $\{W_i\}$ is a cover of $\mathbb{T}^n \times \mathbb{R}^n$, the sum $\sum_i \zeta_i \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is bounded from below by 1; hence, it is invertible in $C_{\mathcal{R}}(T^*\mathbb{T}^n)$, and therefore every function

$$\eta_i := \frac{\zeta_i}{\sum_j \zeta_j},$$

also lies in $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. Now (17) gives us

$$\left\| \Phi_V^* g - \sum_i f_i \eta_i \right\|_\infty < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is norm-closed, the assertion follows. $\square$

3.3. Arbitrary Potentials

Having covered the trigonometric case, we now wish to tackle the general case. The following lemma provides the required approximation of a generic potential by trigonometric ones.

Lemma 15. *Let $V \in C^1(\mathbb{T}^n)$. Then there exists a sequence $(V_m)_m$ of trigonometric polynomials such that $(\nabla V_m)_m$ converges uniformly to ∇V. Furthermore, if V is real-valued, then every V_m can be chosen to be real-valued as well.*

Proof. We construct the sequence (V_m) by taking the convolution of V with the n-dimensional analogues of the family of *Fejér kernels*. We first recall that for each $m \geq 1$, the m-th Fejér kernel is given by

$$F_{1,m} \colon \mathbb{T} \to \mathbb{R}, \quad q = [x] \mapsto \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=-k}^{k} e^{2\pi i j x} = \frac{1}{m} \frac{\sin^2(\pi m x)}{\sin^2(\pi x)},$$

where the most right expression in this definition is understood to be equal to m for $x = 0$. The sequence $(F_{1,m})_{m \geq 1}$ is an approximation to the identity, i.e. for every continuous function f on $\mathbb{T}$, the sequence $(F_{1,m} * f)_{m \geq 1}$ converges uniformly to f, where $*$ denotes the operation of convolution of functions [28, Sects. 2.4 and 2.5.2].

Next, we define the n-dimensional analogues of these functions:

$$F_{n,m} \colon \mathbb{T}^n \to \mathbb{R}, \quad q = (q_1, \ldots, q_n) \mapsto \prod_{l=1}^{n} F_{1,m}(q_l).$$

Using the corresponding fact for one-dimensional kernels, it is elementary to show that the sequence $(F_{n,m})_{m \geq 1}$ is an approximation to the identity.

We now define

$$V_m := F_{n,m} * V,$$

for each $m \geq 1$. Because every $F_{n,m}$ is trigonometric, and $e_k * f = \hat{f}(k)e_k$ for every $f \in C(\mathbb{T}^n)$ and $k \in \mathbb{Z}^n$, the sequence $(V_m)_{m\geq 1}$ consists of trigonometric polynomials. Moreover, by a general property of convolutions, we have

$$\frac{\partial V_m}{\partial q_l} = \frac{\partial}{\partial q_l}(F_{n,m} * V) = F_{n,m} * \frac{\partial V}{\partial q_l}\,,$$

and since $(F_{n,m})_{m\geq 1}$ is an approximation to the identity, the right-hand side converges uniformly to $\frac{\partial V}{\partial q_l}$ as $m \to \infty$, for $l = 1, \ldots, n$. It follows that $(\nabla V_m)_{m\geq 1}$ converges uniformly to ∇V. The final assertion is a consequence of the fact that the family of Fejér kernels (as well as its higher-dimensional analogues) consists of real-valued functions. $\square$

We now extend Proposition 14 to general V, thereby arriving at our final result.

Theorem 16. *Let $V \in C^1(\mathbb{T}^n)_{\mathbb{R}}$, and suppose that ∇V is Lipschitz continuous. Then we have*

$$(\Phi_V^t)^*(C_{\mathcal{R}}(T^*\mathbb{T}^n)) = C_{\mathcal{R}}(T^*\mathbb{T}^n),$$

for every $t \in \mathbb{R}$.

Proof. It suffices to show that $(\Phi_V^t)^*(C_{\mathcal{R}}(T^*\mathbb{T}^n)) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$; we can replace t by $-t$ and note that $(\Phi_V^{-t})^*$ is the inverse of $(\Phi_V^t)^*$ to obtain the reverse inclusion. By Lemma 9, we may assume without loss of generality that $t = 1$.

Let $g \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$. By Lemma 15, there exists a sequence of trigonometric potentials (V_m) on $\mathbb{T}^n$ such that (∇V_m) converges uniformly to ∇V. We show that this implies that $(\Phi_{V_m}^*(g))$ converges uniformly to $\Phi_V^*(g)$; since $\Phi_{V_m}^*(g) \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$ by Proposition 14 and since $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is norm-closed, the theorem will follow from this.

Let $\varepsilon > 0$, and let c be the Lipschitz constant of $(q,p) \mapsto (p, -\nabla V(q))$. Since g is uniformly continuous, there exists $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ for each $x, y \in \mathbb{T}^n \times \mathbb{R}^n$ with $d(x,y) < \delta$. By assumption, there exists an $N \in \mathbb{N}$ such that for each $m \geq N$, we have $\|\nabla V - \nabla V_m\|_\infty < \delta e^{-c}$. It follows from Lemma 10 that $d(\Phi_V(x), \Phi_{V_m}(x)) < \delta$ for each $x \in \mathbb{T}^n \times \mathbb{R}^n$ and each $m \geq N$; hence, $\|\Phi_V^*(g) - \Phi_{V_m}^*(g)\|_\infty \leq \varepsilon$. Thus $(\Phi_{V_m}^*(g))$ converges uniformly to $\Phi_V^*(g)$, as desired. $\square$

4. Quantisation of the Resolvent Algebra

Having shown the nice properties of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$, we now ask whether there exists a quantum version of this algebra. What complicates matters is that, contrary to the resolvent algebra $\mathcal{R}(\mathbb{R}^{2n}, \sigma)$ of Buchholz and Grundling, on the cylinder it is hard, if not impossible, to define an algebra in terms of generators and relations implementing canonical commutation relations. Thus, we must take a different approach.

We will define our quantisation of the algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ as an algebra represented on $L^2(\mathbb{T}^n)$, using a version of Weyl quantisation similar to the definition of Landsman [15, Sect. II.3.4] for general Riemannian manifolds.

By contrast, Rieffel's algebras on cylinders in [25], apart from being quantisations of $C_u(T^*\mathbb{T}^n)$ and subalgebras thereof, are in some sense universal objects from which a physical quantum system is obtained as the image of one of its irreducible representations, and it is not always clear which representation corresponds to the physical system that one wishes to model. These algebras have many inequivalent irreducible representations due to the fact that $\mathbb{T}$ is not simply connected, see e.g. [25, Example 10.6] and the discussion in [16, Sect. 7.7]. By no means do we intend to discount such universal objects, however; we will return to this point in the outlook of this paper. The main advantage of quantising $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ as an algebra of operators on $L^2(\mathbb{T}^n)$ lies in the explicit formula for the quantisations of the generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ that we are able to derive.

This section is structured as follows. In Sect. 4.1, we define the Weyl quantisation map and prove the aforementioned explicit formula. In Sect. 4.2, we show that, except for continuity of the map $\hbar \mapsto \|\mathcal{Q}_\hbar^W(f)\|$ at $\hbar > 0$ for fixed $f \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$, the quantisation is strict.

4.1. Definition of the Quantisation Map

Let us first recall the basics of Weyl quantisation in $\mathbb{R}^{2n}$, the quantisation procedure in [32] conceived by Weyl. Given say, a Schwartz function $f \in \mathcal{S}(\mathbb{R}^{2n})$, one associates an operator $\mathcal{Q}_\hbar^W(f) \in B(L^2(\mathbb{R}^n))$ to it as follows. First, one expresses f in terms of functions of the form

$$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}, \quad (q,p) \mapsto e^{i(a\cdot q + b\cdot p)}\,,$$

where $a, b \in \mathbb{R}^n$, by considering the Fourier transform of f. One subsequently substitutes these exponential functions with the operators

$$e^{i(a\cdot Q + b\cdot P)}\,,$$

where Q, P are vectors whose components are the essentially self-adjoint operators on $\mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$, defined by $Q_j\psi(x) := x_j\psi(x)$ and $P_j\psi(x) := -i\hbar\frac{d\psi}{dx_j}(x)$. Thus, the Weyl quantisation of a function f is informally given by the expression

$$\begin{aligned}(2\pi)^{-2n} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(q,p) e^{ia\cdot(Q-q)+ib\cdot(P-p)}\, dq\, dp\, da\, db \\ &= (2\pi)^{-2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(q,p) e^{i\hbar\frac{a\cdot b}{2}} e^{ia\cdot(Q-q)} e^{ib\cdot(P-p)}\, dq\, dp\, da\, db\,,\end{aligned}$$

where we take $\hbar > 0$. To define the above integrals rigorously, we can insert a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ on the right-hand side of the integrand, and check that the resulting expression is well defined and that it defines a bounded operator on $\mathcal{S}(\mathbb{R}^n)$ viewed as a subspace of $L^2(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the operator has a unique bounded extension to $L^2(\mathbb{R}^n)$, which we define to be $\mathcal{Q}_\hbar^W(f)$. Using standard identities for Fourier transforms of functions, and performing a number of substitutions, it can be shown that

$$(\mathcal{Q}_\hbar^W(f)\psi)(x) = (2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(x + \frac{y}{2}, p\right) e^{-i\frac{y\cdot p}{\hbar}}\psi(x+y)\, dp\, dy\,,$$

for each $\psi \in \mathcal{S}(\mathbb{R}^n)$ and each $x \in \mathbb{R}^n$.

We now adapt the Weyl quantisation formula to $T^*\mathbb{T}^n$ in such a way that we can quantise elements of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. We already identified a dense Poisson algebra of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ in Sect. 2, namely the space $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ of finite linear combinations of functions of the form $e_k \otimes h_{U,\xi,g}$; see Proposition 8. These are the functions that we will quantise. To handle such functions, we take inspiration from Rieffel's work [25], regarding the integrals in the above formula as oscillatory integrals, and regularising the expression by inserting a factor in the integrand in the form of a member of a net of functions that converges pointwise to the constant function $1_{\mathbb{R}^n}$, as in part (1) of the next proposition. Part (2) of this proposition is the analogue of [25, Proposition 1.11].

Proposition 17.

(1) Let $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, let $\hbar > 0$, and let $\psi \in C(\mathbb{T}^n)$. Then for each $[x] \in \mathbb{T}^n$, the limit

$$\lim_{\delta\to 0}(2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(\left[x+\tfrac{1}{2}y\right],p\right)e^{-\frac{\delta}{2}p^2}e^{-i\frac{y\cdot p}{\hbar}}\psi[x+y]\,dp\,dy\,, \tag{18}$$

exists.

(2) Now assume $f = e_k \otimes h_{U,\xi,g}$ is a function as described in Definition 7. Then the expression in equation (18) is equal to

$$(2\pi\hbar)^{-\dim(U)}e^{\pi ik\cdot\hbar\xi}e^{2\pi ik\cdot x}\int_U\int_U g\left(p+\pi\hbar P_U(k)\right)e^{-i\frac{y\cdot p}{\hbar}}\psi[x+y+\hbar\xi]\,dp\,dy\,.$$

For each $l \in \mathbb{Z}^n$, let ψ_l be the function

$$\mathbb{T}^n \to \mathbb{C}, \quad [x] \mapsto e^{2\pi il\cdot x}\,,$$

and regard it as an element of $L^2(\mathbb{T}^n)$.

(3) In addition to the assumptions in the previous part of the proposition, suppose that $\psi = \psi_l$ for some $l \in \mathbb{Z}^n$. Then the expression in equation (18) is equal to

$$h_{U,\xi,g}(\pi\hbar(k+2l))\psi_{k+l}[x]\,,$$

and the map defined on $\mathrm{span}_{l\in\mathbb{Z}^n}\{\psi_l\}$ sending ψ to the function on $\mathbb{T}^n$ that assigns to a point $[x] \in \mathbb{T}^n$ the limit in (18) extends in a unique way to a bounded linear operator on $L^2(\mathbb{T}^n)$ with norm $\leq \|g\|_\infty$.

Proof. We first show that for functions f of the form $e_k \otimes h_{U,\xi,g}$, i.e. f as in part (2) of the proposition, the limit in equation (18) exists, and is equal to the formula in part (2) of the proposition. Since $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is by definition the linear span of such functions, part (1) will follow from this. Thus, take such an f, and note that for any $\delta > 0$, we have

$$\begin{aligned}&(2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left([x+\tfrac{1}{2}y],p\right)e^{-\frac{\delta}{2}p^2}e^{-i\frac{y\cdot p}{\hbar}}\psi[x+y]\,dp\,dy\\ &\quad= (2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\left(\xi\cdot p-\frac{y\cdot p}{\hbar}\right)}g\circ P_U(p)e^{-\frac{\delta}{2}p^2}\,dp\,e^{2\pi ik\cdot\left(x+\frac{y}{2}\right)}\psi[x+y]\,dy\end{aligned}$$

$$= (2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\frac{y\cdot p}{\hbar}} g \circ P_U(p) e^{-\frac{\delta}{2}p^2}\, dp\, e^{2\pi i k\cdot\left(x+\frac{y+\hbar\xi}{2}\right)} \psi[x+y+\hbar\xi]\, dy\,.$$

The inner integral over p can be written as a product of two integrals; one over U and one over $U^\perp$:

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-i\frac{y\cdot p}{\hbar}} g\circ P_U(p) e^{-\frac{\delta}{2}p^2}\, dp \\
&\quad = \int_U g(p_1) e^{-\frac{\delta}{2}p_1^2} e^{-i\frac{P_U(y)\cdot p_1}{\hbar}}\, dp_1 \cdot \int_{U^\perp} e^{-\frac{\delta}{2}p_2^2} e^{-ip_2\cdot\frac{y-P_U(y)}{\hbar}}\, dp_2 \\
&\quad = \int_U g(p_1) e^{-\frac{\delta}{2}p_1^2} e^{-i\frac{P_U(y)\cdot p_1}{\hbar}}\, dp_1 \cdot (2\pi\delta^{-1})^{\frac{\dim(U^\perp)}{2}} e^{-\frac{1}{2\delta\hbar^2}(y-P_U(y))^2}\,.
\end{aligned}$$

Inserting this back into the previous displayed formula, and splitting the outer integral in that formula into an integral over U and an integral over $U^\perp$, we obtain

$$\begin{aligned}
&(2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left([x+\tfrac{1}{2}y], p\right) e^{-\frac{\delta}{2}p^2} e^{-i\frac{y\cdot p}{\hbar}} \psi[x+y]\, dp\, dy \\
&\qquad = (2\pi\hbar)^{-\dim(U)} \int_U h_{1,\delta}(y_1) \int_{U^\perp} h_{2,\delta}(y_1,y_2)\, dy_2\, dy_1\,,
\end{aligned}$$

where

$$\begin{aligned}
h_{1,\delta}\colon\, & U \to \mathbb{C}, \\
& y_1 \mapsto e^{2\pi i k\cdot\left(x+\frac{y_1+\hbar\xi}{2}\right)} \int_U g(p_1) e^{-\frac{\delta}{2}p_1^2} e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1\,,
\end{aligned}$$

and

$$\begin{aligned}
h_{2,\delta}\colon\, & U \times U^\perp \to \mathbb{C}, \\
& (y_1,y_2) \mapsto (2\pi\delta\hbar^2)^{\frac{-\dim(U^\perp)}{2}} e^{-\frac{1}{2\delta\hbar^2}y_2^2} \cdot \psi[x+y_1+y_2+\hbar\xi] e^{\pi i k\cdot y_2}\,.
\end{aligned}$$

Now note that the family of functions

$$U^\perp \to \mathbb{R}, \quad y_2 \mapsto (2\pi\delta\hbar^2)^{\frac{-\dim(U^\perp)}{2}} e^{-\frac{1}{2\delta\hbar^2}y_2^2}\,,$$

indexed by $\delta > 0$ is an approximation to the identity for functions on $U^\perp$. By continuity of ψ, it follows that the functions

$$h_{3,\delta}\colon U \to \mathbb{C}, \quad y_1 \mapsto \int_{U^\perp} h_{2,\delta}(y_1,y_2)\, dy_2\,,$$

converge pointwise to the function

$$U \to \mathbb{C}, \quad y_1 \mapsto \psi[x+y_1+\hbar\xi]\,,$$

as $\delta \to 0$. Moreover, they are bounded, with $\|h_{3,\delta}\|_\infty \le \|\psi\|_\infty$ for each $\delta > 0$. In addition, by the dominated convergence theorem, the functions $h_{1,\delta}$ converge pointwise to the function

$$U \to \mathbb{C}, \quad y_1 \mapsto e^{2\pi i k\cdot\left(x+\frac{y_1+\hbar\xi}{2}\right)} \int_U g(p_1) e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1\,,$$

as $\delta \to 0$. Indeed, the integrands defining these functions are all dominated by the integrable function $|g|$. Furthermore, note that

$$\begin{aligned}
&\int_U g(p_1) e^{-\frac{\delta}{2}p_1^2} e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1 \\
&\quad = \frac{(1+\|y_1\|^2)^{\dim(U)}}{(1+\|y_1\|^2)^{\dim(U)}} \int_U g(p_1) e^{-\frac{\delta}{2}p_1^2} e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1 \\
&\quad = \frac{1}{(1+\|y_1\|^2)^{\dim(U)}} \int_U (1-\hbar^2\Delta_U)^{\dim(U)} (g(p') e^{-\frac{\delta}{2}(p')^2})|_{p'=p_1} e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1\,,
\end{aligned}$$

where Δ_U denotes the standard Laplacian on U, and that for the family of the functions

$$U \to \mathbb{C}, \quad p_1 \mapsto (1-\hbar^2\Delta_U)^{\dim(U)} (g(p') e^{-\frac{\delta}{2}(p')^2})|_{p'=p_1}\,,$$

indexed by $\delta \in (0, C]$, where C is an arbitrary positive real number, there exists a positive function $H_C \in L^1(U)$ dominating the entire family. It follows that for each $\delta \in (0, C]$ and each $y_1 \in U$, we have

$$|h_{1,\delta}(y_1)| \leq \frac{\|H_C\|_1}{(1+\|y_1\|^2)^{\dim(U)}}\,.$$

The (absolute values of the) functions

$$U \to \mathbb{C}, \quad y_1 \mapsto h_{1,\delta}(y_1) \int_{U^\perp} h_{2,\delta}(y_1, y_2)\, dy_2\,,$$

are therefore dominated by the integrable function

$$y_1 \mapsto \frac{\|H_C\|_1 \|\psi\|_\infty}{(1+\|y_1\|^2)^{\dim(U)}}\,,$$

so we may again invoke the dominated convergence theorem to find that

$$\begin{aligned}
&\lim_{\delta\to 0} (2\pi\hbar)^{-\dim(U)} \int_U h_{1,\delta}(y_1) \int_{U^\perp} h_{2,\delta}(y_1,y_2)\, dy_2\, dy_1 \\
&\quad = (2\pi\hbar)^{-\dim(U)} \int_U \left(\lim_{\delta\to 0} h_{1,\delta}(y_1)\right) \left(\lim_{\delta\to 0} \int_{U^\perp} h_{2,\delta}(y_1,y_2)\, dy_2\right) dy_1 \\
&\quad = (2\pi\hbar)^{-\dim(U)} \int_U \int_U g(p_1) e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1\, e^{2\pi i k\cdot\left(x+\frac{y_1+\hbar\xi}{2}\right)} \psi[x+y_1+\hbar\xi]\, dy_1 \\
&\quad = \frac{e^{\pi i k\cdot\hbar\xi} e^{2\pi i k\cdot x}}{(2\pi\hbar)^{\dim(U)}} \int_U \int_U g(p_1) e^{-iy_1\cdot\left(\frac{p_1}{\hbar}-\pi k\right)}\, dp_1\, \psi[x+y_1+\hbar\xi]\, dy_1 \\
&\quad = \frac{e^{\pi i k\cdot\hbar\xi} e^{2\pi i k\cdot x}}{(2\pi\hbar)^{\dim(U)}} \int_U \int_U g(p_1+\pi\hbar P_U(k)) e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1\, \psi[x+y_1+\hbar\xi]\, dy_1\,,
\end{aligned}$$

which completes our proof of part (2).

For part (3), we simply take $\psi = \psi_l \in C(\mathbb{T}^n) \subset L^2(\mathbb{T}^n)$, with $l \in \mathbb{Z}^n$, and apply the formula we just found:

$$\begin{aligned}
&(2\pi\hbar)^{-\dim(U)} e^{\pi i k\cdot\hbar\xi} e^{2\pi i k\cdot x} \int_U \int_U g(p_1+\pi\hbar P_U(k)) e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1\, e^{2\pi i l\cdot(x+y_1+\hbar\xi)}\, dy_1 \\
&\quad = (2\pi\hbar)^{-\dim(U)} e^{\pi i(k+2l)\cdot\hbar\xi} e^{2\pi i(k+l)\cdot x} \int_U \int_U g(p_1+\pi\hbar P_U(k)) e^{-iy_1\cdot\left(\frac{p_1}{\hbar}-2\pi l\right)}\, dp_1\, dy_1
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-\dim(U)} e^{\pi i(k+2l)\cdot\hbar\xi} e^{2\pi i(k+l)\cdot x} \int_U \int_U g(p_1 + \pi\hbar P_U(k+2l)) e^{-iy_1\cdot p_1}\, dp_1\, dy_1 \\
&= e^{\pi i(k+2l)\cdot\hbar\xi} e^{2\pi i(k+l)\cdot x} g \circ P_U(\pi\hbar(k+2l)) \\
&= h_{U,\xi,g}(\pi\hbar(k+2l))\psi_{k+l}[x]\,,
\end{aligned}
$$

which proves the formula in part (3).

We thus see that the linear map on $\mathrm{span}_l\{\psi_l\}$ uniquely determined by

$$
\psi_l \mapsto h_{U,\xi,g}(\pi\hbar(k+2l))\psi_{k+l}\,,
$$

maps an orthonormal basis to an orthogonal system of vectors in $L^2(\mathbb{T}^n)$, and the norm of the image of such a vector ψ_l is less than or equal to $\|g\|_\infty = \|h_{U,\xi,g}\|_\infty = \|f\|_\infty$. (Note that the suprema defining these sup-norms are taken over U, $\mathbb{R}^n$ and $\mathbb{T}^n \times \mathbb{R}^n$, respectively.) Because of this and the fact that the ψ_l's densely span $L^2(\mathbb{T}^n)$, the map extends in a unique way to a bounded operator on $L^2(\mathbb{T}^n)$ with norm $\leq \|g\|_\infty$, which proves the final assertion. □

The proposition justifies the following definitions:

Definition 18. For each $\hbar > 0$ and each $f \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, we define the *Weyl quantisation* $\mathcal{Q}_\hbar^W(f)$ *of* f to be the unique bounded linear extension of the operator on $\mathrm{span}_{l\in\mathbb{Z}^n}\{\psi_l\}$ defined by the formula

$$
(\mathcal{Q}_\hbar^W(f)\psi)[x] := \lim_{\delta\to 0}(2\pi\hbar)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left([x+\tfrac{1}{2}y], p\right) e^{-\frac{\delta}{2}p^2} e^{-i\frac{yp}{\hbar}} \psi[x+y]\, dp\, dy\,.
$$

We thus obtain a map, the *Weyl quantisation map* $\mathcal{Q}_\hbar^W \colon \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n) \to B(L^2(\mathbb{T}^n))$, for each $\hbar > 0$. We define the *quantum resolvent algebra* $A_\hbar$ *on* $\mathbb{T}^n \times \mathbb{R}^n$ to be the C*-subalgebra of $B(L^2(\mathbb{T}^n))$ generated by the image of $\mathcal{S}_\mathcal{R}(\mathbb{T}^*\mathbb{T}^n)$ under $\mathcal{Q}_\hbar^W$.

Part (3) of Proposition 17 can now be phrased as an explicit formula for the Weyl quantisation of a generator $e_k \otimes h \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, namely

$$
\mathcal{Q}_\hbar^W(e_k \otimes h)\psi_l = h(\pi\hbar(k+2l))\psi_{k+l}\,. \tag{19}
$$

Proposition 19. *Let* $\hbar > 0$.

*(1) The Weyl quantisation map is linear and *-preserving;*
(2) For each $\hbar' > 0$, *we have* $A_\hbar = A_{\hbar'}$*;*
(3) The image of

$$
\mathit{span}_\mathbb{C}\{e_k \otimes g \colon k \in \mathbb{Z}^n,\ g \in \mathcal{S}(\mathbb{R}^n)\} \subseteq \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n) \cap C_0(T^*\mathbb{T}^n)\,,
$$

under $\mathcal{Q}_\hbar^W$ *is a dense subspace of* $\mathcal{K}(L^2(\mathbb{T}^n))$*;*
(4) Under the canonical embedding

$$
B(L^2(\mathbb{T}^n)) \hookrightarrow B(L^2(\mathbb{T}^{n+m})) \cong B(L^2(\mathbb{T}^n))\hat{\otimes} B(L^2(\mathbb{T}^m))\,, \quad a \mapsto a \otimes \mathbf{1}\,,
$$

induced by the projection at the level of configuration spaces $\mathbb{T}^{n+m} \to \mathbb{T}^n$ *onto the first* n *coordinates, the image of the quantum resolvent algebra on* $T^*\mathbb{T}^n$ *is a subalgebra of the quantum resolvent algebra on* $T^*\mathbb{T}^{n+m}$*. (Here,* $\hat{\otimes}$ *denotes the von Neumann algebraic tensor product.)*

(5) Let ρ_0 be the group representation of $\mathbb{T}^n$ on $C_b(T^*\mathbb{T}^n)$ given by

$$\rho_0[x]f := \left((q,p) \mapsto f(-x+q,p) \right) ,$$

and let $\rho_\hbar$ be the group representation of $\mathbb{T}^n$ on $B(L^2(\mathbb{T}^n))$ given by

$$\rho_\hbar[x]a := L[x]aL[-x] ,$$

where $L\colon \mathbb{T}^n \to U(L^2(\mathbb{T}^n))$ denotes the left regular representation of $\mathbb{T}^n$. Then both $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ and $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ are invariant under ρ_0. Furthermore, the Weyl quantisation map is equivariant with respect to these representations.

Remark 20. Because of part (2) of this proposition, we will write $A_\hbar$ for the C*-algebra generated by $\mathcal{Q}^W_{\hbar'}(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$ for any value of $\hbar' > 0$ without specifying $\hbar$. Part (3) is the analogue of the first part of [15, Corollary II.2.5.4] in the present setting, while part (5) is the analogue of [15, Theorem II.2.5.1].

Proof.

(1) Linearity of $\mathcal{Q}^W_\hbar$ is obvious from the definition. Now let $e_k \otimes h$ be a generator of $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, and let

$$\mathcal{F}\colon L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n), \quad \psi' \mapsto \left(a \mapsto \langle \psi_a, \psi' \rangle \right) ,$$

be the Fourier transform. Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(\mathbb{T}^n)$. We follow the physicists' convention, taking the inner product to be linear in its second argument. It follows from (19) that

$$\mathcal{Q}^W_\hbar(e_k \otimes h) = \mathcal{F}^{-1} S^k M_{h_1} \mathcal{F} ,$$

where $S^k\colon \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ denotes the shift operator defined by

$$(S^k\phi)(l) := \phi(l-k) ,$$

and M_{h_1} denotes the multiplication operator on $\ell^2(\mathbb{Z}^n)$ associated with the function

$$h_1\colon \mathbb{Z}^n \to \mathbb{C}, \quad l \mapsto h(\pi\hbar(k+2l)) .$$

Next, for each $l \in \mathbb{Z}^n$, we have

$$\begin{aligned}(S^k M_{h_1})^*\delta_l &= M_{\overline{h_1}} S^{-k}\delta_l = \overline{h}(\pi\hbar(k+2(l-k)))\delta_{l-k} \\ &= \overline{h}(\pi\hbar(-k+2l))\delta_{l-k} = S^{-k} M_{h_2}\delta_l ,\end{aligned}$$

where h_2 is defined as $h_2(l) := \overline{h}(\pi\hbar(-k+2\,l))$. Also note that

$$\mathcal{Q}^W_\hbar(\overline{e_k \otimes h}) = \mathcal{Q}^W_\hbar(e_{-k} \otimes \overline{h}) = \mathcal{F}^{-1} S^{-k} M_{h_2} \mathcal{F} ,$$

so by unitarity of the Fourier transform, we have

$$\mathcal{Q}^W_\hbar(\overline{e_k \otimes h}) = \mathcal{F}^{-1}(S^k M_{h_1})^*\mathcal{F} = (\mathcal{F}^{-1} S^k M_{h_1} \mathcal{F})^* = \mathcal{Q}^W_\hbar(e_k \otimes h)^* ,$$

hence $\mathcal{Q}^W_\hbar$ is indeed compatible with the involutions.

(2) For each $\hbar > 0$, each $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ and each $\psi \in L$, we have

$$(\mathcal{Q}^W_\hbar(f)\psi)(x) = \lim_{\delta \to 0} (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left([x+\tfrac{1}{2}y], \hbar p'\right) e^{-\frac{\delta}{2}(p')^2} e^{-iy\cdot p'} \psi[x+y]\, dp'\, dy ,$$

where we have made the substitution $p = \hbar p'$ in the formula defining $\mathcal{Q}_\hbar^W(f)\psi$, and absorbed a factor $\hbar^2$ in δ. Next, we observe that $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ is closed under the map

$$f \mapsto (\,(q,p) \mapsto f(q, Cp)\,),$$

for each $C \in \mathbb{R}$, in particular for $C = \hbar'/\hbar$ for any $\hbar, \hbar' > 0$. It follows that $\mathcal{Q}_\hbar^W(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)) = \mathcal{Q}_{\hbar'}^W(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n))$; hence, $A_\hbar = A_{\hbar'}$, as desired.

(3) Let B be the left-hand side of the displayed formula in the statement. Now let $k \in \mathbb{Z}^n$, and let $g \in \mathcal{S}(\mathbb{R}^n)$. Using notation from the proof of part (1) of this proposition, we have

$$\mathcal{Q}_\hbar^W(e_k \otimes g) = \mathcal{F}^{-1} S^k M_{g_1} \mathcal{F}\,,$$

where g_1 denotes the function

$$\mathbb{Z}^n \to \mathbb{C}, \quad l \mapsto g(\pi\hbar(k + 2l))\,.$$

This function vanishes at infinity, so its corresponding multiplication operator M_{g_1} is compact. All of the other operators that we compose to obtain $\mathcal{Q}_\hbar^W(e_k \otimes g)$ are bounded; hence, $\mathcal{Q}_\hbar^W(e_k \otimes g)$ is compact. Since $\mathcal{Q}_\hbar^W$ is a linear map and $\mathcal{K}(L^2(\mathbb{T}^n))$ is a linear subspace of $B(L^2(\mathbb{T}^n))$, it follows that $\mathcal{Q}_\hbar^W(B) \subseteq \mathcal{K}(L^2(\mathbb{T}^n))$.

To prove the assertion that $\mathcal{Q}_\hbar^W(B)$ is in fact a dense subspace of $\mathcal{K}(L^2(\mathbb{T}^n))$, we note that, given a and b in $\mathbb{Z}^n$, we can fix a $g \in \mathcal{S}(\mathbb{R}^n)$ such that

$$g(\pi\hbar(a - b + 2l)) = \delta_{l,b}\,,$$

for each $l \in \mathbb{Z}^n$. It follows that, in bra-ket notation,

$$\mathcal{Q}_\hbar^W(e_{a-b} \otimes g) = |\psi_a\rangle\langle\psi_b|\,,$$

and from the fact that $a, b \in \mathbb{Z}^n$ were arbitrary and that the family of vectors $(\psi_l)_{l\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$, we infer that $\mathcal{Q}_\hbar^W(B)$ is dense in $\mathcal{K}(L^2(\mathbb{T}^n))$.

(4) From Definition 7 one straightforwardly shows that $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n) \otimes \mathbb{C}\mathbf{1}_{\mathbb{T}^m\times\mathbb{R}^m} \subseteq \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^{n+m})$. From formula (19), one obtains $\mathcal{Q}_\hbar^W(f \otimes \mathbf{1}_{\mathbb{T}^m\times\mathbb{R}^m}) = \mathcal{Q}_\hbar^W(f) \otimes \mathbb{1}$ for all $f \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$. Therefore,

$$\mathcal{Q}_\hbar^W(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)) \otimes \mathbb{1} \subseteq \mathcal{Q}_\hbar^W(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^{n+m})),$$

which implies the same inclusion for the respective generated C*-algebras.

(5) Suppose f is of the form $e_k \otimes h$. Then it is readily seen that

$$\rho_0[x](e_k \otimes h) = e^{-2\pi i k\cdot x} e_k \otimes h \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)\,,$$

for each $[x] \in \mathbb{T}^n$, from which it follows that both $C_\mathcal{R}(T^*\mathbb{T}^n)$ and $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ are invariant subspaces of the representation ρ_0. Furthermore, for each $l \in \mathbb{Z}^n$, we have

$$\begin{aligned}
(\rho_\hbar[x](\mathcal{Q}_\hbar^W(e_k \otimes h)))\psi_l &= L[x]\mathcal{Q}_\hbar^W(e_k \otimes h)L[-x]\psi_l \\
&= e^{2\pi i l\cdot x} L[x]\mathcal{Q}_\hbar^W(e_k \otimes h)\psi_l \\
&= e^{2\pi i l\cdot x} h(\pi\hbar(k + 2l))L[x]\psi_{k+l} \\
&= e^{2\pi i l\cdot x} e^{-2\pi i(k+l)\cdot x} h(\pi\hbar(k + 2l))\psi_{k+l}
\end{aligned}$$

$$= \mathcal{Q}^W_\hbar(e^{-2\pi i k\cdot x} e_k \otimes h)\psi_l\,,$$

from which we conclude that

$$\rho_\hbar[x](\mathcal{Q}^W_\hbar(e_k \otimes h)) = \mathcal{Q}^W_\hbar(\rho_0[x](e_k \otimes h))\,,$$

for each $[x]$ and each generator $e_k \otimes h$ of $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$. Since these generators span $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, and the quantisation map and the maps $\rho_0[x]$ and $\rho_\hbar[x]$ are linear, we may substitute for $e_k \otimes h$ any element of $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ in the above equation. □

4.2. Proof of Strict Quantisation

We now show that Weyl quantisation as defined in the previous section yields a strict quantisation of the dense Poisson subalgebra $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ of the classical resolvent algebra $C_\mathcal{R}(T^*\mathbb{T}^n)$ on $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$, see [15, Sect. II.1.1.1] or Theorem 23. Of these properties, the most difficult one to prove is Rieffel's condition, i.e. convergence of the operator norms of $\mathcal{Q}^W_\hbar(f)$ to the sup-norm of $f \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, which we discuss separately before showing that the other conditions hold. To prepare for the proof, we first make the following observation:

Lemma 21. *For $K \in \mathbb{N}\backslash\{0\}$ let $K\mathbb{Z}^n := K\mathbb{Z} \times \cdots \times K\mathbb{Z}$, and let $\mathbb{Z}^n_K := \mathbb{Z}^n/K\mathbb{Z}^n$. For each $k \in \mathbb{Z}^n_K$, let $S^k_{per}\colon \ell^2(\mathbb{Z}^n_K) \to \ell^2(\mathbb{Z}^n_K)$ be the operator given by*

$$\phi \mapsto (\, l \mapsto \phi(-k+l)\,)\,.$$

Then for any $f \in \ell^\infty(\mathbb{Z}^n_K)$, we have

$$\left\| \sum_{k\in\mathbb{Z}^n_K} f(k) S^k_{per} \right\| = \max_{l\in\mathbb{Z}^n_K} \left| \sum_{k\in\mathbb{Z}^n_K} f(k) e^{2\pi i \sum_{j=1}^n \frac{k_j l_j}{K}} \right|\,.$$

Proof. This is readily seen by conjugating the operator $\sum_{k\in\mathbb{Z}^n_K} f(k) S^k_{\mathrm{per}}$ with the discrete Fourier transform,

$$\phi \mapsto \left(l \mapsto K^{-\frac{n}{2}} \sum_{m\in\mathbb{Z}^n_K} \phi(m) e^{-2\pi i \sum_{j=1}^n \frac{l_j m_j}{K}} \right)\,,$$

yielding the multiplication operator of which the corresponding function is the one within absolute value strokes. □

Proposition 22 (Rieffel's condition). *For each $f \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, we have*

$$\lim_{\hbar\to 0} \|\mathcal{Q}^W_\hbar(f)\| = \|f\|_\infty\,.$$

Before we give a precise proof of this proposition, it is instructive to first give a sketch of the underlying idea. To relate the norm of $\mathcal{Q}^W_\hbar(f)$ to that of f, we conjugate the quantised function with the Fourier transform to obtain an operator on $\ell^2(\mathbb{Z}^n)$. We visualise $\mathbb{Z}^n$ as a lattice of points in $\mathbb{R}^n$, and divide it into identical boxes. In each of these boxes, we identify a slightly smaller box such that all of the smaller boxes are translates of each other in the same

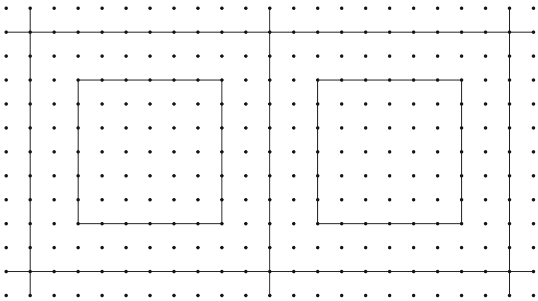

FIGURE 2. Part of the lattice $\mathbb{Z}^2$ with two larger boxes that are adjacent, each of which contains a smaller box

way that the larger boxes that contain them are translates of each other. See Fig. 2.

The difference between the sizes of the small boxes and the sizes of the larger boxes is determined by the values of the various k_j that appear in the function

$$f = \sum_{j=1}^{m} e_{k_j} \otimes h_{U_j,\xi_j,g_j} ,$$

of which we consider the quantisation; specifically, the shift $S_{k_j} \in B(\ell^2(\mathbb{Z}^n))$ should always map elements on $\ell^2(\mathbb{Z}^n)$ supported on points inside the smaller box to functions supported on points inside the larger box containing the small one. The size of the larger box is determined by a chosen value of $\varepsilon > 0$ and a crude estimate of $\|\mathcal{Q}_\hbar^W(f)\|$.

Given a function $\phi \in \ell^2(\mathbb{Z}^n)$, we can now estimate the norm of its image under the conjugated quantised function as follows. First, we consider the projection of ϕ onto the subspace of $\ell^2(\mathbb{Z}^n)$ of elements supported on the set of points inside one of the smaller boxes, and use the fact that its image under the operator consists of elements supported on the set of points inside the larger box. We can then consider a periodic version of the operator, and use the preceding lemma to get an estimate on its norm and relate it to the norm of f. Finally, we sum the contributions of all projections of ϕ onto the subspaces corresponding to the smaller boxes to obtain an estimate on the difference of the norm of f and that of the conjugated version of its quantisation. To control the difference between ϕ and its projection onto the space corresponding to the union of all of the smaller boxes, we note that the partition into boxes can always be offset by some element of $\mathbb{Z}^n$ in such a way that the part of ϕ supported on the complement of this union is small.

Proof. Fix $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ and $\varepsilon > 0$. We first prove the following statement:

(a) There exists an $\hbar_1 \in (0, \infty)$ such that for each $\hbar \in (0, \hbar_1]$, we have

$$\|\mathcal{Q}_\hbar^W(f)\| < \|f\|_\infty + \varepsilon .$$

Write $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ as $f = \sum_{j=1}^m f_j$, where

$$f_j = e_{k_j} \otimes h_j \,,$$

for $k_j \in \mathbb{Z}^n$ and $h_j = h_{U_j,\xi_j,g_j}$ for some U_j, ξ_j and g_j (which are not needed in the proof). Note that by (19), we have a uniform bound on the norms of the operators $(\mathcal{Q}^W_\hbar(f))_{\hbar>0}$, namely

$$\|\mathcal{Q}^W_\hbar(f)\| \leq \sum_{j=1}^m \|h_j\|_\infty = \sum_{j=1}^m \|g_j\|_\infty =: C\,.$$

Since the case $C = 0$ is trivial, we assume that $C > 0$ (which also implies that $m > 0$). Now define $L := \max_{1\leq j\leq m} \|k_j\|_\infty$ and fix $K \in \mathbb{N}\backslash\{0\}$ such that $K \geq 2\,L$ and such that

$$\left(1 - \frac{2L}{K}\right)^n > 1 - \left(\frac{\varepsilon}{4C}\right)^2 . \tag{20}$$

Moreover, for $j = 1, \dots, m$, the function h_j is uniformly continuous; hence, there exists $\hbar_1 \in (0, \infty)$ such that for each $\hbar \in (0, \hbar_1]$, each $a \in \mathbb{Z}^n$ and each $b \in \mathbb{Z}^n$ with $|b_l| < K$ for $l = 1, \dots, n$, we have

$$|h_j(2\pi\hbar a) - h_j(\pi\hbar(k_j + 2(a + b)))| < \frac{\varepsilon}{4m}\,. \tag{21}$$

Now fix $\hbar \in (0, \hbar_1]$, fix $\psi \in L^2(\mathbb{T}^n)$ with $\|\psi\| = 1$, and let ϕ be the image of ψ under the Fourier transform $\mathcal{F}\colon L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$, which we already defined in part (1) of the proof of Proposition 19. Furthermore, we define the set

$$X := \{a \in \mathbb{Z}^n \colon L \leq a_l < K - L \text{ for } l = 1, \dots, n\}\,,$$

and we define $K\mathbb{Z}^n$ and $\mathbb{Z}^n_K$ as in the previous lemma. Then, we have

$$\begin{aligned}
\sum_{b+K\mathbb{Z}^n \in \mathbb{Z}^n_K} \sum_{a \in X + K\mathbb{Z}^n} |\phi(a + b)|^2 &= \sum_{b+K\mathbb{Z}^n \in \mathbb{Z}^n_K} \sum_{a \in X} \sum_{a' \in K\mathbb{Z}^n} |\phi(a + a' + b)|^2 \\
&= \sum_{a \in X} \sum_{b+K\mathbb{Z}^n \in \mathbb{Z}^n_K} \sum_{a' \in K\mathbb{Z}^n} |\phi(a + a' + b)|^2 \\
&= \sum_{a \in X} \sum_{b \in \mathbb{Z}^n} |\phi(a+b)|^2 = |X| \cdot \sum_{b \in \mathbb{Z}^n} |\phi(b)|^2 = |X|\,,
\end{aligned}$$

where

$$|X| = (K - 2L)^n\,,$$

is the cardinality of the set X. It follows that there exists a $b \in \mathbb{Z}^n$ with $0 \leq b_l < K$ for $l = 1, \dots, n$ such that

$$\sum_{a \in X + K\mathbb{Z}^n} |\phi(a + b)|^2 \geq |\mathbb{Z}^n_K|^{-1}(K - 2L)^n = \left(1 - \frac{2L}{K}\right)^n > 1 - \left(\frac{\varepsilon}{4C}\right)^2 .$$

Let $P_{X,b}$ be the orthogonal projection of $\ell^2(\mathbb{Z}^n)$ onto the subspace

$$\{\phi' \in \ell^2(\mathbb{Z}^n) \colon \operatorname{supp}(\phi') \subseteq b + X + K\mathbb{Z}^n\}\,,$$

so that by the above inequality, we have

$$\begin{aligned}\|\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}(1-P_{X,b})\mathcal{F}\psi\| &\le \|\mathcal{Q}^W_\hbar(f)\|\|\mathcal{F}^{-1}(1-P_{X,b})\phi\| \\ &\le C\|(1-P_{X,b})\phi\| \\ &= C\left(1-\|P_{X,b}\phi\|^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{4}\,. \end{aligned} \tag{22}$$

For each $a \in K\mathbb{Z}^n$, let

$$P_{a,b}\colon \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n_K), \quad \phi' \mapsto (\, a' + K\mathbb{Z}^n \mapsto \phi'(a+a'+b)\,)\,,$$

where the representative $a' \in \mathbb{Z}^n$ has been chosen so that $0 \le a'_l < K$ for $l = 1,\dots,n$. Furthermore, for each $a \in \mathbb{Z}^n$, we have a corresponding shift operator

$$S^a\colon \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n), \quad \phi' \mapsto (\, a' \mapsto \phi'(-a+a')\,)\,,$$

and for each $a + K\mathbb{Z}^n \in \mathbb{Z}^n_K$, we define the shift operator $S^{a+K\mathbb{Z}^n}_{\mathrm{per}}$ as in the previous lemma. Finally, for each $a \in K\mathbb{Z}^n$, we define

$$A_{a,b} := \sum_{j=1}^m h_j(2\pi\hbar(a+b))S^{k_j+K\mathbb{Z}^n}_{\mathrm{per}}\,.$$

Using Lemma 21, we obtain

$$\begin{aligned}\|A_{a,b}\| &= \max_{a'+K\mathbb{Z}^n\in\mathbb{Z}^n_K}\left|\sum_{j=1}^m e^{2\pi i\frac{k_j\cdot a'}{K}}h_j(2\pi\hbar(a+b))\right| \\ &\le \sup_{[x]\in\mathbb{T}^n}\left|\sum_{j=1}^m e^{2\pi i k_j\cdot x}h_j(2\pi\hbar(a+b))\right| \\ &\le \sup_{([x],p)\in\mathbb{T}^n\times\mathbb{R}^n}\left|\sum_{j=1}^m e^{2\pi i k_j\cdot x}h_j(p)\right| \\ &= \|f\|_\infty\,. \end{aligned} \tag{23}$$

Moreover, using our explicit formula (19), we find that

$$\begin{aligned}&P_{a,b}\mathcal{F}\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}P_{X,b}\phi \\ &\quad= P_{a,b}\mathcal{F}\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}P_{X,b}\sum_{a'\in\mathbb{Z}^n}\phi(a')\delta_{a'} \\ &\quad= P_{a,b}\sum_{a'\in b+X+K\mathbb{Z}^n}\sum_{j=1}^m h_j(\pi\hbar(k_j+2a'))\phi(a')\delta_{a'+k_j} \\ &\quad= \sum_{a'\in X}\sum_{j=1}^m h_j(\pi\hbar(k_j+2(a+b+a')))\phi(a+b+a')\delta_{a'+k_j+K\mathbb{Z}^n} \\ &\quad= \sum_{j=1}^m S^{k_j+K\mathbb{Z}^n}_{\mathrm{per}}\sum_{a'\in X}h_j(\pi\hbar(k_j+2(a+b+a')))\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n}\,,\end{aligned}$$

where in the third step, we have used the fact that $a' + k_j \in \{0, \ldots, K-1\}^n$ for each $a' \in X$ and $j = 1, \ldots, m$. On the other hand, we have

$$A_{a,b}P_{a,b}P_{X,b}\phi = A_{a,b}P_{a,b}P_{X,b} \sum_{a' \in \mathbb{Z}^n} \phi(a')\delta_{a'} = A_{a,b} \sum_{a' \in X} \phi(a+b+a')\delta_{a'+K\mathbb{Z}^n}$$
$$= \sum_{j=1}^{m} S_{\text{per}}^{k_j + K\mathbb{Z}^n} \sum_{a' \in X} h_j(2\pi\hbar(a+b))\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n} \,.$$

Writing

$$\mu_{a',j} := h_j(2\pi\hbar(a+b)) - h_j(\pi\hbar(k_j + 2(a+b+a'))) \,,$$

for $j = 1, \ldots, m$ and $a' \in X$, we obtain

$$\begin{aligned}
&\|(A_{a,b}P_{a,b}P_{X,b} - P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b})\phi\| \\
&\quad = \left\| \sum_{j=1}^{m} S_{\text{per}}^{k_j + K\mathbb{Z}^n} \sum_{a' \in X} \mu_{a',j}\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n} \right\| \\
&\quad \leq \sum_{j=1}^{m} \left\| \sum_{a' \in X} \mu_{a',j}\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n} \right\| \\
&\quad = \sum_{j=1}^{m} \left(\sum_{a' \in X} |\mu_{a',j}|^2 |\phi(a+b+a')|^2 \right)^{\frac{1}{2}} \\
&\quad \leq m \cdot \max_{a'' \in X} |\mu_{a'',j}| \left(\sum_{a' \in X} |\phi(a+b+a')|^2 \right)^{\frac{1}{2}} \\
&\quad < \frac{\varepsilon}{4} \|P_{a,b}P_{X,b}\phi\| \,,
\end{aligned} \tag{24}$$

where we have used equation (21) in the final step. From equations (23) and (24), we obtain

$$\begin{aligned}
&\|P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\| \\
&\quad \leq \|A_{a,b}P_{a,b}P_{X,b}\phi\| + \|(A_{a,b}P_{a,b}P_{X,b} - P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b})\phi\| \\
&\quad < \left(\|f\|_\infty + \frac{\varepsilon}{4} \right) \|P_{a,b}P_{X,b}\phi\| \,,
\end{aligned}$$

for each $a \in K\mathbb{Z}^n$. It is straightforward to see that for each $\phi' \in \ell^2(\mathbb{Z}^n)$, we have

$$\sum_{a \in K\mathbb{Z}^n} \|P_{a,b}\phi'\|^2 = \|\phi'\|^2 \,,$$

so

$$\begin{aligned}
\|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\|^2 = \|\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\|^2 &= \sum_{a \in K\mathbb{Z}^n} \|P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\|^2 \\
&< \sum_{a \in K\mathbb{Z}^n} \left(\|f\|_\infty + \frac{\varepsilon}{4} \right)^2 \|P_{a,b}P_{X,b}\phi\|^2 \\
&= \left(\|f\|_\infty + \frac{\varepsilon}{4} \right)^2 \|P_{X,b}\phi\|^2 \leq \left(\|f\|_\infty + \frac{\varepsilon}{4} \right)^2 ,
\end{aligned}$$

which together with equation (22) implies

$$\begin{aligned}\|\mathcal{Q}_\hbar^W(f)\psi\| &\leq \|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\mathcal{F}\psi\| + \|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}(1-P_{X,b})\mathcal{F}\psi\| \\ &< \|f\|_\infty + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \|f\|_\infty + \frac{\varepsilon}{2},\end{aligned}$$

and since $\psi \in L^2(\mathbb{T}^n)$ was an arbitrary vector with norm 1, we obtain

$$\|\mathcal{Q}_\hbar^W(f)\| \leq \|f\|_\infty + \frac{\varepsilon}{2} < \|f\|_\infty + \varepsilon,$$

for each $\hbar \in (0, \hbar_1]$ which proves (a).
We now turn to the reverse inequality:

(b) There exists an $\hbar_2 \in (0,\infty)$ such that for each $\hbar \in (0,\hbar_2]$, we have

$$\|f\|_\infty < \|\mathcal{Q}_\hbar^W(f)\| + \varepsilon.$$

Let $(x,p) \in [0,1)^n \times \mathbb{R}^n$ be a point such that

$$\|f\|_\infty < |f([x],p)| + \frac{\varepsilon}{8}.$$

By uniform continuity of f, there exists a $\delta > 0$ such that for each $(x',p') \in (-1,1)^n \times \mathbb{R}^n$ with $\sum_{l=1}^n |x'_l - x_l| + |p'_l - p_l| < \delta$, we have

$$|f([x],p) - f([x'],p')| < \frac{\varepsilon}{8}.$$

Now fix $L \in \mathbb{N}$ as in the proof of part (a), and fix $K \in \mathbb{N}\backslash\{0\}$ in such a way that equation (20) holds, and that we have

$$K > \max\left(2L, \frac{2n}{\delta}\right). \tag{25}$$

Furthermore, fix $\hbar_2 > 0$ such that equation (21) holds for each $\hbar \in (0,\hbar_2]$, and that we have

$$2\pi\hbar_2 K < \frac{\delta}{2n}. \tag{26}$$

Now fix such an $\hbar \in (0,\hbar_2]$. Next, we note that by equation (26) there exists an $a \in K\mathbb{Z}^n$ such that

$$p_l - \frac{\delta}{2n} < 2\pi\hbar a_l \leq p_l,$$

and that by equation (25), there exists a $b \in \{0,\ldots,K-1\}^n$ such that

$$\left|\frac{b_l}{K} - x_l\right| < \frac{\delta}{2n},$$

for $l = 1,\ldots,n$. Fix such a and b. It follows that

$$\sum_{l=1}^n \left|\frac{b_l}{K} - x_l\right| + |2\pi\hbar a_l - p_l| < \delta,$$

so that

$$\left|\sum_{j=1}^m e^{2\pi i \frac{k_j\cdot b}{K}} h_j(2\pi\hbar a) - f([x],p)\right| < \frac{\varepsilon}{8},$$

and therefore, by the triangle inequality and our choice of $([x],p)$,

$$\left|\left|\sum_{j=1}^{m} e^{2\pi i\frac{k_j\cdot b}{K}} h_j(2\pi\hbar a)\right| - \|f\|_\infty\right| < \frac{\varepsilon}{4}\,.$$

Now define $\phi\in\ell^2(\mathbb{Z}^n)$ by

$$\phi(a') := \begin{cases} K^{-\frac{n}{2}}e^{-2\pi i\frac{a'\cdot b}{K}} & \text{if } 0\le a'_l - a_l < K \text{ for } l=1,\dots,n,\\ 0 & \text{otherwise,}\end{cases}$$

and let $\psi := \mathcal{F}^{-1}\phi\in L^2(\mathbb{T}^n)$. Then $\|\psi\| = \|\phi\| = 1$, and

$$A_{a,0}P_{a,0}\phi = \sum_{j=1}^{m} e^{2\pi i\frac{k_j\cdot b}{K}} h_j(2\pi\hbar a)P_{a,0}\phi\,,$$

with $A_{a,b}$ and $P_{a,b}$ as defined in part (a). Since $\|P_{a,0}\phi\| = 1$, we have

$$\|A_{a,0}P_{a,0}\phi\| = \left|\sum_{j=1}^{m} e^{2\pi i\frac{k_j\cdot b}{K}} h_j(2\pi\hbar a)\right| > \|f\|_\infty - \frac{\varepsilon}{4}\,.$$

Defining X in the same way as we did in the proof part (a), it follows that

$$\begin{aligned}\|A_{a,0}P_{a,0}P_{X,0}\phi\| &\ge \|A_{a,0}P_{a,0}\phi\| - \|A_{a,0}\|\|(1-P_{X,0})\phi\|\\ &> \|f\|_\infty - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \|f\|_\infty - \frac{\varepsilon}{2}\,.\end{aligned}$$

Next, we note that the function $\mathcal{F}\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}P_{X,0}\phi : \mathbb{Z}^n\to\mathbb{C}$ is supported in the set of $a'\in\mathbb{Z}^n$ satisfying $a_l\le a'_l < a_l + K$ for $l=1,\dots,n$. Combining this observation with the estimate just obtained and equation (24) yields

$$\begin{aligned}\|\mathcal{F}\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}P_{X,0}\phi\| &= \|P_{a,0}\mathcal{F}\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}P_{X,0}\phi\|\\ &\ge \|A_{a,0}P_{a,0}P_{X,0}\phi\|\\ &\quad - \|(A_{a,0}P_{a,0}P_{X,0} - P_{a,0}\mathcal{F}\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}P_{X,0})\phi\|\\ &> \|f\|_\infty - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \|f\|_\infty - \frac{3\varepsilon}{4}\,.\end{aligned}$$

We use this together with equation (22) to obtain

$$\begin{aligned}\|\mathcal{Q}^W_\hbar(f)\psi\| &= \|\mathcal{F}\mathcal{Q}^W_\hbar(f)\psi\|\\ &\ge \|\mathcal{F}\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}P_{X,0}\phi\| - \|\mathcal{Q}^W_\hbar(f)\mathcal{F}^{-1}(1-P_{X,0})\mathcal{F}\psi\|\\ &> \|f\|_\infty - \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \|f\|_\infty - \varepsilon\,.\end{aligned}$$

Since $\|\psi\| = 1$, this establishes (b).

Finishing up the proof, taking $\hbar_0 := \min(\hbar_1,\hbar_2)$, we infer that for each $\hbar\in(0,\hbar_0]$, we have $|\|\mathcal{Q}^W_\hbar(f)\| - \|f\|_\infty| < \varepsilon$; hence, $\lim_{\hbar\to 0}\|\mathcal{Q}^W_\hbar(f)\| = \|f\|_\infty$, as desired. ☐

We are now ready to prove the main result of this subsection. Let $\mathcal{Q}^W_0 := \mathrm{Id}_{\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)}$, let A_0 be the C*-algebra $C_\mathcal{R}(T^*\mathbb{T}^n)$. In the following theorem, it should be understood that $\|\mathcal{Q}^W_\hbar(f)\| := \|f\|_\infty$ for $\hbar = 0$.

Theorem 23. *Let $I \subset [0,\infty)$ be a subset containing 0 as an accumulation point. Then, except for continuity at $\hbar > 0$, the triple*

$$(I, (A_\hbar)_{\hbar\in I}, (\mathcal{Q}^W_\hbar : \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n) \to A_\hbar)_{\hbar\in I}),$$

is a strict quantisation of the Poisson algebra $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, i.e. it satisfies

(1) *Rieffel's condition at $\hbar = 0$: for each $f \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, the function $\hbar \mapsto \|\mathcal{Q}^W_\hbar(f)\|$ is continuous at 0.*

(2) *Von Neumann's condition: for each $f, g \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, we have*

$$\lim_{\substack{\hbar\to 0\\ \hbar\in I}} \|\mathcal{Q}^W_\hbar(f)\mathcal{Q}^W_\hbar(g) - \mathcal{Q}^W_\hbar(fg)\| = 0\,.$$

(3) *Dirac's condition: for each $f, g \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$, we have*

$$\lim_{\substack{\hbar\to 0\\ \hbar\in I}} \|(-i\hbar)^{-1}[\mathcal{Q}^W_\hbar(f), \mathcal{Q}^W_\hbar(g)] - \mathcal{Q}^W_\hbar(\{f,g\})\| = 0\,.$$

(4) *Completeness: for each $\hbar \in I$, the set $\mathcal{Q}^W_\hbar(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n))$ is dense in $A_\hbar$.*

Proof.

(1) This was shown in Proposition 22.

(2) First suppose that f_j is a generator $e_{k_j} \otimes h_{U_j,\xi_j,g_j}$ of $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ for $j = 1, 2$. As in Proposition 22, we will write h_j instead of h_{U_j,ξ_j,g_j}. Let $k := k_1 + k_2$. Then

$$f_1 \cdot f_2 = (e_{k_1} \otimes h_1) \cdot (e_{k_2} \otimes h_2) = e_k \otimes (h_1 \cdot h_2)\,.$$

Applying part (3) of Proposition 17 yields

$$\mathcal{Q}^W_\hbar(f_1 f_2)\psi_a = (h_1 \cdot h_2)(\pi\hbar(k + 2a))\psi_{k+a}\,.$$

for each $a \in \mathbb{Z}^n$. On the other hand, we have

$$\begin{aligned}\mathcal{Q}^W_\hbar(f_1)\mathcal{Q}^W_\hbar(f_2)\psi_a &= h_2(\pi\hbar(k_2 + 2a))\mathcal{Q}^W_\hbar(f_1)\psi_{k_2+a}\\ &= h_1(\pi\hbar(k_1 + 2(k_2 + a)))h_2(\pi\hbar(k_2 + 2a))\psi_{k+a}\\ &= h_1(\pi\hbar(k + k_2 + 2a)) \cdot h_2(\pi\hbar(k - k_1 + 2a)) \cdot \psi_{k+a}\,,\end{aligned} \tag{27}$$

so

$$\begin{aligned}(\mathcal{Q}^W_\hbar(f_1)\mathcal{Q}^W_\hbar(f_2) - \mathcal{Q}^W_\hbar(f_1 f_2))\psi_a = (&h_1(\pi\hbar(k + k_2 + 2a)) \cdot h_2(\pi\hbar(k - k_1 + 2a)))\\ &-(h_1 \cdot h_2)(\pi\hbar(k + 2a)))\,\psi_{k+a}\,,\end{aligned}$$

for each $a \in \mathbb{Z}^n$. Now let $c^{(1)}_{a,\hbar}$ be the scalar in front of ψ_{k+a} on the right-hand side of the last equation. It is not hard to see from this equation that

$$\|\mathcal{Q}^W_\hbar(f_1)\mathcal{Q}^W_\hbar(f_2) - \mathcal{Q}^W_\hbar(f_1 f_2)\| \le \sup_{a\in\mathbb{Z}^n} |c^{(1)}_{a,\hbar}|\,,$$

for each $\hbar > 0$. Now note for $j = 1, 2$, all derivatives of $h_j \in \mathcal{W}^0_\mathcal{R}(\mathbb{R}^n)$ are bounded, and so h_j is Lipschitz continuous. This implies that

$$\begin{aligned}h_1(\pi\hbar(k + k_2 + 2a)) &= h_1(\pi\hbar(k + 2a)) + \mathcal{O}(\hbar)\,,\\ h_2(\pi\hbar(k - k_1 + 2a)) &= h_2(\pi\hbar(k + 2a)) + \mathcal{O}(\hbar)\,,\end{aligned}$$

where big O notation signifies a limit of $\hbar \to 0$, *uniformly in a*, analogous to the notation in §3.2. When plugging the above formulas into the definition of $c^{(1)}_{a,\hbar}$ and using the fact that h_1 and h_2 are bounded functions, we find that

$$c^{(1)}_{a,\hbar} = \mathcal{O}(\hbar),$$

and therefore

$$\lim_{\substack{\hbar\to 0\\ \hbar\in I}} \|\mathcal{Q}^W_\hbar(f_1)\mathcal{Q}^W_\hbar(f_2) - \mathcal{Q}^W_\hbar(f_1 f_2)\| = 0\,.$$

By bilinearity, this result extends to arbitrary $f_1, f_2 \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$.

(3) As in the previous part of the proof, we prove the statement for $f_j = e_{k_j} \otimes h_j$, from which the general case readily follows. We have

$$\begin{aligned}
\{f_1, f_2\} &= \sum_{l=1}^n \left(\frac{\partial f_1}{\partial p_l}\frac{\partial f_2}{\partial q_l} - \frac{\partial f_1}{\partial q_l}\frac{\partial f_2}{\partial p_l}\right)\\
&= \sum_{l=1}^n \left(e_{k_1}\otimes\frac{\partial h_1}{\partial p_l}\right)\cdot\left(\frac{\partial e_{k_2}}{\partial q_l}\otimes h_2\right) - \left(\frac{\partial e_{k_1}}{\partial q_l}\otimes h_1\right)\cdot\left(e_{k_2}\otimes\frac{\partial h_2}{\partial p_l}\right)\\
&= 2\pi i e_k \otimes \left((\nabla_{k_2} h_1) h_2 - h_1 \nabla_{k_1} h_2\right)\,,
\end{aligned}$$

where $k = k_1 + k_2$, as in part (2) of this theorem, and $\nabla_v h$ is the directional derivative of h in the direction of v. Applying part (3) of Proposition 17 yields

$$\mathcal{Q}^W_\hbar(\{f_1, f_2\})\psi_a = 2\pi i \left((\nabla_{k_2} h_1) h_2 - h_1\nabla_{k_1} h_2\right)(\pi\hbar(k+2a))\psi_{k+a}\,,$$

while equation (27) yields

$$\begin{aligned}
[\mathcal{Q}^W_\hbar(f_1), \mathcal{Q}^W_\hbar(f_2)]\psi_a = \big(&h_1(\pi\hbar(k+k_2+2a))\cdot h_2(\pi\hbar(k-k_1+2a))\\
&- h_1(\pi\hbar(k-k_2+2a))\cdot h_2(\pi\hbar(k+k_1+2a))\big)\,\psi_{k+a}\,.
\end{aligned}$$

It follows that

$$\left((-i\hbar)^{-1}[\mathcal{Q}^W_\hbar(f_1), \mathcal{Q}^W_\hbar(f_2)] - \mathcal{Q}^W_\hbar(\{f_1, f_2\})\right)\psi_a = c^{(2)}_{a,\hbar}\psi_{k+a}\,,$$

where for each $a \in \mathbb{Z}^n$ and each $\hbar > 0$, we define

$$\begin{aligned}
c^{(2)}_{a,\hbar} := (-i\hbar)^{-1}\big(&h_1(\pi\hbar(k+k_2+2a))\cdot h_2(\pi\hbar(k-k_1+2a))\\
&- h_1(\pi\hbar(k-k_2+2a))\cdot h_2(\pi\hbar(k+k_1+2a))\big)\\
&- 2\pi i\left((\nabla_{k_2} h_1)h_2 - h_1\nabla_{k_1} h_2\right)(\pi\hbar(k+2a))\,.
\end{aligned}$$

It is readily seen that

$$\left\|(-i\hbar)^{-1}[\mathcal{Q}^W_\hbar(f_1), \mathcal{Q}^W_\hbar(f_2)] - \mathcal{Q}^W_\hbar(\{f_1, f_2\})\right\| \leq \sup_{a\in\mathbb{Z}^n} |c^{(2)}_{a,\hbar}|\,.$$

We claim that the right-hand side of this inequality converges to 0 as $\hbar \in I$ goes to 0; evidently, this will show that Dirac's condition holds.

Because the second order derivatives of h_j are bounded, Taylor's theorem gives

$$h_j(\pi\hbar(k+v+2a)) - h_j(\pi\hbar(k+2a)) - \pi\hbar\nabla_v h_j(\pi\hbar(k+2a)) = \mathcal{O}(\hbar^2)\,, \quad (28)$$

for each $v \in \mathbb{R}^n$ and $j = 1, 2$. Dividing the expression on the left-hand side of (28) by $-i\hbar$ yields

$$(-i\hbar)^{-1}(h_j(\pi\hbar(k+v+2a)) - h_j(\pi\hbar(k+2a))) - \pi i \nabla_v h_j(\pi\hbar(k+2a)) = \mathcal{O}(\hbar)\,.$$

This can be used to show that $c^{(2)}_{a,\hbar} \to 0$ uniformly in $a \in \mathbb{Z}^n$ as $\hbar \to 0$, which proves the claim.

(4) According to part (2) of Proposition 8, the space $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is a *-subalgebra of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. According to part (1) of Proposition 19 the Weyl quantisation map is linear and compatible with the involutions on the algebras involved. Moreover, it is readily seen from our computation of $\mathcal{Q}^W_\hbar(f_1)\mathcal{Q}^W_\hbar(f_2)$ in the proof of part (2) of this theorem that $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$ is closed under multiplication. Thus $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$ is a *-algebra. It follows that $A_\hbar$, which is by definition the smallest C*-algebra that contains $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$, is the closure of $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$. □

Remark 24. The statement that for arbitrary $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, the map

$$[0,\infty) \to [0,\infty)\,, \quad \hbar \mapsto \|\mathcal{Q}^W_\hbar(f)\|\,,$$

is continuous at points other than $\hbar = 0$ is false. As a counterexample, let $\hbar_0 > 0$ be arbitrary, and consider the function $f = e_0 \otimes h$, where the function h is defined as follows:

$$h\colon \mathbb{R}^n \to \mathbb{R}, \quad p = (p_1, p_2, \ldots, p_n) \mapsto \sin\left(\frac{p_1}{\hbar_0}\right)\,.$$

Note that h can be written as the sum of two generators of $\mathcal{W}^0(\mathbb{R}^n) \subseteq \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, so $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$. Furthermore, h vanishes at each point in $2\pi\hbar_0 \cdot \mathbb{Z}^n$; hence, $\mathcal{Q}^W_{\hbar_0}(f) = 0$ by the explicit formula (19), or equivalently, $\|\mathcal{Q}^W_{\hbar_0}(f)\| = 0$. On the other hand, for each $N \in \mathbb{N}\backslash\{0\}$, let

$$\hbar_N := \hbar_0\left(1 + \frac{1}{4N}\right)\,.$$

Then $\|\mathcal{Q}^W_{\hbar_N}(f)\| = 1$; indeed, we have $\|\mathcal{Q}^W_{\hbar_N}(f)\| \leq \|h\|_\infty = 1$, and equality holds since

$$\mathcal{Q}^W_{\hbar_N}(f)\psi_{(N,0,0,\ldots,0)} = \psi_{(N,0,0,\ldots,0)}\,.$$

Thus, while $\lim_{N\to\infty} \hbar_N = \hbar_0$, we also have

$$\lim_{N\to\infty} \|\mathcal{Q}^W_{\hbar_N}(f)\| = 1 \neq 0 = \|\mathcal{Q}^W_{\hbar_0}(f)\|\,,$$

so the function $\hbar \to \|\mathcal{Q}^W_\hbar(f)\|$ fails to be continuous at $\hbar_0$.

The issue of continuity of the norm of the quantisation of a given function at points $\hbar \neq 0$ is often sidestepped in the literature for reasons related to geometric quantisation, which imposes the condition that $\hbar$ be of the form $\hbar_0/m$, $m \in \mathbb{N}\backslash\{0\}$ for some fixed $\hbar_0 > 0$ (cf. [12] for a discussion of this point, and also a nice overview of various notions of quantisation throughout the literature). In such cases the set $I\backslash\{0\}$ in the above theorem is a discrete subset of $(0,\infty)$, so the restriction of $\hbar \to \|\mathcal{Q}^W_\hbar(f)\|$ to I is trivially continuous at all points outside of 0, and the family of quantisation maps constitutes an actual strict quantisation.

Injectivity is the second and last property barring our quantisation map from being a strict deformation quantisation in the sense of [15, Definition II.1.1.2]. Injectivity fails precisely because (19) depends only on the values of h at $\pi\hbar\mathbb{Z}^n \subseteq \mathbb{R}^n$. A solution to both problems is proposed in [20], as discussed in the outlook: Sect. 6.

Moreover, despite the fact that the norm of the quantisation of a function is not continuous for $\hbar > 0$, we still have continuity of quantisation in another way:

Proposition 25. *Let $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$. Then the map*

$$(0,\infty) \to A_\hbar \subseteq B(L^2(\mathbb{T}^n))\,, \quad \hbar \mapsto \mathcal{Q}^W_\hbar(f)\,,$$

is continuous with respect to the strong operator topology on the codomain.

Proof. By linearity of the quantisation map and the fact that $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is the linear span of generators of $\mathcal{C}_{\mathcal{R}}(T^*\mathbb{T}^n)$, we may assume without loss of generality that there exists a $k \in \mathbb{Z}^n$ and a generator h of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ such that $f = e_k \otimes h$. Then, by our explicit formula (19), we find

$$\left\|\mathcal{Q}^W_\hbar(f)\psi_l - \mathcal{Q}^W_{\hbar_0}(f)\psi_l\right\| = |h(\pi\hbar(k+2l)) - h(\pi\hbar_0(k+2l))| \to 0,$$

whenever $\hbar \to \hbar_0$ in $(0,\infty)$. This convergence also holds when we replace ψ_l by a vector in $\mathrm{span}_l\{\psi_l\}$. Furthermore, in part (3) of Proposition 17, we have seen that

$$\left\|\mathcal{Q}^W_\hbar(f)\right\| \leq \|h\|_\infty\,.$$

Now let $\psi \in L^2(\mathbb{T}^n)$ be arbitrary. Fix $\varepsilon > 0$. Since $\mathrm{span}_l\{\psi_l\}$ is dense in $L^2(\mathbb{T}^n)$, there exists $\tilde{\psi} \in \mathrm{span}_l\{\psi_l\}$ such that $\|\tilde{\psi} - \psi\| < \varepsilon/(4(\|h\|_\infty + 1))$. By the discussion above, there exists $\delta > 0$ such that $\left\|\mathcal{Q}^W_\hbar(f)\tilde{\psi} - \mathcal{Q}^W_{\hbar_0}(f)\tilde{\psi}\right\| < \varepsilon/2$ whenever $\hbar > 0$ satisfies $|\hbar - \hbar_0| < \delta$. Then for any such $\hbar$, we have

$$\begin{aligned}
&\left\|\mathcal{Q}^W_\hbar(f)\psi - \mathcal{Q}^W_{\hbar_0}(f)\psi\right\| \\
&\quad\leq \left\|\mathcal{Q}^W_\hbar(f)(\psi - \tilde{\psi})\right\| + \left\|\mathcal{Q}^W_\hbar(f)\tilde{\psi} - \mathcal{Q}^W_{\hbar_0}(f)\tilde{\psi}\right\| + \left\|\mathcal{Q}^W_{\hbar_0}(f)(\tilde{\psi} - \psi)\right\| \\
&\quad\leq 2\,\|h\|_\infty \left\|\tilde{\psi} - \psi\right\| + \left\|\mathcal{Q}^W_\hbar(f)\tilde{\psi} - \mathcal{Q}^W_{\hbar_0}(f)\tilde{\psi}\right\| < \varepsilon\,,
\end{aligned}$$

which concludes the proof of the proposition. □

5. Quantum Time Evolution

Our next task is to show that $A_\hbar = C^*\big(\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))\big)$ is invariant under time evolution for each Hamiltonian with potential $V \in C(\mathbb{T}^n)$. The general proof strategy resembles that of Buchholz and Grundling in [9, Proposition 6.1]. However, the present setting differs from theirs in two important ways, each of which introduces its own technical problems. First of all, our configuration space is $\mathbb{T}^n$ rather than $\mathbb{R}^n$. Secondly, we consider the problem of invariance under time evolution for arbitrary $n \in \mathbb{N}$, whereas Buchholz and Grundling only discuss the case $n = 1$. We start with the simplest type of time evolution:

Lemma 26. *Let $\hbar > 0$. The algebra $A_\hbar$ is closed under the quantum time evolution corresponding to the free Hamiltonian H_0 that is the unique self-adjoint extension of the essentially self-adjoint operator $-\frac{\hbar^2}{2}\sum\frac{d^2}{dx_j^2}$ with domain $C^\infty(\mathbb{T}^n)$.*

Remark 27. The fact that for any compact Riemannian manifold M the Laplace–Beltrami operator on $C^\infty(M)$ has a unique self-adjoint extension is due to Gaffney [10].

Proof. We show that the quantum time evolution corresponding to H_0 maps the set of quantisations of the generators $e_k \otimes h_{U,\xi,g}$ of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ into itself; since the time evolution consists of a family of automorphisms of C*-algebras, the lemma will follow from this.

Let $e_k \otimes h_{U,\xi,g}$ be such a generator. Note that for each $a \in \mathbb{Z}^n$, we have

$$e^{-\frac{itH_0}{\hbar}}\psi_a = e^{-2\pi^2 it\hbar\|a\|^2}\psi_a\,. \tag{29}$$

Using part (3) of Proposition 17, we obtain

$$\begin{aligned}
&e^{\frac{itH_0}{\hbar}}\mathcal{Q}^W_\hbar(e_k\otimes h_{U,\xi,g})\,e^{-\frac{itH_0}{\hbar}}\psi_a\\
&\quad= e^{2\pi^2 it\hbar(\|a+k\|^2-\|a\|^2)}e^{\pi\hbar i(k+2a)\cdot\xi}g\circ P_U(\pi\hbar(k+2a))\psi_{k+a}\\
&\quad= e^{\pi i\hbar(k+2a)\cdot(\xi+2\pi tk)}g\circ P_U(\pi\hbar(k+2a))\psi_{k+a}\\
&\quad= \mathcal{Q}^W_\hbar\left(e_k\otimes h_{U,\tilde\xi,\tilde g}\right)\psi_a\,,
\end{aligned}$$

for each $a \in \mathbb{Z}^n$, where

$$\tilde\xi := \xi + 2\pi t P_{U^\perp}(k) \in U^\perp\,,$$

and

$$\tilde g\colon U \to \mathbb{C}\,,\quad p \mapsto e^{2\pi it P_U(k)\cdot p}g(p)\,,$$

is again a Schwartz function on U, so $e_k \otimes h_{U,\tilde\xi,\tilde g}$ is a generator of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. It follows that the set of generators of $A_\hbar$ is indeed invariant under the free quantum time evolution. □

Remark 28. Comparing the proof of Lemma 26 with the proof of the analogous Lemma 11, we see that (for $t = 1$) $\tilde\xi$ and $\tilde g$ are both the same. Indeed, one can easily obtain

$$\mathcal{Q}^W_\hbar \circ (\Phi^t_0)^* = \tau^0_t \circ \mathcal{Q}^W_\hbar\,,$$

which is analogous to a known result for Weyl quantisation on $\mathbb{R}^{2n}$ (proved in higher generality in [15, Theorem II.2.5.1]). There is generally no such result for non-free time evolution.

In order to deal with the general quantum time evolution, we recall some basic theory about lattices that we need due to the appearance of the lattice $\mathbb{Z}^n$ in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. A set of linearly independent vectors $v_1, \ldots, v_l$ in a lattice Λ is called *primitive in* Λ if $\mathrm{span}_{\mathbb{Z}}(v_1, \ldots, v_l) = \mathrm{span}_{\mathbb{R}}(v_1, \ldots, v_l) \cap \Lambda$. For instance, every $\mathbb{Z}$-basis of a lattice Λ is primitive in Λ. Furthermore, we have the following result:

Lemma 29. *Let $\Lambda \subset \mathbb{R}^m$ be a lattice. Every primitive set $v_1, \ldots, v_l$ in Λ can be extended to a $\mathbb{Z}$-basis $v_1, \ldots, v_l, v_{l+1}, \ldots, v_m$ of Λ.*

Proof. This is exactly [17, §1.3, Theorem 5]. □

This will help us prove the main theorem of this section:

Theorem 30. *Let $V \in C(\mathbb{T}^n)$. Then the operator $H = -\frac{\hbar^2}{2}\sum \frac{d^2}{dx_j^2} + M(V)$ with domain domH_0 (see Lemma 26) is self-adjoint. Let $\left(e^{\frac{-itH}{\hbar}}\right)_{t\in\mathbb{R}}$ be the corresponding one-parameter group implementing the quantum mechanical time evolution on $L^2(\mathbb{T}^n)$, and let $(\tau_t)_{t\in\mathbb{R}}$ be the associated one-parameter group of automorphisms on $B(L^2(\mathbb{T}^n))$. Then $(\tau_t)_{t\in\mathbb{R}}$ preserves $A_\hbar$.*

Proof. Self-adjointness of H is a consequence of the Kato–Rellich theorem. We claim that for each $t \in \mathbb{R}$, we have

$$e^{\frac{itH_0}{\hbar}} e^{\frac{-itH}{\hbar}} \in A_\hbar \,.$$

Suppose for the moment that this claim holds true. Then for each $a \in A_\hbar$ and each $t \in \mathbb{R}$, we have

$$\tau_t(a) = e^{\frac{itH}{\hbar}} a e^{\frac{-itH}{\hbar}} = \left(e^{\frac{itH_0}{\hbar}} e^{\frac{-itH}{\hbar}}\right)^* \tau_t^0(a) \left(e^{\frac{itH_0}{\hbar}} e^{\frac{-itH}{\hbar}}\right).$$

By assumption, the first and the third factors within parentheses are elements of $A_\hbar$, and the second factor is an element of $A_\hbar$ by Lemma 26. It then follows that $\tau_t(a) \in A_\hbar$.

Thus, it remains to prove the claim. As in the proof of [9, Proposition 6.1], we use the fact that the product of two of the elements of different one-parameter groups can be written as a norm-convergent Dyson series, i.e.

$$e^{\frac{itH_0}{\hbar}} e^{\frac{-itH}{\hbar}} = \sum_{m=0}^{\infty} (i\hbar)^{-m} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} \tau_{t_1}^0(M(V)) \cdots \tau_{t_m}^0(M(V))\, dt_m \cdots dt_2\, dt_1. \tag{30}$$

The integrals in the above expression can be defined in the following way. First, observe that the function

$$\mathbb{R} \to B(L^2(\mathbb{T}^n)), \quad t \mapsto \tau_t^0(M(V)),$$

is bounded and strongly continuous. It follows that the function

$$\mathbb{R}^m \to B(L^2(\mathbb{T}^n)), \quad (t_1, \ldots, t_m) \mapsto \tau_{t_1}^0(M(V)) \cdots \tau_{t_m}^0(M(V)),$$

is bounded and strongly continuous. For each $\psi \in L^2(\mathbb{T}^n)$, one can therefore define the integral

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} \tau_{t_1}^0(M(V)) \cdots \tau_{t_m}^0(M(V))\psi\, dt_m \cdots dt_2\, dt_1 \,, \tag{31}$$

using Bochner integration, and it is easy to check that the norm of the corresponding operator is less than or equal to $(m!)^{-1}|t|^m\|V\|_\infty^m$, so that the Dyson series is indeed norm-convergent. As in [9], because (31) is continuous in V it suffices to prove the claim for potentials V that lie in a dense subset of $C(\mathbb{T}^n)$. If we assume that V is in the span of $\{e_k : k \in \mathbb{Z}^n\}$, we can write (31) as a sum of relatively explicit expressions. Thus, we are left to show that for each $t \in \mathbb{R}$ and each $k_1, \ldots, k_m \in \mathbb{Z}^n$, the operator

$$a := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} \tau_{t_1}^0(M(e_{k_1})) \cdots \tau_{t_m}^0(M(e_{k_m}))\, dt_m \cdots dt_1\,,$$

lies in $A_\hbar$. A quick computation using (29) gives us

$$\begin{aligned}\tau_t^0(M(e_k))\psi_a &= M(e_k)e^{2\pi^2 it\hbar(\|a+k\|^2-\|a\|^2)}\psi_a\\ &= M(e_k)e^{2\pi^2 it\hbar\|k\|^2}e^{4\pi^2 it\hbar k\cdot a}\psi_a\,,\end{aligned}$$

which shows that, for any $\psi \in L^2(\mathbb{T}^n)$ and $[x] \in \mathbb{T}^n$, we have

$$(\tau_t^0(M(e_k))\psi)[x] = e^{2\pi i x\cdot k}e^{2\pi^2 i\hbar t\|k\|^2}\psi\,[x + 2\pi\hbar t k]\,.$$

Applying this formula many times, we find a function $f_0 \in C_b(\mathbb{R}^m)$ that takes values on the unit circle such that

$$\tau_{t_1}^0(M(e_{k_1})) \cdots \tau_{t_m}^0(M(e_{k_m}))\psi[x] = e^{2\pi i x\cdot\sum k_i} f_0(t_1, \ldots, t_m)\psi\left[x + 2\pi\hbar \sum t_i k_i\right]\,.$$

The operator a looks like an integral operator, in the sense that we perform an integral over the variables t_i that appear as $\sum t_i k_i$ in the argument of ψ. However, the k_i's may both fail to constitute a linearly independent and a complete set of vectors in $\mathbb{R}^n$. Still, we can relate a to an integral operator, which will be the subject of the rest of the proof.

We use a special case of Lemma 29 (extending an empty primitive set) to find a $\mathbb{Z}$-basis $v_1, \ldots, v_l$ of $\operatorname{span}_{\mathbb{R}}(k_1, \ldots, k_m) \cap \mathbb{Z}^n$. Because the k_i's are integral, this is also an $\mathbb{R}$-basis of $\operatorname{span}_{\mathbb{R}}(k_1, \ldots, k_m)$. Expressing the k_i's in terms of v_j's as

$$k_i = \sum_{j=1}^{l} c_{ij} v_j,$$

we obtain

$$\begin{aligned}\psi\left[x + 2\pi\hbar\sum_{i=1}^m t_i k_i\right] &= \psi\left[x + 2\pi\hbar\sum_{j=1}^l\sum_{i=1}^m t_i c_{ij} v_j\right]\\ &= \psi\left[x + 2\pi\hbar\sum_{j=1}^l T_0(t_1, \ldots, t_m)_j v_j\right],\end{aligned}$$

for a unique surjective linear map $T_0 \colon \mathbb{R}^m \to \mathbb{R}^l$. By surjectivity, the map T_0 admits a lift to an invertible linear map $T \colon \mathbb{R}^m \to \mathbb{R}^m$ with respect to the projection $\mathbb{R}^m \to \mathbb{R}^l$ onto the first l coordinates. Fix such a T, and perform a change of variables, replacing $(t_1, \dots, t_m)$ with $T^{-1}(s)$. We get

$$a\psi[x] = e^{2\pi i x \cdot \sum k_i} \, |\det T|^{-1} \int_K f_0\big(T^{-1}s\big)\psi\left[x + 2\pi\hbar \sum_{j=1}^{l} s_j v_j\right] ds\,,$$

for some compact subset $K \subseteq \mathbb{R}^m$. Let K' be the image of K under the projection $\mathbb{R}^k \to \mathbb{R}^l$ onto the first l coordinates, and define the function $f_1 \colon \mathbb{R}^l \to \mathbb{C}$ by

$$f_1 \colon s_{(1)} \mapsto |\det T|^{-1} \int_{\mathbb{R}^{m-l}} 1_K(s_{(1)} \oplus s_{(2)}) f_0\big(T^{-1}(s_{(1)} \oplus s_{(2)})\big)\, ds_{(2)}\,.$$

One easily finds that $f_1 \in L^\infty(\mathbb{R}^l)$. We are now left with the integral

$$a\psi[x] = e^{2\pi i x \cdot \sum k_i} \int_{K'} f_1(s)\psi\left[x + 2\pi\hbar \sum_{j=1}^{l} s_j v_j\right] ds\,.$$

We want to relate the above integral to an integral over the first l components in $\mathbb{T}^n$. For this purpose, we apply Lemma 29 once more to extend $v_1, \dots, v_l$ to a $\mathbb{Z}$-basis $v_1, \dots, v_n$ of $\mathbb{Z}^n$, and let S be the matrix whose columns are the vectors $v_1, \dots, v_n$. Since S and its inverse are matrices in $GL_n(\mathbb{Z})$, we find that $\det S = \pm 1$. Moreover, S induces the group automorphism $[x] \mapsto [Sx]$ of $\mathbb{T}^n$, which we can pull back to the unitary map

$$U \colon L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n), \quad U\psi[x] := \psi[Sx],$$

for which it is straightforward to check (on generators of $A_\hbar$) that $U^{-1}A_\hbar U \subseteq A_\hbar$. For $\varphi = \varphi_1 \otimes \varphi_2 \in L^2(\mathbb{T}^l) \otimes L^2(\mathbb{T}^{n-l})$ we have, denoting $k := \sum_i k_i$,

$$\begin{aligned}
UM(e_{-k})aU^{-1}\varphi[x] &= \int_{K'} f_1(s) U^{-1}\varphi\left[S(x) + 2\pi\hbar \sum_{j=1}^{l} s_j S(e_j)\right] ds \\
&= \int_{K'} f_1(s)\varphi\left[x + 2\pi\hbar(s \oplus 0)\right]\, ds \\
&= \int_{K'} f_1(s)\varphi_1\left(x_{(1)} + 2\pi\hbar s + \mathbb{Z}^l\right)\varphi_2\left(x_{(2)} + \mathbb{Z}^{n-l}\right)\, ds \\
&= \int_{\mathbb{T}^l} f_2\left(x_{(1)} + \mathbb{Z}^l, s\right)\varphi_1(s)\, ds\, \varphi_2\left(x_{(2)} + \mathbb{Z}^{n-l}\right)\,,
\end{aligned}$$

where $x = x_{(1)} \oplus x_{(2)}$ and $f_2 \in L^\infty(\mathbb{T}^l \times \mathbb{T}^l) \subseteq L^2(\mathbb{T}^l \times \mathbb{T}^l)$ denotes the function

$$f_2(r, s) := \sum_{M \in \mathbb{Z}^l} f_1\left(\frac{\iota(s - r) + M}{2\pi\hbar}\right)\,,$$

where ι denotes the canonical map $\mathbb{T}^l \to [0,1)^l$. Note that the above sum has only finitely many nonzero terms since f_1 is compactly supported.

In conclusion, we have proved that

$$a = M(e_k)U^{-1}(F \otimes \mathbf{1})U\,,$$

for an integral operator $F \in L^2(L^2(\mathbb{T}^l))$. By part (3) of Proposition 19, any compact operator, like F, is inside the quantum resolvent algebra on $\mathbb{T}^l \times \mathbb{R}^l$. By part (4) of Proposition 19, this implies that $F \otimes \mathbf{1} \in A_\hbar$, and hence, $U^{-1}(F \otimes \mathbf{1})U \in A_\hbar$. As $M(e_k)$ is the quantisation of $e_k \otimes \mathbf{1}_{\mathbb{R}^n}$, we find $a \in A_\hbar$. As we have seen, linearity and continuity of the Dyson series imply that $e^{\frac{itH_0}{\hbar}} e^{\frac{-itH}{\hbar}} \in A_\hbar$, and this implies the theorem itself. $\square$

6. Discussion and Outlook

We have constructed C*-algebras that extend, respectively, the commutative resolvent algebra and the Buchholz–Grundling resolvent algebra to the case that the phase space is the cotangent bundle of the torus. Here we will show that our algebras have the properties P1–P4 stated in the introduction, briefly reiterating the main results of this paper.

With regard to P1, on quantisation, the respective algebras are the classical and quantum resolvent algebras of the cylinder, $A_0 := C_{\mathcal{R}}(T^*\mathbb{T}^n)$ and $A_\hbar := C^*(\mathcal{Q}_\hbar^W(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)))$. We stress that the latter is a C*-subalgebra of $\mathcal{B}(L^2(\mathbb{T}^n))$ that is independent of $\hbar$ (cf. Proposition 19), and can easily be defined without reference to the quantisation map. Thus, one is allowed to view the quantisation map as merely a tool to help one guess an appropriate quantum algebra, with the added bonus that one quite directly obtains the correct classical limit. How well this classical limit behaves is indicated by the properties that the quantisation map satisfies.

P2, closure under time evolution, is satisfied because of Theorem 16 (classical time evolution) and Theorem 30 (quantum time evolution).

P3, or closure under embedding of algebras corresponding to subsystems, is satisfied in the classical case by Corollary 6, and, in the quantum case, in the sense of Proposition 19(4). In particular, the nice behavior with respect to 'tensoring with the identity' allows one to define a thermodynamic limit; the associated inductive limit of C*-algebras commutes with quantisation.

The final property P4 on the 'size' of the algebras, requires on the one hand that the C*-algebras are sufficiently large, in the sense that $C_0(T^*\mathbb{T}^n) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$ and $\mathcal{K}(L^2(\mathbb{T}^n)) \subseteq A_\hbar$, which follows from Theorem 5 and Proposition 19(3), in addition to them satisfying P3. On the other hand, the classical algebra has been constructed with the goal of minimality in mind, namely as the smallest C*-algebra containing the canonical embedding of each algebra of C_0-functions on the quotient of $T^*\mathbb{T}^n$ by hyperplanes of the acting group $\mathbb{R}^{2n}$. Accepting that these embeddings are 'necessary' with regard to P3, the classical algebra fulfills P4. Considering also the Weyl quantisation map as a given, this implies that the quantum algebras are no larger than necessary either, and it is in this sense that P4 is fulfilled. It is, however, an open problem whether the given algebras are minimal with respect to satisfying the aforementioned inclusions, as well as properties P1–P3.

While this paper was under review, two papers were written that may be considered follow-up papers, which demonstrate that the methods of this paper have at least two promising applications, and which offer specific suggestions for future research. Let us briefly compare the results of these papers with the present article. The first [20] deals with the continuum limit in lattice gauge theory, both classically and quantum mechanically. On the quantum side, the limit is constructed through a direct system of operator systems rather than C*-algebras, temporarily loosening the conditions on multiplicativity of embedding maps in P3. The limit object, however, is naturally closed under multiplication. Furthermore, the quantisation map between the limit algebras is a strict *deformation* quantisation in the sense of [15, Definition II.1.1.2], meaning it has better properties than the one in the current paper, specifically:

- The norm of the quantisation map depends continuously on $\hbar$ for values greater than or equal to 0, rather than only at 0;
- The quantisation map is nondegenerate, i.e. for fixed $\hbar$, the map is injective;

The second paper [21] deals with the thermodynamic limit of classical interacting particle systems and contains results on closure under time evolution, demonstrating another variant of P2.

It would be good to compare our approach with the one initiated by Rieffel [25]. As the manifold $T^*\mathbb{T}^n$ has a canonical action of $\mathbb{R}^{2n}$, there is an associated strict deformation quantisation of $C_u(T^*\mathbb{T}^n)$ as defined in [25] and extended in [2]. These algebras also behave very well under embeddings, see [26, Theorem 1.4 and Proposition 1.5]. The fact that $C_u(T^*\mathbb{T}^n)$ contains more observables than $C_\mathcal{R}(T^*\mathbb{T}^n)$ may be considered beneficial, but may at the same time complicate discussions of representations, states, etc. Due to the multiply-connectedness of the torus, the strict deformation quantisation of Rieffel has many inequivalent irreducible representations, and it is unclear how each of them naturally corresponds to a quantum system that is realised in nature. We conclude that, to relate the work in this paper to that of Rieffel, a better understanding of the irreducible representations of his algebras, preferably in relation to some natural faithful representation on some Hilbert space, is desirable.

Finally, we would like to generalise the theory to that of compact, connected Lie groups G, with the case $G = \mathrm{SU}(3)$ and powers thereof being of particular interest because of its application to QCD, as stated in the introduction. Let us sketch three possible directions for generalisations based on this paper:

- a geometric approach, similar to our original definition of the classical resolvent algebra $C_\mathcal{R}(T^*\mathbb{T}^n)$ on the torus based on functions that are constant on orbits of some action. It is, however, not so clear what this action should be in the case of more general Lie groups;
- an algebraic approach, relying on the characterisation of the resolvent algebra from Theorem 5, for instance replacing $C(\mathbb{T}^n)\hat{\otimes}\mathcal{W}^0_\mathcal{R}(\mathbb{R}^n)$ with $C(G)\hat{\otimes}\mathcal{W}^0_\mathcal{R}(\mathfrak{g}^*)$. A reasonable guess for the quantisations of the functions $(q,p) \mapsto e^{ip\cdot\xi}$ would be the operators that pull back by left translations $g \mapsto \exp(\hbar\xi)g$. However, it could very well be that the momentum part

of the algebra, $\mathcal{W}^0_{\mathcal{R}}(\mathfrak{g}^*)$, must be replaced with something else in order to satisfy P1–P4.

- a categorical approach, expanding on the idea of looking for the smallest subfunctor of C_b that accommodates embeddings of algebras that arise as pullbacks of surjective maps between phase spaces, as alluded to in the introduction. It is unclear though how to define such maps between phase spaces in a way that their pullbacks restricted to the corresponding spaces of Schwartz functions both remain physically interesting, and are Poisson with respect to the canonical Poisson structure on $C^\infty(T^*G^n)$ for non-abelian G (cf. [31, Theorem 2.9]). It also leaves the most work in terms of characterisation of the resulting algebras like the one provided for the torus in Theorem 5;

Inspiration might be taken from other sources as well, such as [22], possibly adapting the Fedosov construction and the star product constructed in that paper to the C*-algebraic setting to obtain a strict deformation quantisation. For an example of passing from formal to strict deformation quantisation, the reader can consult [27], of which the introduction contains a nice overview of various approaches to deformation quantisation.

One might also consider postponing the non-abelian generalisation, given that the torus $\mathbb{T}$ is the structure group of quantum electrodynamics (QED), of which a mathematically rigorous description is still lacking. We express the hope that the results in this paper may contribute to the development of such a theory.

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